Lesson No. 3: Graphs continued

1. Graph colorings
   - Vertex-colorings (Brook's theorem, Mycielski's construction)
   - Edge-colorings (Vizing's theorem, König's theorem, snarks)
2. Matchings (Hall's theorem)

Vertex-colorings of graphs

- Let $\mathrm{G} = (V, E)$ be a graph and C a set of "colors".

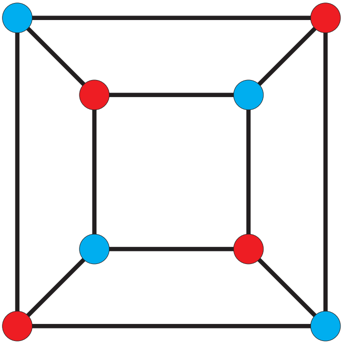
Definition

A **vertex-coloring** (barvanje točk) of G is a function $c \colon V \to C$. The coloring is **proper** (pravilno) if $u \sim v \Rightarrow (v) \neq c(u)$. G is k-vertex-colorable (točkovno k-obarvljiv) if there exists a proper vertex-coloring with $|C| = k$.

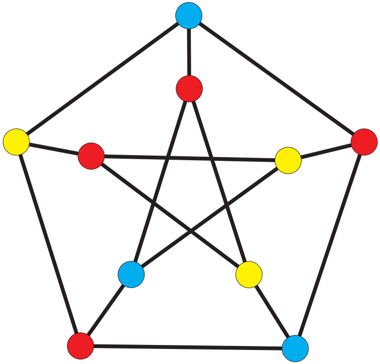
- G is k-colorable $\Rightarrow$ G is ℓ-colorable for any $\ell \geq k$.
- **Chromatic number** (kormatično število) of G:

$$\chi(\mathrm{G}) = \min\{k : \mathrm{G} \text{ is } k\text{-vertex-colorable}\}$$

Vertex-colorings – examples



2-coloring of the cube.



3-coloring of the Petersen.

… more examples

- $\chi(K_n) = n$.
- $\chi(C_{2n}) = 2$
- $\chi(C_{2n+1}) = 3$.
- If $\mathrm{H} \leq \mathrm{G}$, then $\chi(\mathrm{H}) \leq \chi(\mathrm{G})$.

Corollary

If G contains a cycle of odd length, then $\chi(\mathrm{G}) \geq 3$.

Graphs with $\chi \leq 2$

- Clearly, $\chi(\mathrm{G}) = 1$ if and only if $\mathrm{G} \cong K_n^C$.

Lemma

$\chi(\mathrm{G}) \leq 2$ if and only if G is bipartite.

PROOF: ... think of color classes as bipartition sets ...

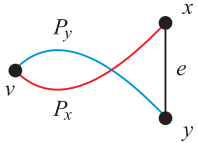
- We know that graphs with $\chi \leq 2$ cannot have cycles of odd length We will now show that the converse holds as well:

Lemma

If G contains no cycles of odd length, then $\chi(\mathrm{G}) \leq 2$.

...

- PROOF. WLOG: G is connected.
- Choose $v \in V(\mathrm{G})$. For $u \in V(\mathrm{G})$ let:
  - $c(u) =$ "blue" if $d(v,u)$ is even;
  - $c(u) =$ "red" if $d(v,u)$ is odd.
- If this is **not** a proper coloring, then there are two adjacent vertices x, y that are both at even or both at odd distance from v.
- Find shortest paths P_x, P_y from v to x and to y. Then $P_x(xy)P_y^{-1}$ is a closed walk of odd length.



- To complete the proof, we need to show the following:
- **Exercise.**: If a graph contains a **closed walk** of odd length, then it also contains a **cycle** of odd length.

Characterization of bipartite graphs

This proves the following characterization of bipartite graphs.

Theorem

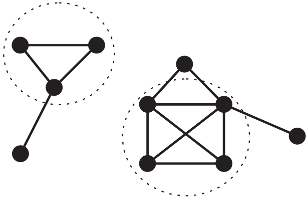
If G *is a graph, then the following statements are equivalent:*

- G *is bipartite.*
- $\chi(\mathrm{G}) \leq 2$.
- G *contains no cycles of odd length.*

Cliques

Definition

A subset $U \subseteq V(\mathrm{G})$ is called a **clique** (klika), if the induced subgraph $\mathrm{G}[U]$ is a complete graph.



