Lesson No. 4: Group Actions and Cayley Graphs

1. Group Actions (Orbits, Stabilisers, Burnside's lemma);
2. Transitivity and Regularity of actions;
3. Cayley graphs

Symmetric groups

- Let Ω be a (finite) set, and let $\mathrm{Sym}(\Omega)$ be the set of all permutation on Ω.
- There are two standard ways to define a binary operation on $\mathrm{Sym}(\Omega)$:
  - **left multiplication** = composition:

$$g \circ h \colon \omega \mapsto g(h(\omega)).$$

  - **right multiplication** = inverse composition:

$$g \cdot h \colon \omega \mapsto h(g(\omega)).$$

- Example: Let $g = (1,3,4)(2,5)$, $h = (1,3)(2,4)(5)$

$$g \circ h = (1,4,5,2)(3), \quad g \cdot h = (1)(2,5,4,3).$$

Symmetric groups II

- Both $(\mathrm{Sym}(\Omega), \circ)$ and $(\mathrm{Sym}(\Omega), \cdot)$ are groups.

Definition

The groups $(\mathrm{Sym}(\Omega), \circ)$ and $(\mathrm{Sym}(\Omega), \cdot)$ are called the **left symmetric group** and **the right symmetric group** on Ω.

- **H1:** Show these two groups are isomorphic.
- We will mainly work with the right symmetric group:

$$S_\Omega = (\mathrm{Sym}(\Omega), \cdot)$$

- We write ω^g instead of $g(\omega)$. With this notation, we have:

$$\omega^{(gh)} = (\omega^g)^h$$

  for all $\omega \in \Omega$ and $g, h \in S_\Omega$.
- Subgroups of S_Ω are called **permutation groups**.

Group actions

- Group action generalize the notion of permutation groups.
- Let G be a group, let Ω be a set, and let

$$\Phi\colon \Omega \times G \to \Omega, \qquad (\omega, g) \mapsto \omega^g,$$

  be a mapping which satisfies:
  - $\omega^1 = \omega$ for all $\omega \in \Omega$;
  - $\omega^{(gh)} = (\omega^g)^h$ for all $\omega \in \Omega$ and $g, h \in G$.
- In this case we say that Φ is an **action** (delovanje) of G on Ω.

Group actions vs. permutation representations

- Let $\Phi\colon \Omega \times G \to \Omega$ be a group action.
- Define a mapping:

$$\bar{\Phi}\colon G \to \mathrm{Sym}_\Omega, \qquad \bar{\Phi}(g) = (\omega \mapsto \omega^g)$$

- $\bar{\Phi}$ is a homomorphism of groups.
- Conversely, let $\Phi\colon G \to \mathrm{Sym}_\Omega$ be any group homomorphism. Define can define an action:

$$\bar{\Phi}\colon \Omega \times G \to \Omega, \qquad (\omega, g) \mapsto \omega^{\bar{\Phi}(g)}$$

- Note that $\bar{\bar{\Phi}} = \Phi$.
- "Groups actions on Ω" $\equiv$ "homomorphism to S_Ω".

Kernel

- The set of all $g \in G$ which fix **every** elelemnt of Ω is called the **kernel** (jedro) of the action.

$$\mathrm{Ker} = \{g \in G : \omega^g = \omega \text{ for all } \omega \in \Omega\}$$

- If the action is viewed as a homomorphism $\Psi \colon G \to S_\Omega$, then the kernel of the action is simply the kernel of Ψ.

$$\mathrm{Ker} = \{g \in G : \Psi(g) = \mathrm{id}\}.$$

- If $\mathrm{Ker} = 1$, then the action is **faithful** (zvesto).
- If G acts faithfully on Ω, then $G \to S_\Omega$ is injective.
- "Faithfull actions" $\equiv$ "permutation groups".

Orbits

- Let G act on Ω and let $\omega \in \Omega$. The set

$$\omega^G = \{\omega^g : g \in G\}$$

is called the *orbit* (orbita) of ω.

- The set of all orbits is a partition of Ω.
- If there is only one orbit, then the action is **transitive**.

Lemma

An action of G on Ω is transitive if and only for any $\alpha, \beta \in \Omega$ there exists $g \in G$ s.t. $\alpha^g = \beta$.

