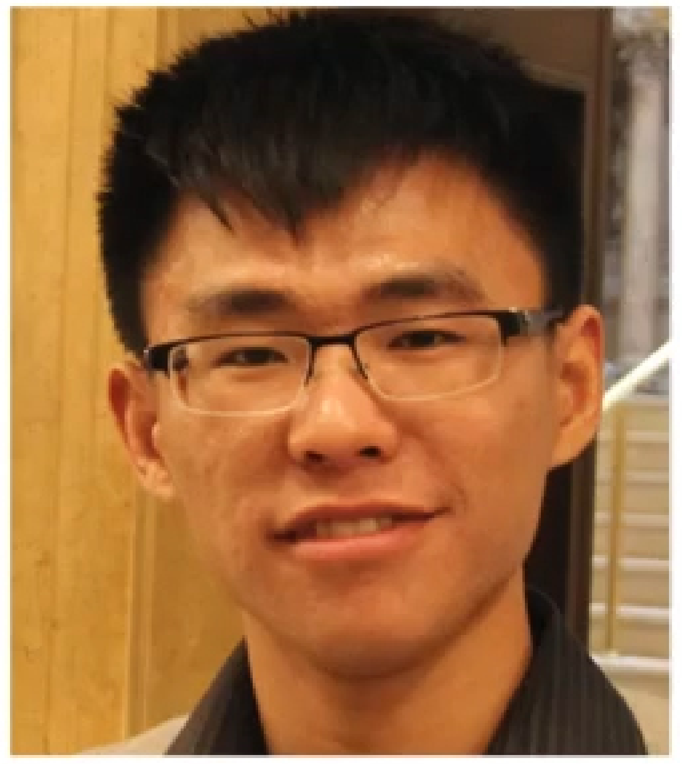
Complexity of Discrete Energy Minimization Problems



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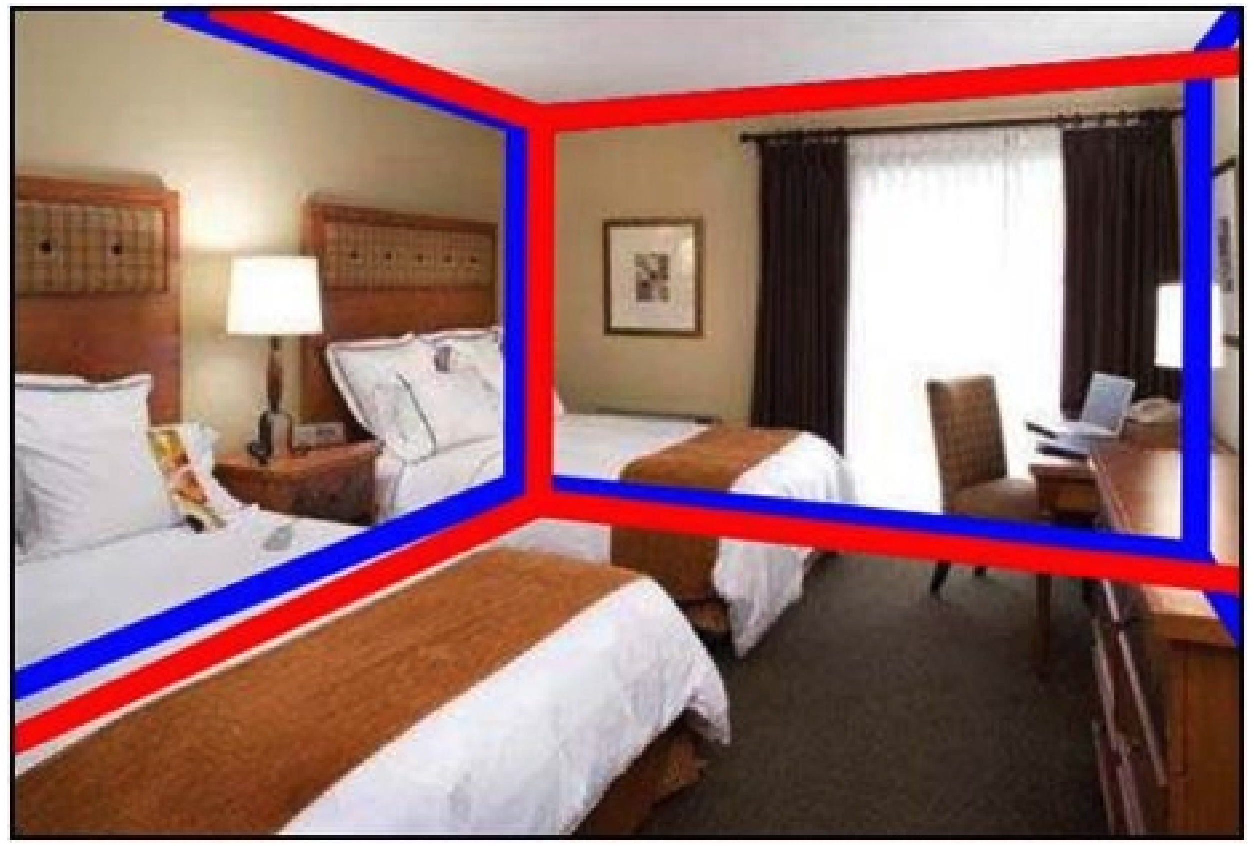
Energy Minimization

- Used in many CV applications



3D Room Layout
[Schwing and Urtasun, 2012]