Maximal Clique

Definition

A **maximal clique** (maksimalna klika) is a clique that is not contained in any other clique. A **largest clique** (največja klika) is a clique with the largest number of vertices among all cliques.

$$\omega(\mathrm{G}) = \text{"the size of the largest clique in G"}.$$

- Since $\chi(K_n) = n$, it follows that $\chi(\mathrm{G}) \geq \omega(\mathrm{G})$.

The Brooks theorem

Theorem

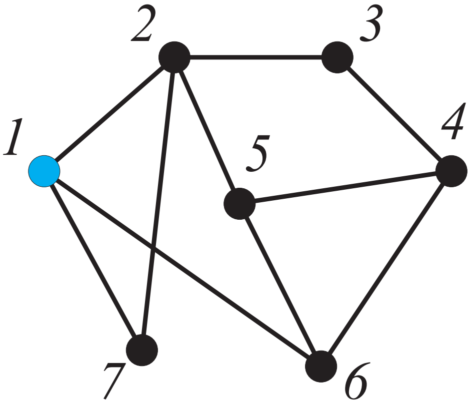
(Brooks) Let G *be a graph. Then*

$$\omega(\mathrm{G}) \leq \chi(\mathrm{G}) \leq \Delta(\mathrm{G}) + 1.$$

Moreover, $\chi(\mathrm{G}) \leq \Delta(\mathrm{G})$ *unless* G *is a complete graph or a cycle of odd length.*

Proof of the Brooks theorem

- WLOG: G is connected.
- We already know that $\omega(\mathrm{G}) \leq \chi(\mathrm{G})$. So we need to show two things:
  1. $\chi(\mathrm{G}) \leq \Delta(\mathrm{G}) + 1$.
  2. If $\mathrm{G} \ncong K_n$ or C_{2m+1}, then $\chi(\mathrm{G}) \leq \Delta(\mathrm{G})$.
- Finding a $(\Delta + 1)$-coloring is easy:
  - Let $\{1, \ldots, \Delta + 1\}$ be the set of colors. Order the vertices of G in some linear order. Color the first vertex with color 1.
  - Suppose that we have already colored the first m vertices. Let v be the next vertex, and let $c \in \{1, \ldots, \Delta + 1\}$ be the smallest integer that does not appear as a color of some neighbor of v. Color v with the color c.
  - Repeat this procedure until all vertices are colored.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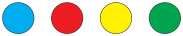

