# Guaranteed Non-convex Machine Learning Using Tensor Methods 

## Anima Anandkumar


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## Regime of Modern Machine Learning

Massive datasets, growth in computation power, challenging tasks

## Success of Supervised Learning

- Learn $p(y \mid x)$ from labeled samples $\left\{\left(x_{i}, y_{i}\right)\right\}$.
- Extract relevant features from large amounts of labeled data.


Image classification


Speech recognition


Text processing

## Regime of Modern Machine Learning

Massive datasets, growth in computation power, challenging tasks

## Missing Link in AI: Unsupervised Learning

- Learn $p(x)$ from unlabeled samples $\left\{x_{i}\right\}$.
- Discover latent variables related to observed variable $x$.
- Human vs. Machine Learning: Make discoveries automatically.



## Unsupervised Learning via Probabilistic Models

Data $\rightarrow$ Model $\rightarrow$ Learning Algorithm $\rightarrow$ Predictions


Challenges in High dimensional Learning

- Dimension of $x \gg \mathrm{dim}$. of latent variable $h$.
- Learning is like finding needle in a haystack.

- Computationally \& statistically challenging.


## Overview of Unsupervised Learning Methods

$$
\text { Goal: learn model parameters } \theta \text { from observations } x \text {. }
$$

- Maximum likelihood: $\max _{\theta} p(x ; \theta)$.
- Non-convex: stuck in local optima.
- Curse of dimensionality: Exponential
 no. of critical points.
- Heuristics: Expectation Maximization, Variational Inference ....
- Other mechanisms such as Generative Adversarial Nets also non-convex.


## Guaranteed Learning through Tensor Methods

Replace the objective function Max Likelihood vs. Best Tensor decomp.


Preserves Global Optimum (infinite samples)
$\arg \max _{\theta} p(x ; \theta)=\arg \min _{\theta}\|\widehat{T}(x)-T(\theta)\|_{\mathbb{F}}^{2}$
$\widehat{T}(x)$ : empirical tensor, $T(\theta)$ : low rank tensor based on $\theta$.

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Finding globally opt tensor decomposition
Simple algorithms succeed under mild and natural conditions for many learning problems.

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Dataset 1

## Outline

(1) Introduction
(2) Tensor Decomposition Algorithms
(3) Tensors for Probabilistic Models

4 Tensors in Deep Learning
(5) Steps Forward

## Matrix Decomposition: Discovering Latent Factors



- List of scores for students in different tests
- Learn hidden factors for Verbal and Mathematical Intelligence [C. Spearman 1904]

$$
\begin{aligned}
\text { Score }(\text { student }, \text { test })= & \text { student }_{\text {verbal-intlg }} \times \text { test }_{\text {verbal }} \\
& + \text { student }_{\text {math-intlg }} \times \text { test }_{\text {math }}
\end{aligned}
$$

## Matrix Decomposition: Discovering Latent Factors



- Identifying hidden factors influencing the observations
- Characterized as matrix decomposition


## Matrix Decomposition: Discovering Latent Factors



- Decomposition is not necessarily unique.
- Decomposition cannot be overcomplete.


## Tensor: Shared Matrix Decomposition



- Shared decomposition with different scaling factors
- Combine matrix slices as a tensor


## Tensor Decomposition



- Outer product notation:

$$
\begin{aligned}
& T=u \otimes v \otimes w+\tilde{u} \otimes \tilde{v} \otimes \tilde{w} \\
& \hat{\mathbb{y}} \\
& T_{i_{1}, i_{2}, i_{3}}=u_{i_{1}} \cdot v_{i_{2}} \cdot w_{i_{3}}+\tilde{u}_{i_{1}} \cdot \tilde{v}_{i_{2}} \cdot \tilde{w}_{i_{3}}
\end{aligned}
$$

## Tensor Decomposition



Uniqueness of Tensor Decomposition [J. Kruskal 1977]

- Above tensor decomposition: unique when rank one pairs are linearly independent
- Matrix case: when rank one pairs are orthogonal


## Tensor Decomposition



Finding Best Tensor Decomposition? Overcome Non-convexity?

## Notion of Tensor Contraction

Extends the notion of matrix product

Matrix product
$M v=\sum_{j} v_{j} M_{j}$


Tensor Contraction

$$
T(u, v, \cdot)=\sum_{i, j} u_{i} v_{j} T_{i, j,:}
$$



## Symmetric Tensor Decomposition


A., R. Ge, D. Hsu, S. Kakade, M. Telgarsky, "Tensor Decompositions for Learning Latent Variable Models," JMLR 2014.