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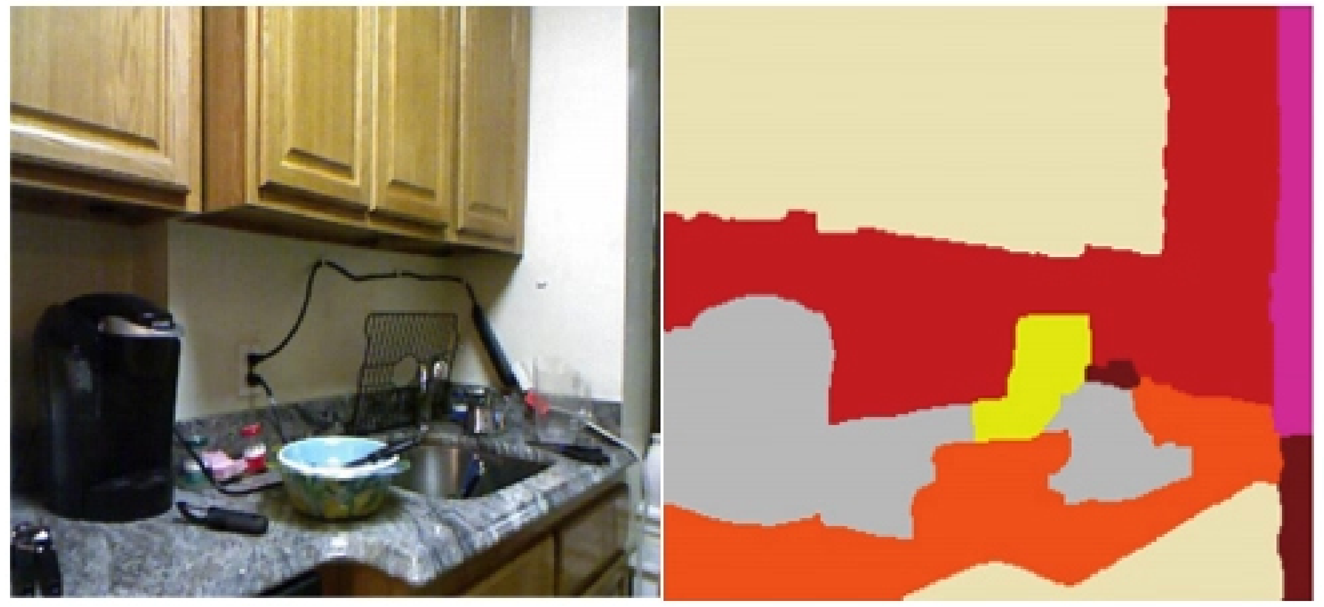

Semantic Segmentation
[Ren et al, 2012]

Energy Minimization

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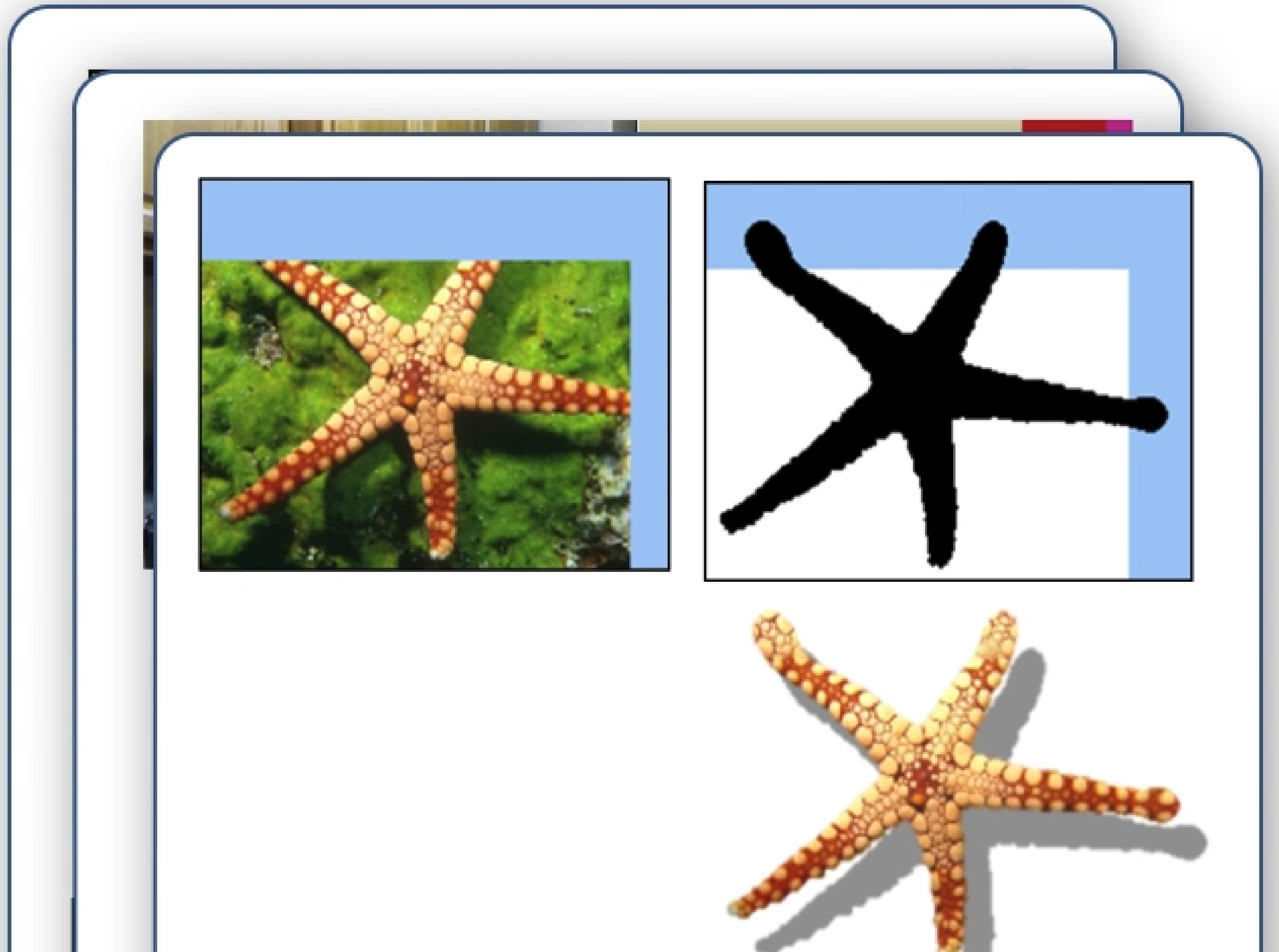