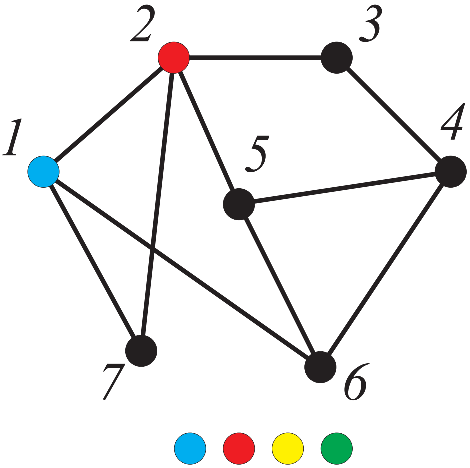
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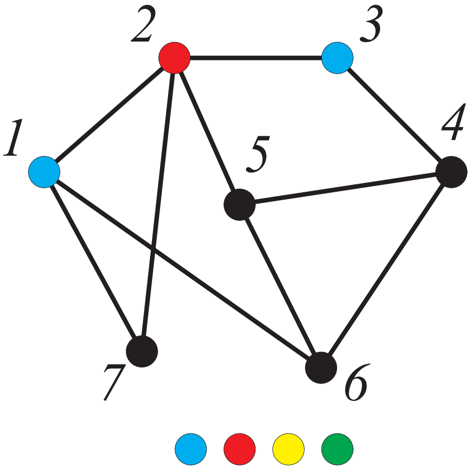
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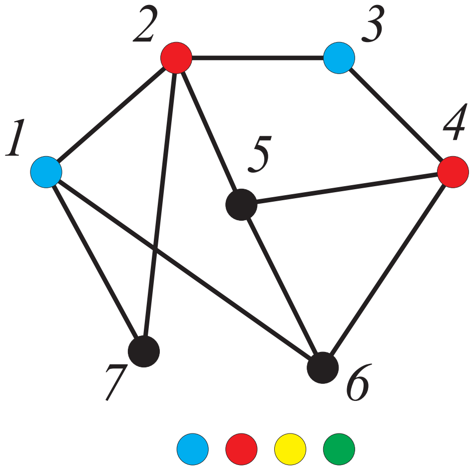
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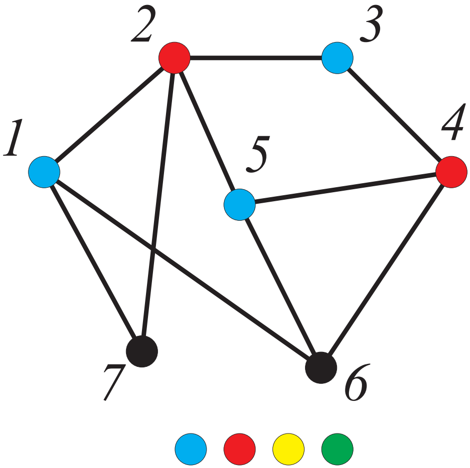
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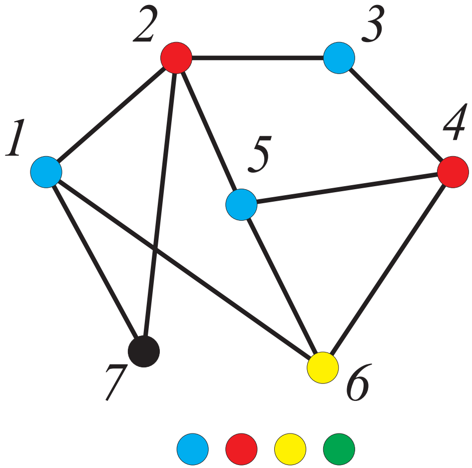
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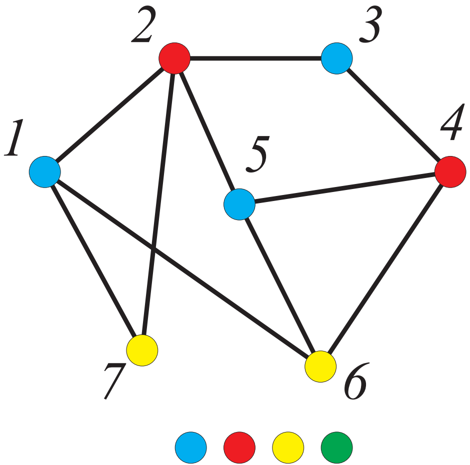
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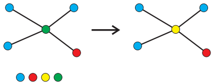
Proof of the Brooks theorem – killing unessential greens

- In the rest of the proof, we may assume that $G \ncong K_n$ or C_{2m+1}. It remains to show, that we may change the $(\Delta + 1)$-coloring in such a way that one of the colors "disappears".
- For the rest of the proof: $\Delta + 1 =$ "green".
- Let $S =$ "the set of green vertices".
- If there is $v \in S$ such that one of the non-green colors **does not** appear among its neighbors, then we may use this color for v. Apply this throughout S. This procedure is called "killing unessential greens".

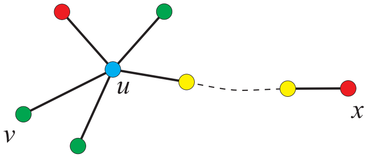


- After unessential greens are "killed", we get a $(\Delta + 1)$-coloring in which each green vertex v has valence Δ, and no two neighbors of v are of the same color.

Proof of the Brooks theorem – pushing the green color

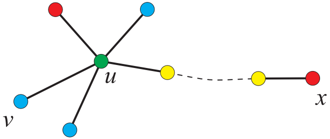
The second procedure allow us to "push" the green color from any $v \in S$ to any other $x \in V(\mathrm{G})$ along any path P from v to x.

1. Kill unessential $(\Delta + 1)$s. If $S = \emptyset$ or $S = \{x\}$, then stop.
2. Let u be the first vertex on the path from v to x. Let c be the color of u. Since we killed unessential greens, no green neighbours of u have any other neighbours of color c.

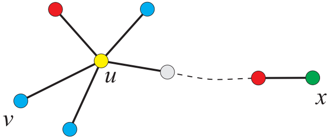


Proof of the Brooks theorem – pushing the green color

1. Change the color of green neighbors of u to c, and change the color of u to green.



2. Go to step 1 with u in place of v, and with P being the part of old P from u to x.



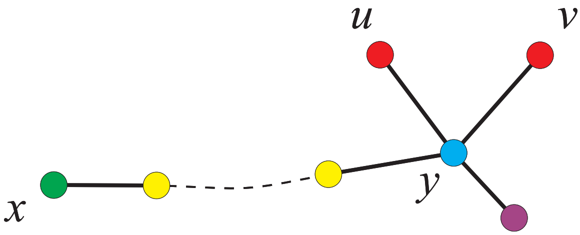
This procedure changed the color of some old green vertices, and cyclically rotated the colors along P.

