Inference in Ising Models

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11th December, 2015

The Ising Model

- The data is a vector of dependent ± 1 random variables

$$\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N).$$

- The dependence between the coordinates of σ are modeled by a one-parameter exponential family on $S_N := \{-1, 1\}^N$

$$\mathbb{P}_\beta(\sigma = \tau) = 2^{-N} \exp\left\{\frac{1}{2}\beta H_N(\tau) - F_N(\beta)\right\}.$$

- Here
  - $\beta > 0$ is the *natural parameter (inverse temperature)*,
  - the sufficient statistic is a quadratic form:

$$H_N(\tau) = \tau' J_N \tau = \sum_{1 \le i,j \le N} J_N(i,j)\tau_i\tau_j$$

    for a symmetric matrix J_N with zeros on the diagonals.
  - $F_N(\beta)$ is the *log-normalizing constant* which is determined by the condition $\sum_{\tau \in S_N} \mathbb{P}\{\sigma = \tau\} = 1$.

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- *Maximum Likelihood Estimation*: Very difficult due to appearance of an intractable normalizing constant $F_N(\beta)$ in the likelihood.
- *Maximum Pseudo-Likelihood Estimation* (MPLE): The pseudo-likelihood is obtained by multiplying all of the conditional likelihoods (Besag 1974).
    - The distribution of σ_i given $\{\sigma_j, j \neq i\}$ takes the values $+1$ and -1 with probabilities $\frac{e^{\beta m_i}}{e^{\beta m_i}+e^{-\beta m_i}}$ and $\frac{e^{-\beta m_i}}{e^{\beta m_i}+e^{-\beta m_i}}$ respectively, where $m_i = \sum_{j=1}^{n} J_N(i,j)\sigma_j$.
    - The value of β which maximizes the pseudo-likelihood is the pseudo-likelihood estimator $\hat{\beta}_N$.

Consistency of the MPLE

Theorem (Chatterjee (2007))

If $\sup_N ||J_N|| < \infty$ *and*

$$\liminf_{N \to \infty} \frac{1}{N} F_N(\beta_0) > 0,$$

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Consistency of the MPLE: Our Results

Theorem (B.-Mukherjee (2015))

Let $\sup_{N \geq 1} ||J_N|| < \infty$, *and* $\beta_0 > 0$ *be fixed. Suppose* $\{a_N\}_{N \geq 1}$ *is a sequence of positive reals diverging to* ∞ *such that*

$$0 < \lim_{\delta \to 0} \liminf_{N \to \infty} \frac{1}{a_N} F_N(\beta_0 - \delta) \leq \lim_{\delta \to 0} \limsup_{N \to \infty} \frac{1}{a_N} F_N(\beta_0 + \delta) < \infty.$$

Then (under technical conditions) the MPLE $\hat{\beta}_N$ *is* $\sqrt{a_N}$*-consistent for* $\beta = \beta_0$.

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- (*Erdős-Renyi Graphs*) Suppose $G_N \sim \mathcal{G}(N, p_N)$ with $p_N \gg \frac{\log N}{N}$. Then $\hat{\beta}_N$ is $\sqrt{\frac{1}{p_N}}$ consistent if $\beta < 1$, and $\sqrt{N}$ consistent if $\beta > 1$.

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- (*Regular Graphs*) Suppose G_N is a sequence of d_N-regular graphs. If $d_N \to \infty$, then $\hat{\beta}_N$ is $\sqrt{\frac{N}{d_N}}$ consistent if $\beta < 1$, and $\sqrt{N}$ consistent if $\beta > 1$.

Numerical examples

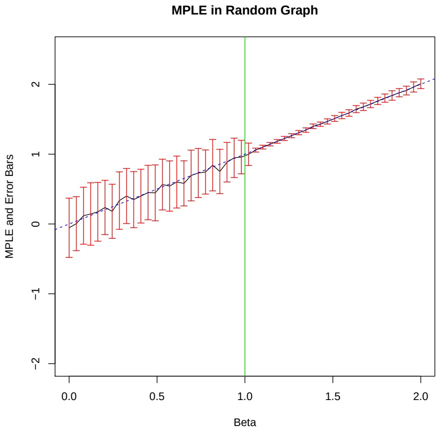

Figure: The MPLE and the 1-standard deviation error bar in an Ising model on $G_N \sim \mathcal{G}(N, p(N))$ with $N = 2000$ and $p(N) = N^{-\frac{1}{3}}$, averaged over 100 repetitions for a sequence of values of $\beta \in [0, 2]$.

When Consistent Estimation is Impossible?

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Suppose the log-normalizing constant $F_N(\beta_0) = O(1)$. *Then there does not exist any consistent sequence of estimators in the interval* $[0, \beta_0]$.

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- (*Dense Graphs*) Suppose G_N is a sequence of dense graphs converging a graphon W with maximum eigenvalue of $\lambda_1(W)$ (Borgs et al. (2008)).
  - Then $\hat{\beta}_N$ is *inconsistent* for $\beta < \frac{1}{\lambda_1(W)}$, and $\sqrt{N}$ consistent for $\beta > 1$.
  - Moreover, there exists *no sequence of consistent estimators* for $\beta < \frac{1}{\lambda_1(W)}$, where $\lambda_1(W)$ denotes the largest eigenvalue of W.

Example: Erdős-Renyi Graphs

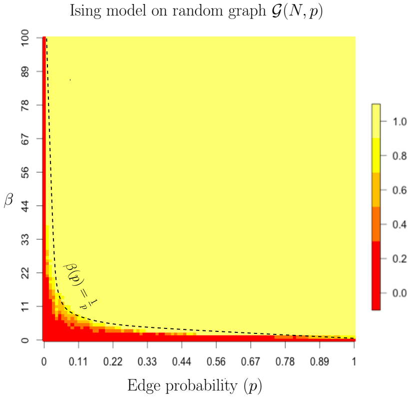
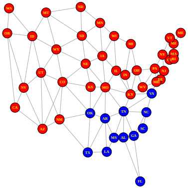


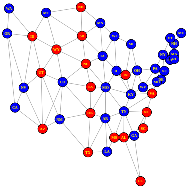
Figure: The power of the MP-test for the Ising model on an Erdős-Rényi random graph $\mathcal{G}(N, p)$ as a function of p and β, with $N = 500$. In this case, the limiting graphon is $W \equiv p$, and $\lambda_1(W) = p$. *Note the phase transition curve $\beta(p) = \frac{1}{p}$ above which the MP-test has power 1.*

US Presidential Elections Data

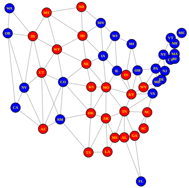
- The dataset consists of political colors of the 48 states in the continental US (excluding Alaska, Hawai, and Washington D.C.) in the last 26 presidential elections held during 1912-2012.
- Each state is assigned 1 for Democratic (colored blue) and -1 for Republican (colored red).
- The vertices of the *neighborhood graph* are the states and there is an edge between two states if they share a border.



1924 Elections: $\hat{\beta} = 2.84$



1992 Elections: $\hat{\beta} = 0.37$



2012 Elections: $\hat{\beta} = 0.96$

Thank

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