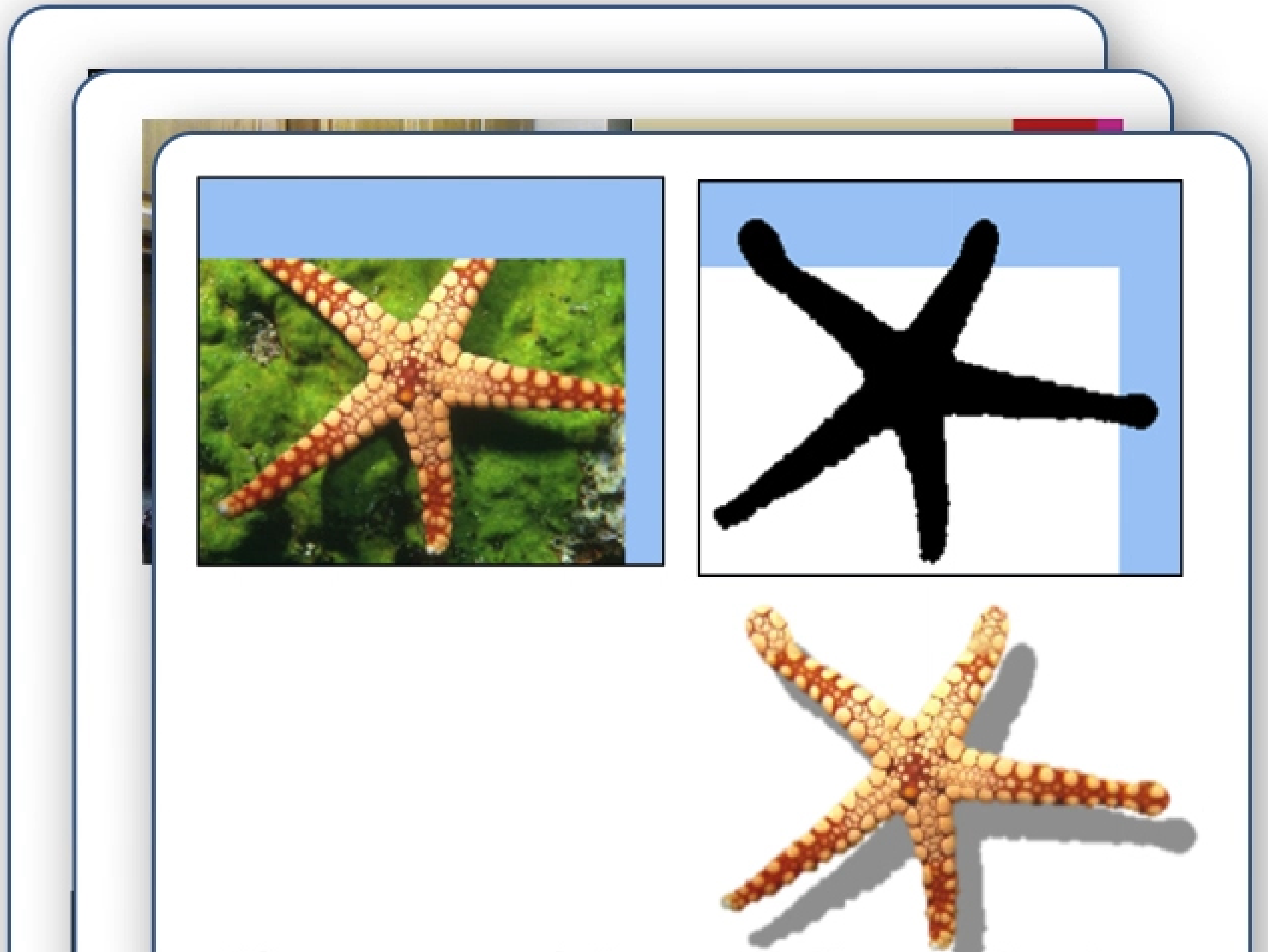
Shape and Image In-painting
[Shekhovtsov et al, 2012]

Energy Minimization

- Used in many CV applications
- Pairwise and higher order

$$\min_{x \in \mathcal{L}^{\mathcal{V}}} \sum_{u \in \mathcal{V}} f_u(x_u) + \sum_{(u,v) \in \mathcal{E}} f_{uv}(x_u, x_v)$$

- But is it efficient?



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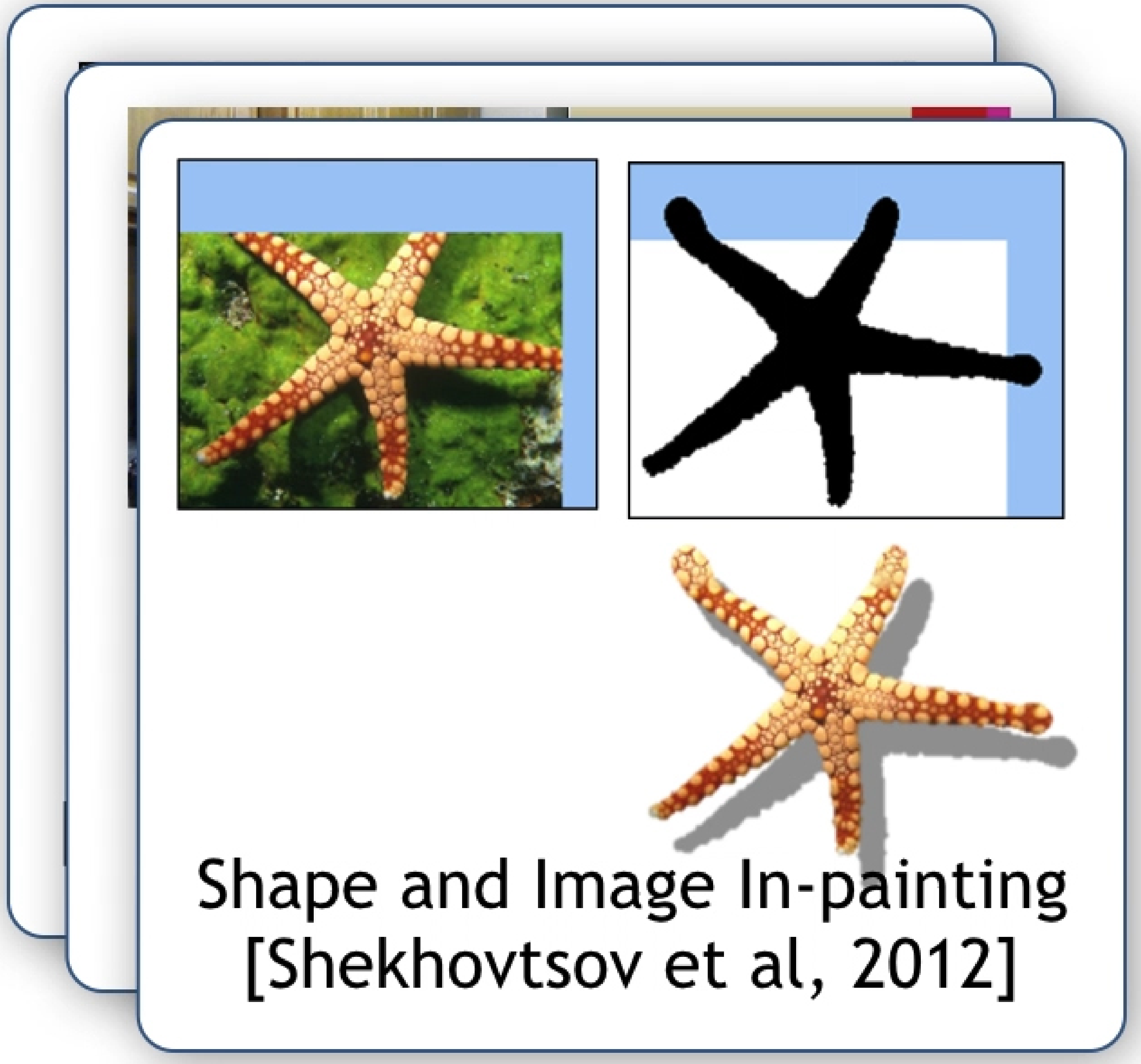
- In general, NP-hard
- There are tractable classes