The proof of Brook's theorem – reduction to regular graphs

- Let x be a vertex of smallest valence. For each green v, choose a shortest path from v to x, and push the green color to x. Now x is the only green vertex.
- If $\mathrm{val}(x) < \Delta$, then there is a non-green color which does not appear among neighbors of x. Hence we can kill the green color at x, and finish.
- It follows: We may thus assume that G is regular (all vertices have valence Δ).
- The proof now splits into two cases:
  - G is 3-connected;
  - G is not 3-connected.

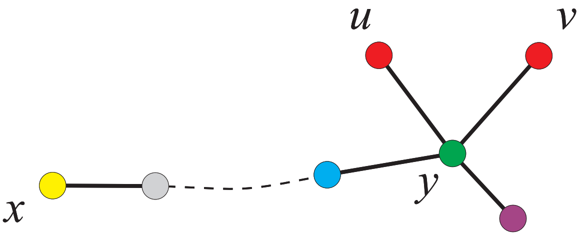
The proof of Brook's theorem – the 3-connected case

- Suppose G is 3-connected.
- Since G is not complete, there exist $x \not\sim y$. Push the green color from all green vertices to x.
- Since there in no green color in $N(y)$, there exist $u, v \in N(y)$ of the same color.



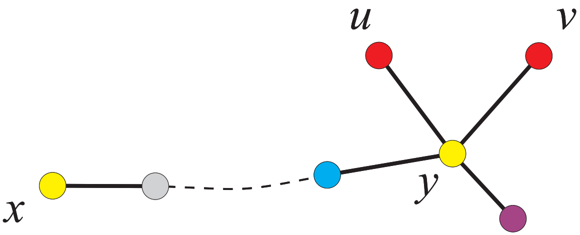
The proof of Brook's theorem – the 3-connected case II

- Consider the graph $G' = G - u - v$. Since G is 3-connected, G' is connected. Choose a shortest path from x to y in G' and push the green color from x to y along this path.



The proof of Brook's theorem – the 3-connected case III

- This results in a proper coloring of G where the only green vertex is y, where u and v (two neighbors of y) have the same color. Therefore, the green color of y can be "killed", giving a Δ-coloring of G.

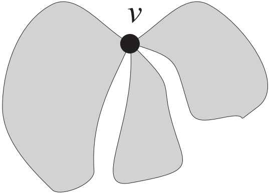


The proof of Brook's theorem – $\kappa \leq 2$

- Suppose now that G is not 3-connected.
- The rest of the proof of is by induction on $n = |V(\mathrm{G})|$. By inspection, we see that the theorem holds for $n \leq 4$. Assume now that $n \geq 5$ and that theorem holds for all graphs with less than n vertices.
- If $\Delta(\mathrm{G}) = 1$, then $\mathrm{G} \cong K_n^C$, and so $\chi(\mathrm{G}) = 1$.
- If $\Delta(\mathrm{G}) = 2$, then $\mathrm{G} \cong C_n$ or P_n, and the theorem holds.
- Assume henceforth that $\Delta(\mathrm{G}) \geq 3$.

...$\kappa = 1$

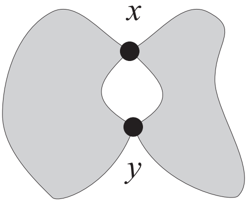
- Suppose that G has a cut-vertex $\{v\}$, and let $X_1, \ldots, X_m$ be the components of $\mathrm{G} - v$.



- By induction, each $X_i + v$ is $\Delta(\mathrm{G})$-colorable. By renaming colors in each X_i if necessary, we may assume that in all X_i, the vertex v has the same color. This gives a $\Delta(\mathrm{G})$-coloring of G.

...$\kappa = 2$

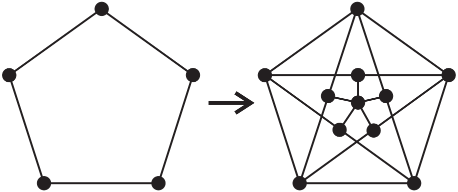
- Suppose now that G has a vertex-cut of size two: $\{x, y\}$.