## Symmetric Tensor Decomposition

## Tensor Power Method

$$
v \mapsto \frac{T(v, v, \cdot)}{\|T(v, v, \cdot)\|}
$$



$$
T(v, v, \cdot)=\left\langle v, v_{1}\right\rangle^{2} v_{1}+\left\langle v, v_{2}\right\rangle^{2} v_{2}
$$

A., R. Ge, D. Hsu, S. Kakade, M. Telgarsky, "Tensor Decompositions for Learning Latent Variable Models," JMLR 2014.

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## Orthogonal Tensors

- $\vec{v}_{1} \perp \vec{v}_{2}$.
- $T\left(v_{1}, v_{1}, \cdot\right)=\lambda_{1} v_{1}$.

A., R. Ge, D. Hsu, S. Kakade, M. Telgarsky, "Tensor Decompositions for Learning Latent Variable Models," JMLR 2014.


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## Tensor Power Method

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## Symmetric Tensor Decomposition

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Exponential no. of stationary points for power method: $T(v, v, \cdot)=\lambda v$
A., R. Ge, D. Hsu, S. Kakade, M. Telgarsky, "Tensor Decompositions for Learning Latent Variable Models," JMLR 2014.

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Exponential no. of stationary points for power method:
$T(v, v, \cdot)=\lambda v \quad$ Stable


Other statitionary points

A., R. Ge, D. Hsu, S. Kakade, M. Telgarsky, "Tensor Decompositions for Learning Latent Variable Models," JMLR 2014.

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Exponential no. of stationary points for power method: $T(v, v, \cdot)=\lambda v$

For power method on orthogonal tensor, no spurious stable points
A., R. Ge, D. Hsu, S. Kakade, M. Telgarsky, "Tensor Decompositions for Learning Latent Variable Models," JMLR 2014.

## Non-orthogonal Tensor Decomposition


A., R. Ge, D. Hsu, S. Kakade, M. Telgarsky, "Tensor Decompositions for Learning Latent Variable Models," JMLR 2014.

## Non-orthogonal Tensor Decomposition

Orthogonalization


Input tensor $T$
A., R. Ge, D. Hsu, S. Kakade, M. Telgarsky, "Tensor Decompositions for Learning Latent Variable Models," JMLR 2014.

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## Non-orthogonal Tensor Decomposition

Orthogonalization

$\tilde{T}=T(W, W, W)={\tilde{v_{1}}}^{\otimes 3}+{\tilde{v_{2}}}^{\otimes 3}+\cdots$,

A., R. Ge, D. Hsu, S. Kakade, M. Telgarsky, "Tensor Decompositions for Learning Latent Variable Models," JMLR 2014.

## Non-orthogonal Tensor Decomposition

Orthogonalization


Find $W$ using SVD of Matrix Slice

$$
M=T(\cdot, \cdot, \theta)=
$$


A., R. Ge, D. Hsu, S. Kakade, M. Telgarsky, "Tensor Decompositions for Learning Latent Variable Models," JMLR 2014.

## Non-orthogonal Tensor Decomposition

Orthogonalization


Orthogonalization: invertible when $v_{i}$ 's linearly independent.

Guaranteed tensor decomposition: when $v_{i}$ 's linearly independent.
A., R. Ge, D. Hsu, S. Kakade, M. Telgarsky, "Tensor Decompositions for Learning Latent Variable Models," JMLR 2014.

## Outline

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4 Tensors in Deep Learning
(5) Steps Forward

## Extracting Topics from Documents


A., D. P. Foster, D. Hsu, S.M. Kakade, Y.K. Liu. "Two SVDs Suffice: Spectral decompositions for probabilistic topic modeling and latent Dirichlet allocation," NIPS 2012.

## Tensor Methods for Topic Modeling


campus
police
witness

- Topic-word matrix $\mathbb{P}[$ word $=i \mid$ topic $=j]$
- Linearly independent columns

Moment Tensor: Co-occurrence of Word Triplets


## Extracting Communities in Social Networks



Moment Tensor: Common Friends among Node Triplets

A., R. Ge, D. Hsu, S.M. Kakade. "A Tensor Spectral Approach to Learning Mixed Membership Community Models" COLT 2013.