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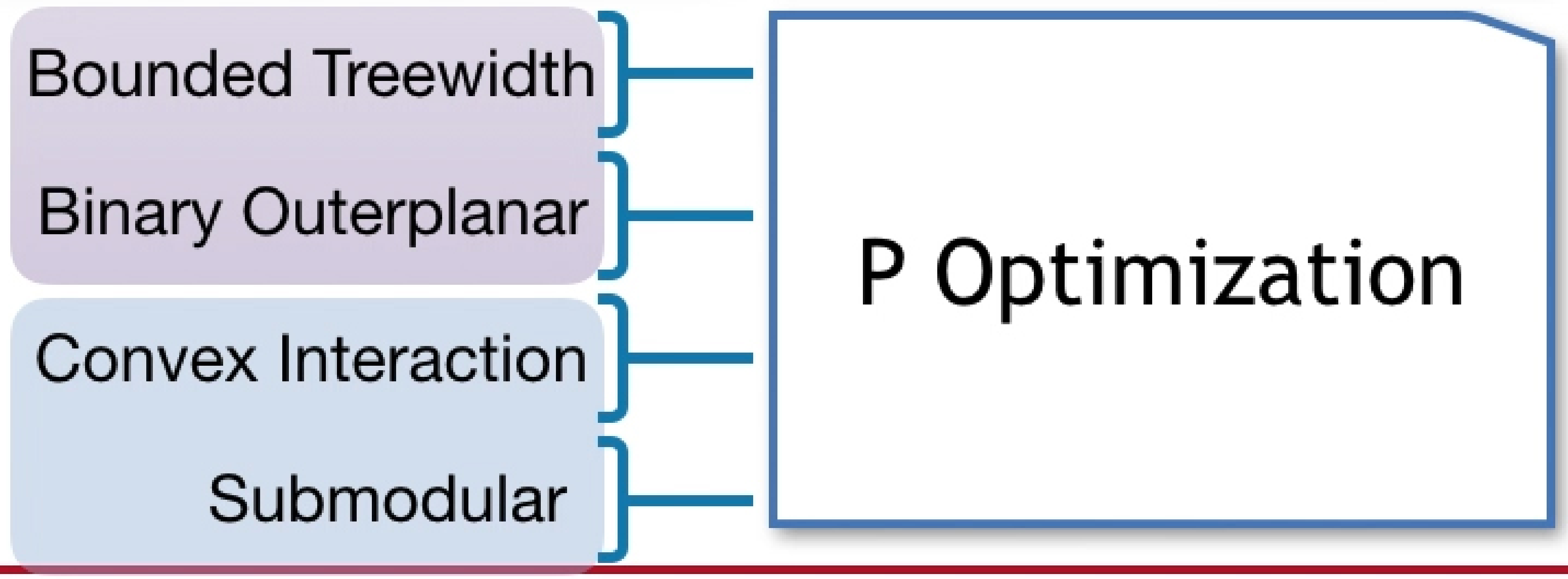

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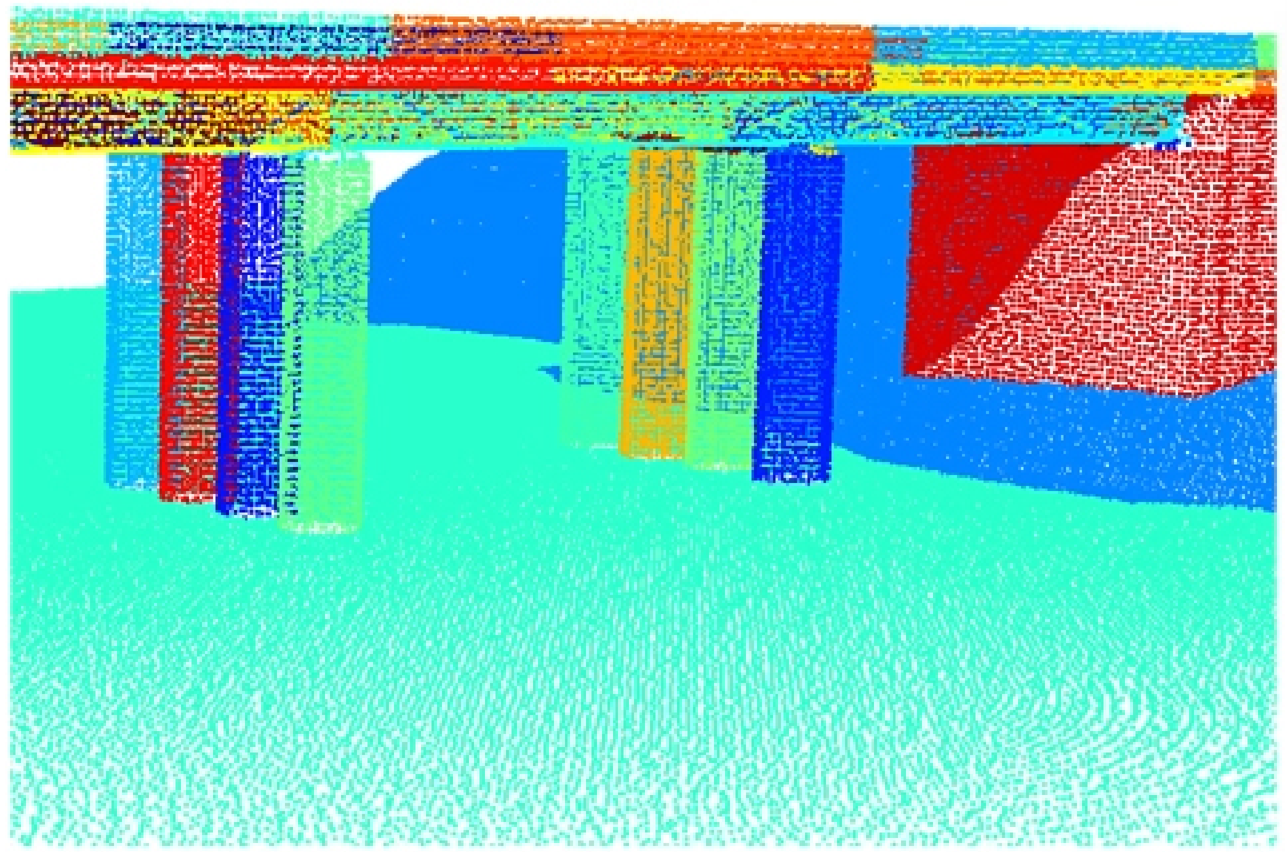
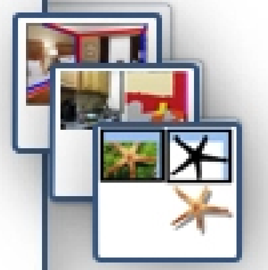
- There are tractable classes
- Growing corpus of general "approximate" inference methods
- Learning Project with QPBO