Stabilisers

- The set
$$G_\omega = \{g \in G : \omega^g = \omega\}$$
is called the **stabiliser** (stabilizator) of ω. G_ω is a subgroup of G.
- For $\Delta \subseteq \Omega$, let
  - $G_\Delta = \{g \in G : \delta^g \in \Delta$ for all $\delta \in \Delta\}$ (setwise-stabiliser);
  - $G_{(\Delta)} = \{g \in G : \delta^g = \delta$ for all $\delta \in \Delta\}$ (pointwise-stabiliser).

$$G_{(\Delta)} = \cap_{\delta \in \Delta} G_\delta.$$

Examples of actions

- Let $G \leq S_\Omega$ be any permutation group. Then G acts on Ω in a natural way.
- Let $G \leq S_\Omega$, let k be an integer, and let $\Omega^{(k)}$ be the set of all k-element subsets of Ω. For $g \in G$ and $\{\omega_1, \ldots, \omega_k\} \in \Omega^{(k)}$ let

$$\{\omega_1, \ldots, \omega_k\}^g = \{\omega_1^g, \ldots, \omega_k^g\}.$$

  This defines an action of G on $\Omega^{(k)}$ (uordered k-tuples).
- Similarly can be defined an action of G on

$$\Omega^{[k]} = \{(\omega_1, \ldots, \omega_k) : \omega_i \in \Omega, \omega_i \neq \omega_j \text{ for } i \neq j\}.$$

Groups acting on groups

Let G be a group. Then there are several ways to define an action of G on itself.

- For $\omega, g \in G$ let $\omega^g = g^{-1}\omega g$. We say that G acts on itself by conjugation.
  - Stabiliser: $G_\omega = \{g \in G : g\omega = \omega g = C_G(\omega)$ (centralizer of ω);
  - Orbits: $\omega^G =$ "conjugacy class of ω";
  - Kernel: $\mathrm{Ker} = \{g \in G : g^{-1}\omega g = \omega$ for all $\omega \in G\} = C(G)$.
- For $\omega, g \in G$ let $\omega^g = \omega g$. We say that G acts on itslef by right multiplication.
  - Stabiliser: $G_\omega = 1$ ... the stabilisers are trivial;
  - Orbits: $\omega^G = G$ ... the action is transitive;
  - Kernel: $\mathrm{Ker} = 1$ ... the action is faithful.

Regular actions

Corollary

(Cayley) Every group is isomorphic to some permutation group.

PROOF. Let a group G act on itself by right multiplication. This action is faithful, hence G is embedded in to S_G.

Definition

Let G act on Ω. If $G_\omega = 1$ for every $\omega \in \Omega$, the the action is called **semiregular**. A transitive semiregular action is called **regular**.

Homework 2: Show that every faithful action of an abelian group is regular.

Action on cosets

- Let $H \leq G$, and let $G/H = \{Ha : a \in G\}$ be the set of all right cosets (desni odseki) of G by H. For $g \in G$ let

$$(Ha)^g = H(ag).$$

We say that G acts on right cosets ny right multiplication.

  - Stabiliser:

$$g \in G_{Ha} \Leftrightarrow Hag = Ha \Leftrightarrow aga^{-1} \in H \Leftrightarrow g \in a^{-1}Ha$$

    Hence $G_{Ha} = H^a$. In particular $G_H = H$.
  - Orbits: $H^G = G/H$ ... the action is transitive;
  - Kernel:

$$\mathrm{Ker} = \cap_{a \in G} G_{Ha} = \cap_{a \in G} H^a = \text{core}_G(H).$$

Conjugate stabilisers

Let G acto on Ω.

- Take $\alpha \in \Omega$, $h \in G$, and let $\beta = \alpha^h$. Consider G_β:

  $g \in G_\beta \Leftrightarrow \beta^g = \beta \Leftrightarrow \alpha^{hg} = \alpha^h \Leftrightarrow hgh^{-1} \in G_\alpha \Leftrightarrow g \in h^{-1}G_\alpha h$

  Therefore we have:

$$G_{(\alpha^h)} = (G_\alpha)^h.$$

Lemma

Let G act on Ω and let $\alpha, \beta \in \Omega$. Then α and β belong to te same orbit of G if and only if the stabilisers G_α and G_β are conjugate in G.

Orbit-stabiliser lemma

Lemma

Let G act on Ω and let $\omega \in \Omega$. Then

$$|G_\omega|\,|\omega^G| = |G|.$$

- PROOF: Define a mapping

$$\varphi\colon G/G_\omega \to \omega^G, \quad G_\omega g \mapsto \omega^g.$$

  - this definition is independent of the choice of the representative of $G_\omega g$.
  - the mapping is 1-1.
  - the mapping is onto.
- Hence $|G/G_\omega| = |\omega^G|$, and so $|G_\omega|\,|\omega^G| = |G|$.

Burnside's lemma

- Also called Cauchy-Frobenius lemma, or non-Burnside lemma.