- In a similar way as in case $\kappa = 1$ we may use induction to show that G is Δ-colorable.
- **Homework H2:** Finish the proof of the theorem in this case.

The Mycielski construction

- Let G be a graph on n vertices with at least one edge. Construct a new graph G^+ on $2n+1$ vertices in the following way:
- $V(\mathrm{G}^+) = V(\mathrm{G}) \cup \{v' : v \in V(\mathrm{G})\} \cup \{\infty\}$ (a disjoint union).
- $E(\mathrm{G}^+) = E(\mathrm{G}) \cup \{v'u : vu \in E(\mathrm{G})\} \cup \{v'\infty : v \in V(\mathrm{G})\}$.
- Homework H3: Show that $\chi(\mathrm{G}^+) = \chi(\mathrm{G}) + 1$.
- Example: The graph, obtained in this way from C_5 is called the Grötch graph.



Edge-colorings

- Let $\mathrm{G} = (V, E)$ be a graph and C a set of "colors". We define edge-colorings in a similar way as vertex-colorings:

Definition

An *edge-coloring* (barvanje povezav) of G is a function $c\colon E \to C$. The coloring is *proper* if incident edges receive different collors. The graph G is k-edge-colorable (povezavno k-obarvljiv) if there exists a proper edge-coloring with $|C| = k$.

- The minimal integer k for which G is k-edge-colorable is called the **chromatic index** (kormatični indeks) of G.

$$\chi'(\mathrm{G}) = \min\{k : \mathrm{G} \text{ is } k - \text{edge–colorable}\}$$

- Note that $\chi'(\mathrm{G}) = \chi(L(\mathrm{G}))$.

Vizing's theorem

- There is an obvious natural lower bound: $\chi'(\mathrm{G}) \geq \Delta(\mathrm{G})$.
- The upper bound is given by Vizing's theorem.

Theorem

(Vizing) $\Delta(\mathrm{G}) \leq \chi'(\mathrm{G}) \leq \Delta(\mathrm{G}) + 1$

- We skip the proof.

Class 1 vs. Class 2, Königs theorem

- Graphs with $\chi'(\mathrm{G}) = \Delta(\mathrm{G})$ are **graphs of class 1**, the others are of **class 2**.
- C_{2m} is of class 1, C_{2m+1} is of class 2.
- Hypercubes are of class 1
- The Petersen graph is of class 2.
- In general, determining χ' is difficult.
- For some graphs, this task is easier. For example, bipartite graphs.

Theorem

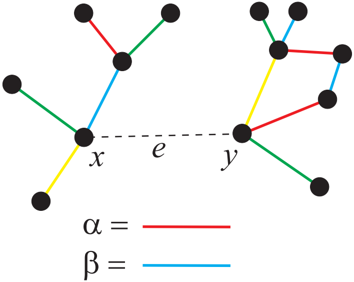
(König) If G is bipartite, then $\chi'(\mathrm{G}) = \Delta(\mathrm{G})$.

Proof of König's theorem

- By contradiction: Let k be a positive integer. Among all graphs with $\Delta = k$ choose a counterexample with the least number of edges.
- Choose an edge $e = xy$, such that $\Delta(\mathrm{G} - e) = \Delta(\mathrm{G})$ (What if such and edge does not exist?).
- By hypothesis, $\chi'(\mathrm{G} - e) = \Delta(\mathrm{G} - e) = k$. Color the edges with k colors.

Proof of König's theorem II

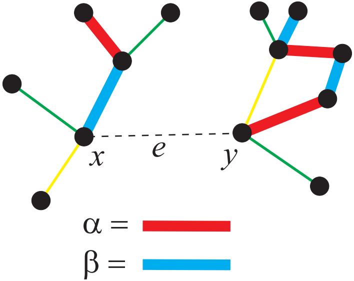
- There is a color α, which does not appear at x, and a color β, which does not appear at y.



- If $\alpha = \beta$, color e with α.

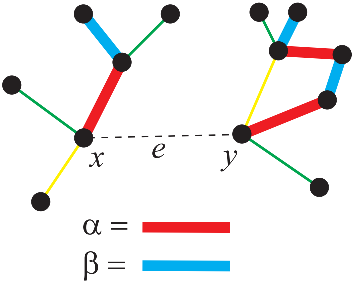
Proof of König's theorem III

- Assume now that $\alpha \neq \beta$.
- Consider the subgraph H induced by the edges of colors α and β. Clearly $\Delta(\mathrm{H}) \leq 2$, so the connected components of H are paths or cycles.