3D scene labeling



NP-hard

Bounded Treewidth

Binary Outerplanar

Convex Interaction

Submodular

P Optimization

Energy Minimization

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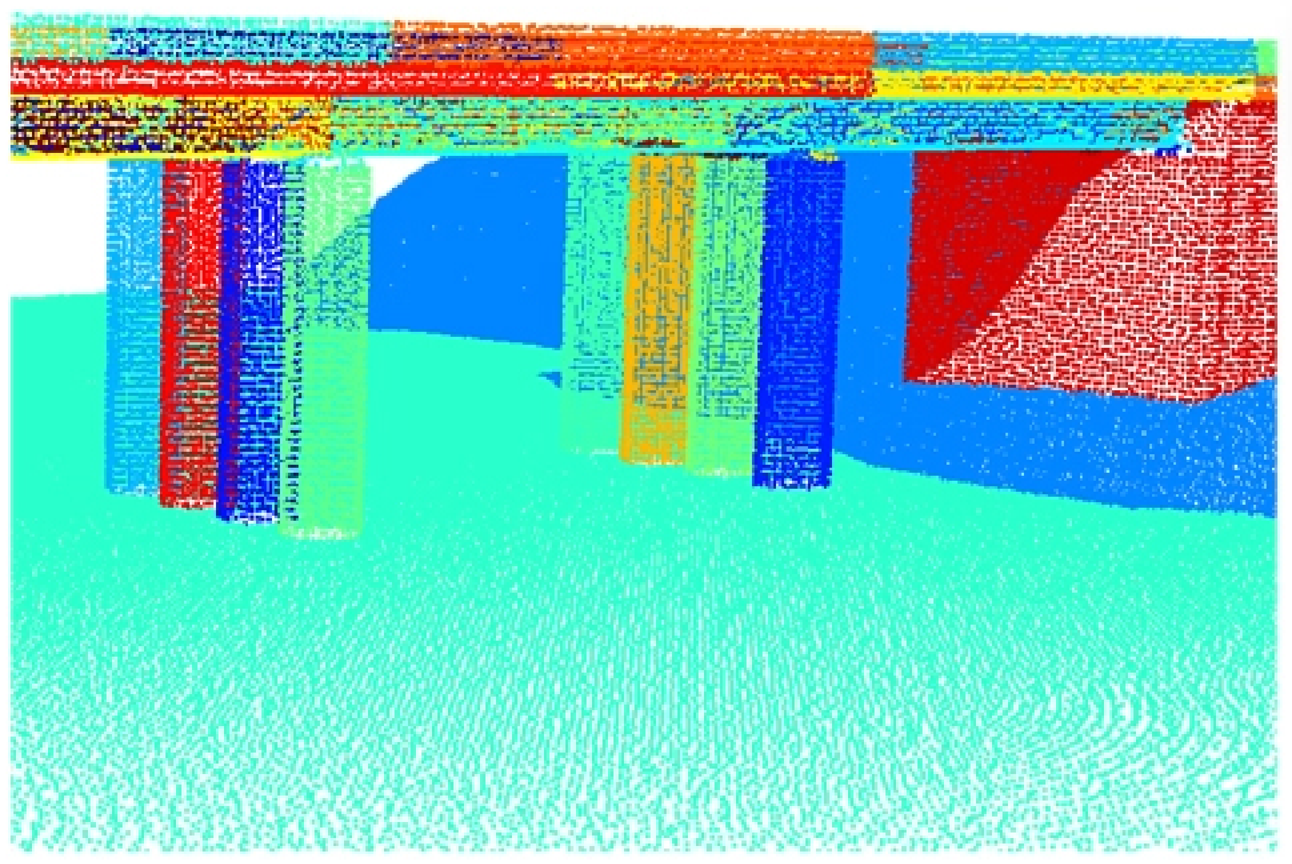
$$\min_{x \in \mathcal{L}^{\mathcal{V}}} \sum_{u \in \mathcal{V}} f_u(x_u) + \sum_{(u,v) \in \mathcal{E}} f_{uv}(x_u, x_v)$$

Quadratic Pseudo-Boolean Optimization (QPBO):

$$\min_{x \in \{0,1\}^{\mathcal{V}}} \sum_{u \in \mathcal{V}} a_u x_u + \sum_{(u,v) \in \mathcal{E}} a_{uv} x_u x_v$$