Lemma

Let m denote the number of orbits of G acting on Ω. Then

$$m = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|.$$

- Here

$$\mathrm{Fix}(g) = \{\omega \in \Omega : \omega^g = \omega\}.$$

Proof of Burnside's lemma

- PROOF: Let $\Omega_1, \ldots, \Omega_m$ be the orbits of G on Ω.
- Consider the set

$$\mathcal{M} = \{(\omega, g) : \omega \in \Omega, g \in G_\omega\} = \{(\omega, g) : g \in G, \omega \in \mathrm{Fix}(g)\}.$$

- Count the number of elements in M in two ways. On one hand:

$$|\mathcal{M}| = \sum_{g \in G} |\mathrm{Fix}(g)|.$$

- On the other hand

$$|\mathcal{M}| = \sum_{\omega \in \Omega} |G_\omega| = \sum_{i=1}^{m} \sum_{\omega \in \Omega_i} |G_\omega| = \sum_{i=1}^{m} \sum_{\omega \in \Omega_i} |G|/|\Omega_i| = m|G|.$$

Cayley graphs

- Let G be a group, and let $S \subseteq G$ be s.t.
  - $1 \notin S$,
  - $s \in S \rightarrow s^{-1} \in S$.

  Such a set S is called a **Cayley subset** of G.
- Examples:
  - $G = \mathbb{Z}_9$, the cyclic group of order 9, $S = \{1, 3, 6, 8\}$;
  - $G = \mathbb{Z}_p^d$, the additive group of a vector space, $S = \{e_i, -e_i : i = 1, \ldots, d\}$;
  - $G = A_4$, $S = \{(1, 2, 3), (1, 3, 2), (1, 2)(3, 4)\}$.

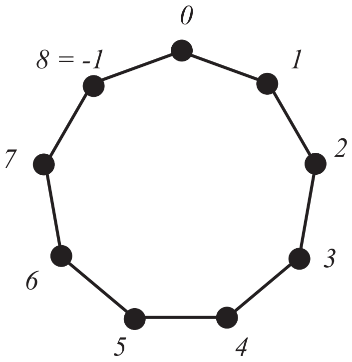
Cayley graphs

- Let S be a Cayley suGset of G. Define the graph Γ by
  - $V(\Gamma) = G$,
  - $u \sim_\Gamma v \Leftrightarrow vu^{-1} \in S$.
- **Exercise.** Show that $\sim$ is indeed an ireflexive, symmetric relation.
- Note that $N(u) = \{us : sinS\}$.
- The graph Γ is called **the Cayley graph** of G relative to S.

$$\Gamma = \mathrm{Cay}(G, s).$$

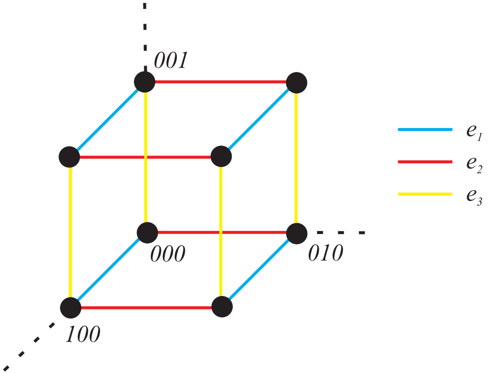
Examples of Cayley graphs

- $\mathrm{Cay}(\mathbb{Z}_n, \{-1, 1\}) \cong C_n$.



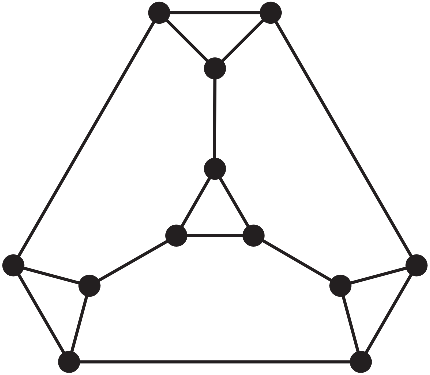
Examples of Cayley graphs

- $\mathrm{Cay}(\mathbb{Z}_2^d, \{e_1, \ldots, e_d\}) \cong Q_d$, the d-dimensional cube.



Examples of Cayley graphs

- $\mathrm{Cay}(A_4, \{(1,2,3),(1,3,2),(1,2)(3,4)\})$.