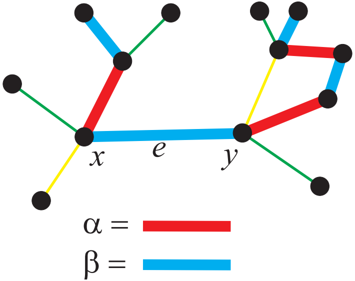
Proof of König's theorem IV

- Note that swappings colors α and β in any component of H gives a different proper coloring.
- Since all paths from x to y are of odd length (G is bipartite!), x and y are in different components of H. Swap the colors α and β in a component containing x.



Proof of König's theorem V

- Finally, color e with β.



Snarks

- A regular graph of valence 3 is called **cubic graph** (kubičen graf).
- **Homework H3.** Show that every connected cubic graph with $\chi' = 3$ is 2-edge-connected.
- On the other hand, it is not easy to find 2-edge-connected cubic graphs with $\chi' = 4$.
- Such a graph is called a **snark**. (The name comes from a poem "The hunting of the Snark" by Lewis Carol.)
- The smallest such graph is the Petersen graph.
- Constructing new families of snarks is still a difficult task.

Matchings

- Consider a proper edge-coloring of G. Consider the set M of edges colored with a fixed color. No vertex of G is incident with more than one edge from M.

Definition

A *matching* (prirejanje) in a graph G is a set $M \subseteq E(\mathrm{G})$ such that each $v \in V(\mathrm{G})$ is incident with at most one $e \in M$.

- Vertices, that **are** incident with some $e \in M$ are **saturated** (nasičen).
- If every vertex of G is saturated, then the matching is **perfect** (popolno prirejanje).
- A matching is **maximal** if it is the largest among all matching.

Stable sets and covers

- Maximal matchings are related to "stable sets", "vertex covers" and "edge covers"

Definition

A **stable set** in G is a set $U \subseteq V(\mathrm{G})$ such that no two vertices in U are adjacent in G. A **vertex cover** in G is a set $U \subseteq V(\mathrm{G})$ such that every edge of G is incident with at least one vertex in U. An **edge cover** in G is a set $F \subseteq E(\mathrm{G})$ such that every vertex of G is incident with at least edge vertex in F.

- $\nu(\mathrm{G}) :=$ "the size of a maximal matching G";
- $\alpha(\mathrm{G}) :=$ "the size of a largest stable set G";
- $\tau(\mathrm{G}) :=$ "the size of a smallest vertex cover of G";
- $\rho(\mathrm{G}) :=$ "the size of a smallest edge cover of G".

Gallai's theorem and the König-Egerváry theorem

Theorem

(Gallai, 1959) If G *has no isolated vertices, then* $\nu(\mathrm{G}) + \rho(\mathrm{G}) = |V(\mathrm{G})|$.

Theorem

(König, Egerváry) If G *is bipartite, then* $\nu(\mathrm{G}) = \tau(\mathrm{G})$.

We skip the proofs.

Hall's theorem

- It is often difficult to decide, what is the size of a largest matching. For bipartite graphs, we have the following nice result:

Theorem

(Hall) Let G be a bipartite graph with bipartition $V(\mathrm{G}) = X \cup Y$. Then G has a matching in which every vertex of X is saturated if and only if $|N(S)| \geq |S|$ for every set $S \subseteq X$.

- Here $N(S)$ is the set of vertices that are adjacent to some vertex in S.

Proof of Hall's theorem

- PROOF. One direction is obvious.
- For the other direction, we need the König-Egerváry theorem. Suppose that there is no matching in which every $v \in X$ is saturated. Then $\nu(\mathrm{G}) < |X|$.
- By the König-Egerváry theorem, $\nu(\mathrm{G}) = \tau(\mathrm{G})$. Therefore, there is a vertex cover K with $|K| < |X|$.
- Let $S = X \setminus K$. Then $N(S) \subseteq Y \cap K$, and so

$$|S| = |X| - |K \cap X| = |X| - |K| + |Y \cap K| > N(S).$$

Homework

H1 Finish the proof of the Brooks theorem in the case where the vertex-connectivity of the graph is 2.

H2 Show that $\chi(\mathrm{G}^{+}) = \chi(\mathrm{G}) + 1$.

H3 Show that every connected cubic graph with $\chi' = 3$ is 2-edge-connected.