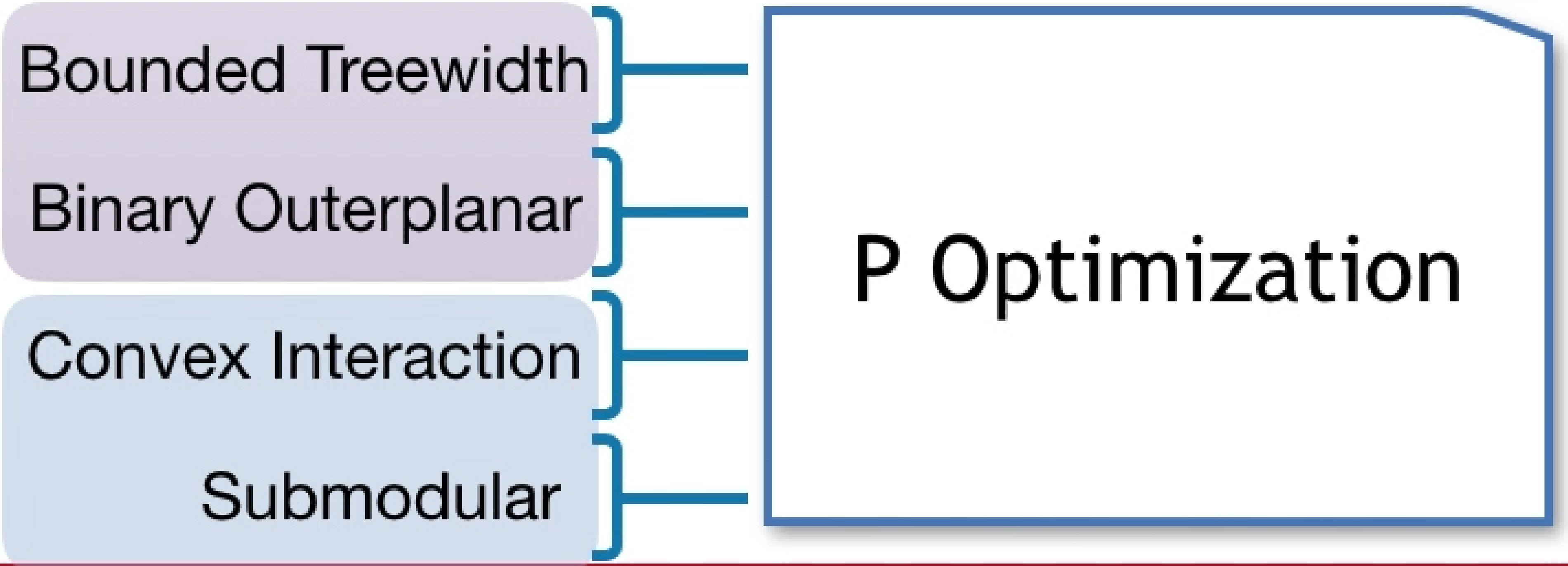
rence methods

- Learning Project with QPBO
  - Difficult instances during learning



3D scene labeling

NP-hard



Energy Minimization

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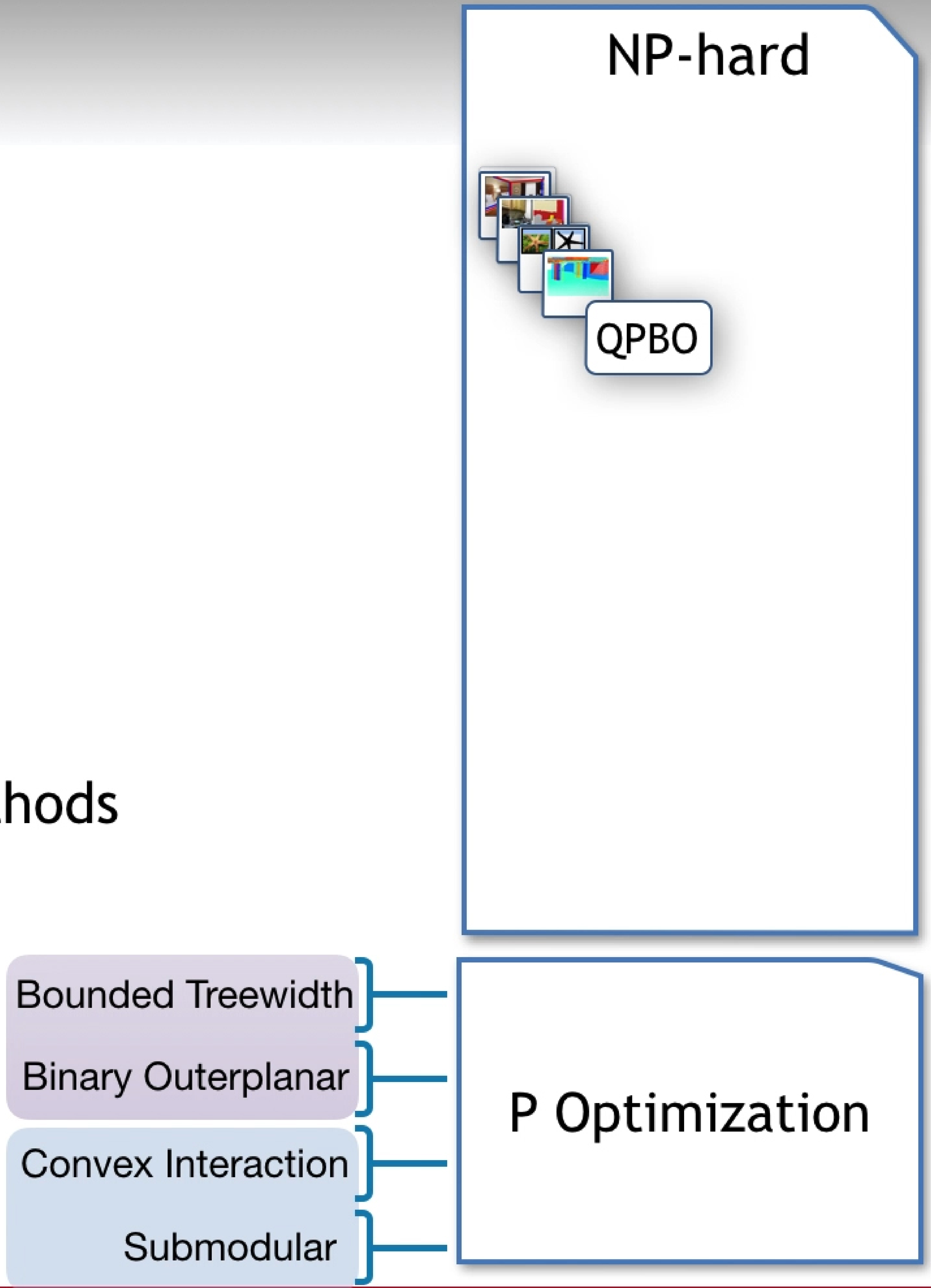
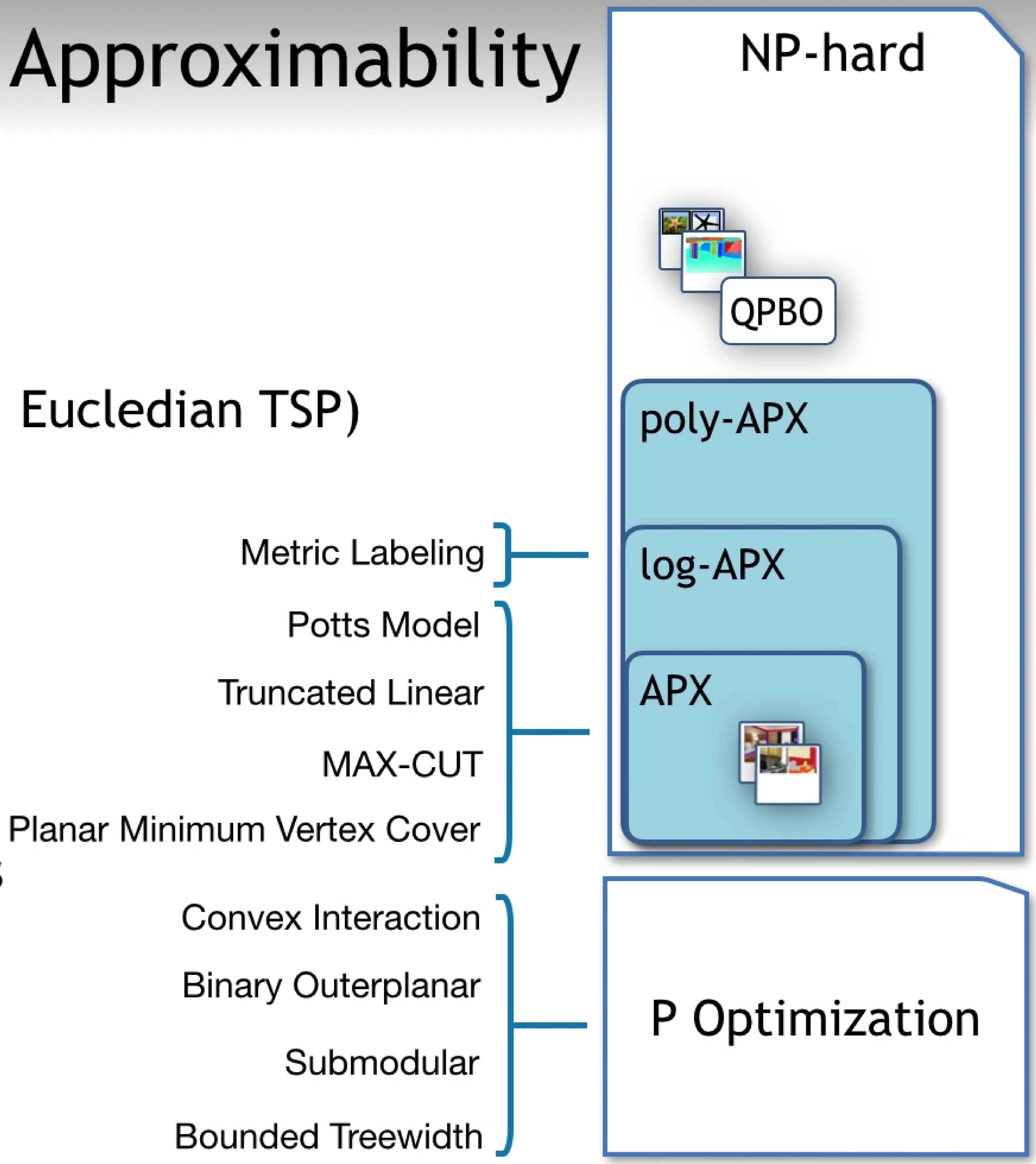
$$\min_{x \in \mathcal{L}^{\mathcal{V}}} \sum_{u \in \mathcal{V}} f_u(x_u) + \sum_{(u,v) \in \mathcal{E}} f_{uv}(x_u, x_v)$$