## Tensors vs. Variational Inference

Criterion: Perplexity $=\exp [-$ likelihood].
Learning Topics from PubMed on Spark, 8mil articles



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Criterion: Perplexity $=\exp [-$ likelihood].
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Learning network communities on single workstation Facebook $n \sim 20 k$, Yelp $n \sim 40 k$, DBLP-sub $n \sim 1 e 5$, DBLP $n \sim 1 e 6$.


F. Huang, U.N. Niranjan, M. Hakeem, A, "Online tensor methods for training latent variable models," JMLR 2014.

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## Learning Representations

## Sparse coding prevalent in neural signaling.


A. Agarwal, A, P. Jain, P. Netrapalli, "Learning Sparsely Used Overcomplete Dictionaries," COLT 2014.

A, M. Janzamin, R. Ge, "Overcomplete Tensor Decomposition, " COLT 2015.
Huang, A., "Convolutional Dictionary Learning through Tensor Factorization", Proc. of JMLR 2015.

## Learning Representations

## Sparse coding prevalent in neural signaling.

Neural sparse coding [Papadopoulou11]


## Linear Model with Overcomplete Dictionary


A. Agarwal, A, P. Jain, P. Netrapalli, "Learning Sparsely Used Overcomplete Dictionaries," COLT 2014.

A, M. Janzamin, R. Ge, "Overcomplete Tensor Decomposition, " COLT 2015.
Huang, A., "Convolutional Dictionary Learning through Tensor Factorization", Proc. of JMLR 2015.

## Learning Representations

Contribution: learn overcomplete incoherent dictionaries


## Learning Representations

## Shift-invariant Dictionary


A. Agarwal, A, P. Jain, P. Netrapalli, "Learning Sparsely Used Overcomplete Dictionaries," COLT 2014.

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## Learning Representations

## Efficient Tensor Decomposition with Shifted Components



## Shift-invariant Dictionary



Image

## Convolutional Model


A. Agarwal, A, P. Jain, P. Netrapalli, "Learning Sparsely Used Overcomplete Dictionaries," COLT 2014.

A, M. Janzamin, R. Ge, "Overcomplete Tensor Decomposition, " COLT 2015.
Huang, A., "Convolutional Dictionary Learning through Tensor Factorization", Proc. of JMLR 2015.

## Fast Text Embeddings through Tensor Methods



Word Embeddings

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Word Embeddings


Sentence Embeddings

## Fast Text Embeddings through Tensor Methods

Paraphrase Detection on MSR corpus with $\sim 5000$ Sentences


## Fast Text Embeddings through Tensor Methods

Paraphrase Detection on MSR corpus with $\sim 5000$ Sentences

| Method | F score | No. of samples |
| :--- | :--- | :--- |
| Vector Similarity (Baseline) | $75 \%$ | $\sim 4 k$ |
| Tensor (Proposed) | $\mathbf{8 1 \%}$ | $\sim 4 k$ |
| Skipthought (RNN) | $82 \%$ | $\sim 74 \mathrm{mil}$ |

- Unsupervised learning of embeddings.
- No outside info for tensor vs. large book corpus ( 74 million) for skipthought
- Similar story with holographic embeddings for knowledge bases by M. Nickel et al.


## Reinforcement Learning of Partially Observable Markov Decision Process

Learning in Adaptive Environments

- Learner changes environment
- Hidden state estimation.



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Partially Observable Markov Decision Process

- Design of tensor algorithms under memoryless policies
- Guaranteed regret bounds: comparable to fully observed



## Reinforcement Learning of Partially Observable Markov Decision Process

Playing Atari Game


Average Reward vs. Time.

K. Azizzadenesheli, A. Lazaric, A, "Reinforcement Learning of POMDPs using Spectral Methods," 2016.

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K. Azizzadenesheli, A. Lazaric, A, "Reinforcement Learning of POMDPs using Spectral Methods," 2016.

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## Local Optima in Backpropagation

"..few researchers dare to train their models from scratch.. small miscalibration of initial weights leads to vanishing or exploding gradients.. poor convergence..*"


Exponential (in dimensions) no. of local optima for backpropagation

[^0]
## Moments of a Neural Network

$$
\mathbb{E}[y \mid x]:=f(x)=\left\langle a_{2}, \sigma\left(A_{1}^{\top} x\right)\right\rangle
$$


"Score Function Features for Discriminative Learning: Matrix and Tensor Framework" by M. Janzamin, H. Sedghi, and A. , Dec. 2014.