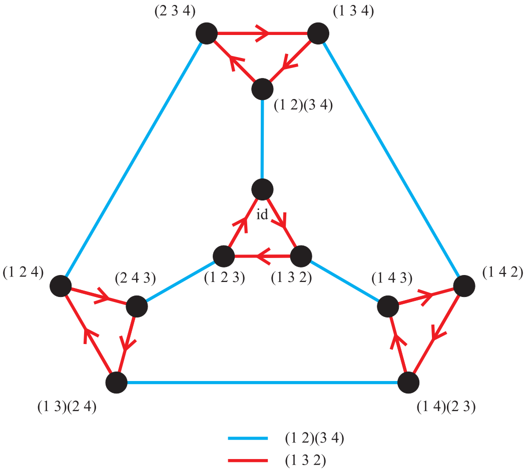


Colored Cayley graphs

- Historically, Cayley graphs were used for "graphical" presentations of groups.
- The Cayley set S was assumed to generate G. **Exercise.** The Cayley set S generates G if and only if $\mathrm{Cay}(G, S)$ is connected.
- Exactly one element from each pair $\{s, s^{-1}\}$ was chosen, and given by its own color, $c(s)$.
- An edge $uv \in E(\Gamma)$ was colored with $c(s)$ where $s = uv^{-1}$.
- The edge was directed "from u to v" if and only $s = uv^{-1}$ was "chosen".

Example

- **Example:** $G = A_4$, $a = (1,2,3) \in G$, $b = (1,2)(3,4)$, $S = \{a, a^{-1}, b\}$.



Automorphisms of graphs

- An automorphism of a graph Γ is a permutataion of $V(\Gamma)$ which preserves adajcency.
- The set of all automorphisms of Γ forms a permutation group on $V(\Gamma)$.

$$\mathrm{Aut}(\Gamma) = \{g : g \text{ is an automorphism of } \Gamma\}.$$

- $\mathrm{Aut}(\Gamma)$ "measures" the symmetry of Γ.
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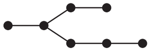
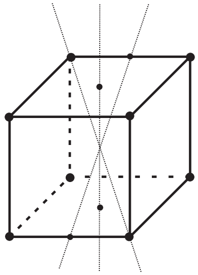
Examples

Examples:

- Some graphs have no symmmetry (are completely asymmetric).



- Complete graphs have the full symmetric group as their automorphism group.
- Interesting examples are in between.



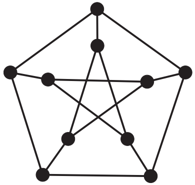
Vertex transitivity

Definition

Let Γ be a graph.If $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$, then we say that Γ is **vertex-transitive** (točkovno tranzitiven). More generally, if a subgroup $G \leq \mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$, then we say that Γ is G-vertex-transitive.

Examples:

- C_n is vertex transitive.
- Q_3 is vertex transitive.
- The Petersen graph is vertex transitive.



Vertex transitivity of Cayley graphs

Lemma

The automorphism group of a Cayley graph $\mathrm{Cay}(G, S)$ *contains a subgroup* $\bar{G}$*, isomorphic to* G*, which acts regularly on* $V(\Gamma)$

PROOF.

- For each $g \in G$ let

$$\rho_g \colon G \to G, \quad x \mapsto xg.$$

- $\rho : G \to S_G$, $g \mapsto \rho_g$ is an isomorphism of groups. (right regular action!)
- It remains to show that $\rho_g \in \mathrm{Aut}(\mathrm{Cay}(G, S))$. Take $x, y \in G$. Then:

$$x \sim y \Leftrightarrow yx^{-1} \in S \Leftrightarrow y(gg^{-1})x^{-1} \in S \Leftrightarrow (yg)(xg)^{-1} \in S \Leftrightarrow y^{\rho_g} \sim x^{\rho_g}.$$

Sabidussi's characterization of Cayley graphs

Theorem

A graph Γ is isomorphic some Cayley graph on a group G if and only if $\mathrm{Aut}(\Gamma)$ contains a subgroup isomorphic to G which acts regularly on $V(\Gamma)$.

PROOF.

- One direction is already shown.
- Suppose now that $\mathrm{Aut}(\Gamma)$ contains a regular subgroup G.
- Choose a vertex $v \in V(\Gamma)$. By regularity, for each $u \in V(\Gamma)$, there exists a unique $g_u \in G$ such that $v^{g_u} = u$. This shows that
$$\varphi \colon V(\Gamma) \to G, \quad u \mapsto g_u$$
is a bijection.
- Let $S = \varphi(N(v)) = \{g_u : u \sim v\}$. Consider $\mathrm{Cay}(G, S)$. Then

$\varphi : V(\Gamma) \to V(\mathrm{Cay}(G, S))$ is an isomorphism of graphs.

Sabidussi's characterization of Cayley graphs

- Sabidussi's characterization helps answering the question, which graphs are Cayley graphs.
- **Question:** Are all vertex-transitive graphs Cayley graphs?
- NO! For example, the Petersen graph is vertex-transitive but not Cayley
- **H3:** Show that the Petersen graph is not a Cayley graph.

Homework

- **H1:** Show that the left and the right symmetric groups are isomorphic.
- **H2:** Show that every faithful action of an abelian group is regular.
- **H3:** Show that the Petersen graph is not a Cayley graph.