- But is it efficient?
- In general, NP-hard
- There are tractable classes
- Growing corpus of general “approximate” inference methods
- Learning Project with QPBO
    - Difficult instances during learning
    - Could benefit from approximation guarantees
- <u>**We find QPBO and general energy minimization to be inapproximable**</u>



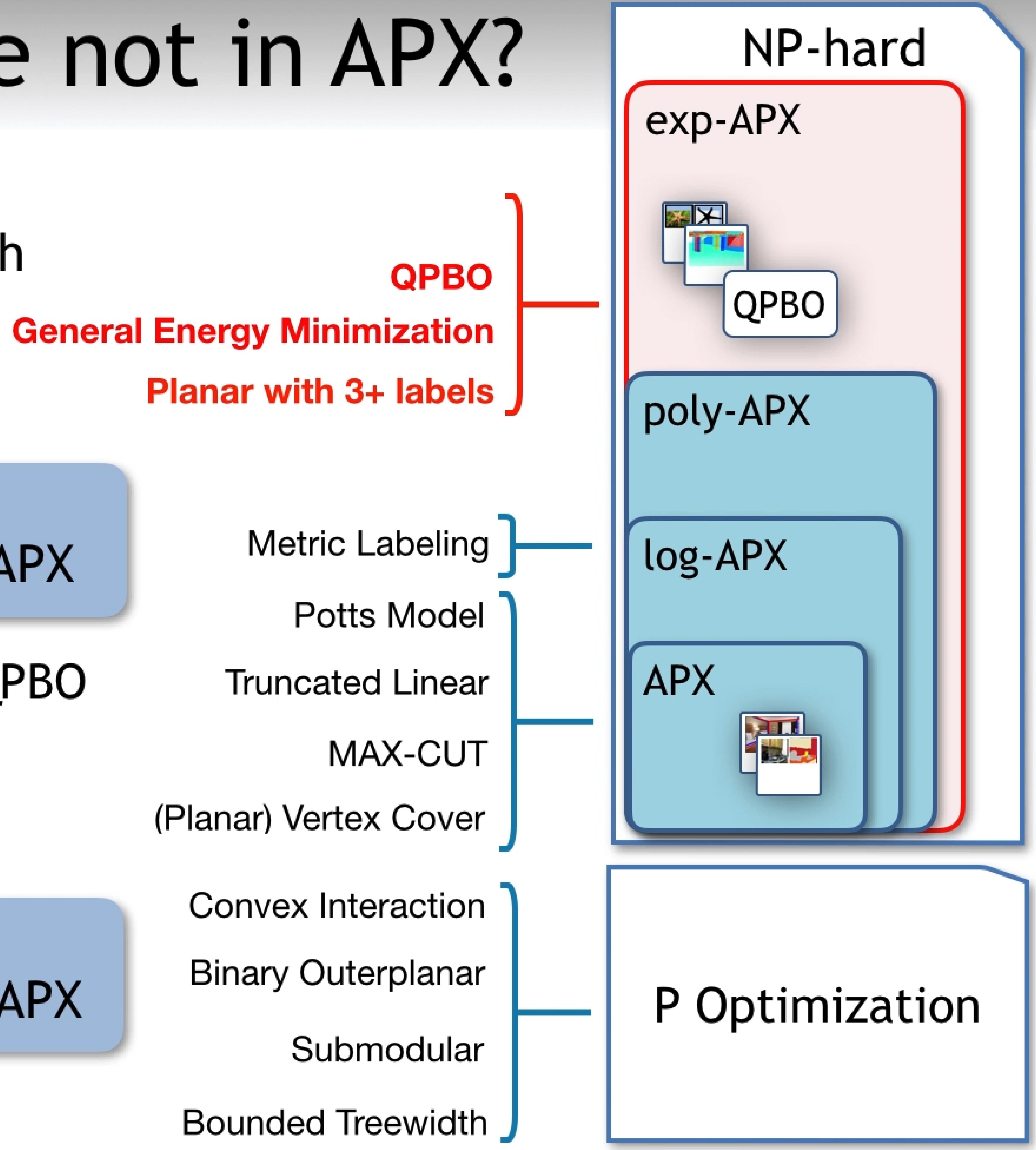
NP-hard Problems Vary Greatly in Approximability