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Moments using score functions $\mathcal{S}(\cdot)$

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$$
\mathbb{E}\left[y \cdot \mathcal{S}_{1}(x)\right]=\square \quad+\square
$$


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Moments using score functions $\mathcal{S}(\cdot)$


Given input pdf $p(\cdot), \mathcal{S}_{m}(x):=(-1)^{m} \frac{\nabla^{(m)} p(x)}{p(x)}$.
Gaussian $x \Rightarrow$ Hermite polynomials.

"Score Function Features for Discriminative Learning: Matrix and Tensor Framework" by M. Janzamin, H. Sedghi, and A., Dec. 2014.

## Tensorizing Neural Networks

- Multi-linear representation of dense layers of CNNs.
- Tensor train format for low rank approximation of weight matrix.
- Compact representation: solves memory problem.

$$
\begin{aligned}
& Y\left(i_{1}, i_{2} \ldots\right)= \\
& \sum_{j_{1}, j_{2} \ldots} G\left(i_{1}, j_{1}\right) G\left(i_{2}, j_{2}\right) \ldots X\left(j_{1}, j_{2} \ldots\right)
\end{aligned}
$$



Results on ImageNet

- Compression rate 200, 000!
- Negligible performance loss.
A. Novikov, D. Podoprikhin, A. Osokin, D. Vetrov, "Tensorizing Neural Networks", NIPS 2015.


## Tensor Analysis for Expressive Power

- Hierarchical Tucker tensors for representing arithmetic conv nets.
- Employs locality, sharing and pooling.
- Exponentially more parameters in shallow net vs. deep net.

First level


Second level

N. Cohen, O. Sharir, A. Shashua, "On the Expressive Power of Deep Learning: A Tensor Analysis" COLT 2016.

# Tensors in Memory Embeddings <br> Human Memory Model. Semantic decoding through Tensor Tucker. 


V. Tresp, C. Esteban, Y. Yang, S. Baier and D. Krompab, "Learning with Memory Embeddingss " 0015

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## Scaling up and Deploying Tensor Methods

## Scaling up

- Dimensionality reduction through sketching.
- Communication efficient methods.


Deployment

- Multi-platform support: CPU, GPU, Cloud, FPGA ...
- Extended BLAS kernels: Beyond linear algebra.
- Many deep learning operations involve tensor contractions.

Wang, Tung, Smola, A. " Guaranteed Tensor Decomposition via Sketching", NIPS'15. Cecka, Niranjan, Shi, A," Tensor Contractions with Extended BLAS kernels on CPU and GPU", under preparation.

## Innovations in Non-Convex Methods

Smoothing and Continuation Methods

- Global approach vs. local search.
- Unified guarantees for non-convex problems?

H. Mobahi, "Training RNNs by Diffusion" .


## Innovations in Non-Convex Methods

Learning to add using RNN


H. Mobahi, "Training RNNs by Diffusion".

## Innovations in Non-Convex Methods

- Escaping saddle points in high dimensions?
- Can SGD escape in bounded time?
- Degeneracy of saddle points in various non-convex problems?


Saddle points


Efficient approaches for escaping higher order saddle points in non-convex optimization by A., R. Ge, COLT 2016.

## Innovations in Non-Convex Methods

## Contribution: First method to escape third order saddle

- Escaping saddle points in high dimensions?
- Can SGD escape in bounded time?
- Degeneracy of saddle points in various non-convex problems?


Saddle points


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## Research Connections and Resources

Collaborators
Jennifer Chayes, Christian Borgs, Prateek Jain, Alekh Agarwal \& Praneeth Netrapalli (MSR), Srinivas Turaga (Janelia), Michael Hawrylycz \& Ed Lein (Allen Brain), Allesandro Lazaric (Inria), Alex Smola (CMU), Rong Ge (Duke), Daniel Hsu (Columbia), Sham Kakade (UW), Hossein Mobahi (MIT).


- Podcast/lectures/papers/software available at http://newport.eecs.uci.edu/anandkumar/


[^0]:    P. Krahenbhl, C. Doersch, J. Donahue, T. Darrell "Data-dependent Initializations of Convolutional Neural Networks", ICLR 2016.