- Approximation Ratio: $f(x)/f(x^*), \quad f(x^*) > 0$
- Classes of approximation:
  - PTAS - ratio 1+ε in polynomial time (knapsak, Eucledian TSP)
  - APX - constant approximation ratio
  - log-APX - logarithmic in bit-length
  - poly-APX - polynomial
- APX / log-APX indicate more practical clases
  - Algorithms can build on achieving guarantees
  - Much better ratios per instance
- APX-hard is not too bad! How hard is QPBO?



Which Energy Problems are not in APX?

- class **exp-APX**:
  - Approximation ratio exponential in bit length
  - Suffices to find **any** feasible solution



Theorem:
QPBO (energy with 2 labels) is **complete** in exp-APX

- Any problem from exp-APX can be reduced to QPBO
- In polynomial time
- While preserving approximation ratio

Theorem:
Planar energy with 3+ labels is **complete** in exp-APX

Proof Scheme

Turing Machine

↓ AP-reduction [Opronen & Mannila 09]

Weighted 3-SAT

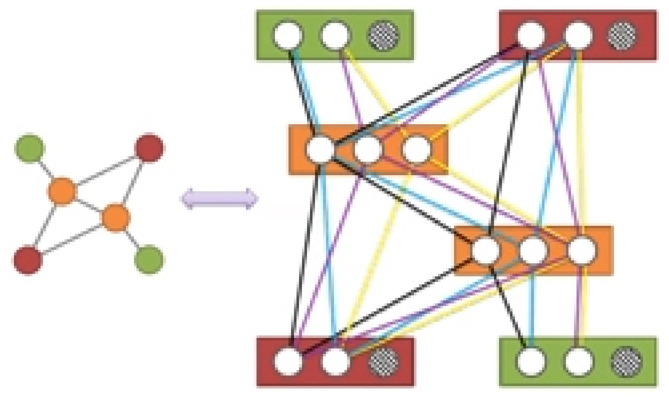
↓ AP-reduction (using Ishikawa's reduction)

QPBO

↓ AP-reduction

Planar Energy with 3 labels

(reduction gadgets)

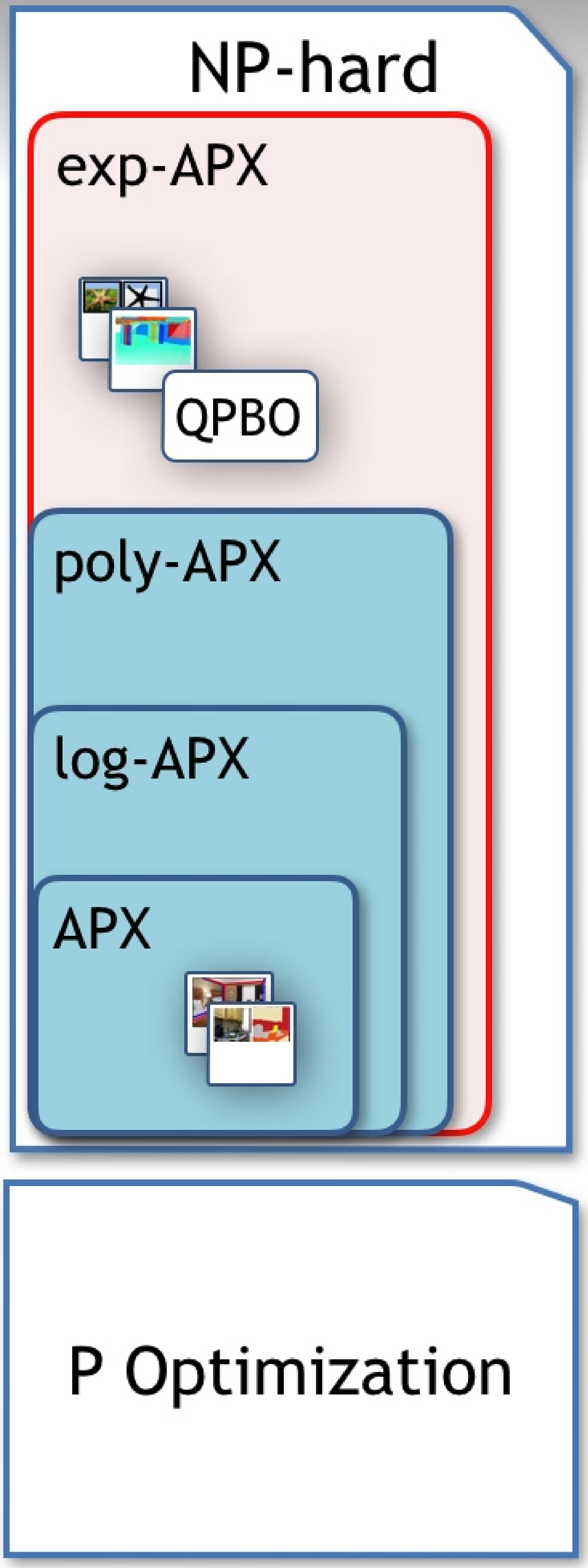


Weighted 3-SAT
QPBO
General Energy Minimization
Planar with 3+ labels

Metric Labeling

Potts Model
Truncated Linear
MAX-CUT
Planar Vertex Cover

Convex Interaction
Binary Outerplanar
Submodular
Bounded Treewidth



Take Away Message

- Energy minimization problems vary in approximation ratio
- Bounded approximation ratio
  - Indicates a class of practical interest
  - Useful for algorithm design (Primal-Dual)
- Do not try to prove approximation guarantee if
  - Model includes QPBO/ planar 3-label / general energy minimization
  - Or you can build AP-reduction from them