Geometry of Partial Cubes

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Outline

Definitions

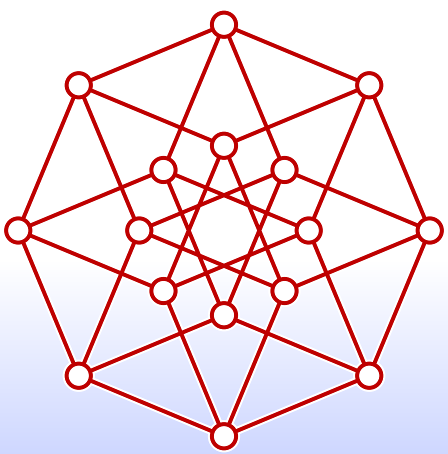
Examples

Dimension

Graph Drawing

Cubic Partial Cubes

Flip Distance



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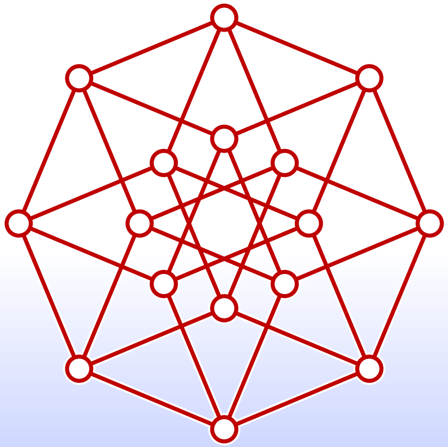


Context: Geometric graphs and metric embedding

Graph theory:

Unweighted graphs

Weighted graphs

Finite metric spaces

Geometry:

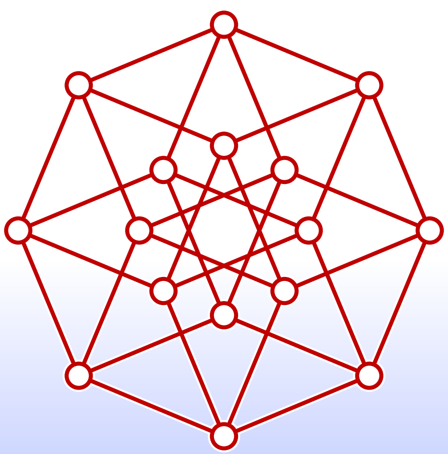
Real vector spaces

Integer lattices

Euclidean distances

L_1 distances

L_∞ distances



Probabilistic tree embedding

Bourgain's theorem

Johnson-Lindenstrauss lemma ...

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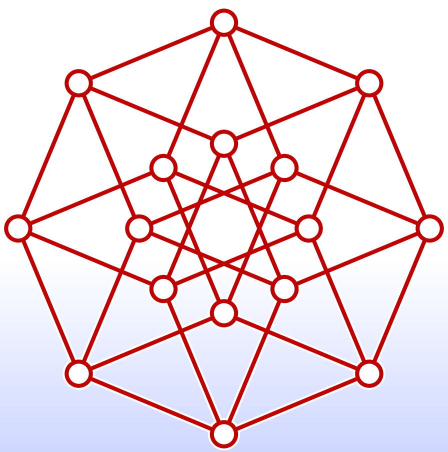
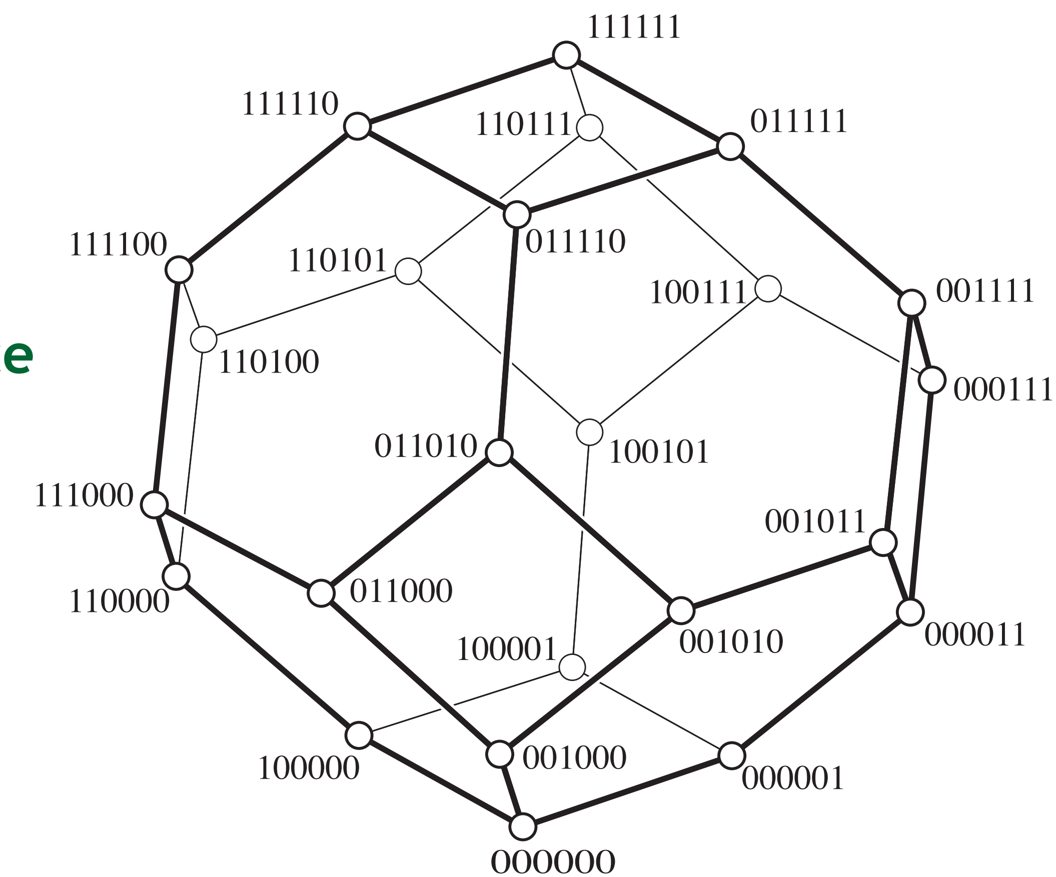


Partial cubes as geometric graphs

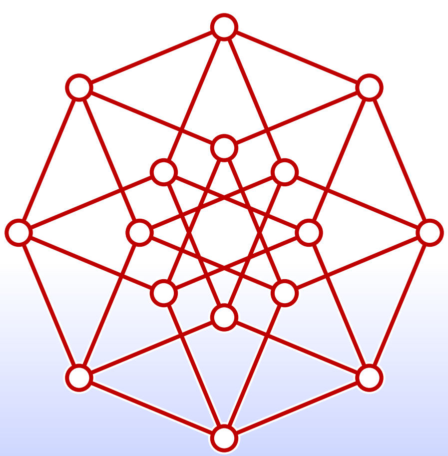
Partial cube:

Undirected graph that can be embedded into an **integer lattice** so that **graph distance = L_1 distance**

Without loss of generality, all coordinates 0 or 1, L_1 distance = Hamming distance: **isometric hypercube subgraph**



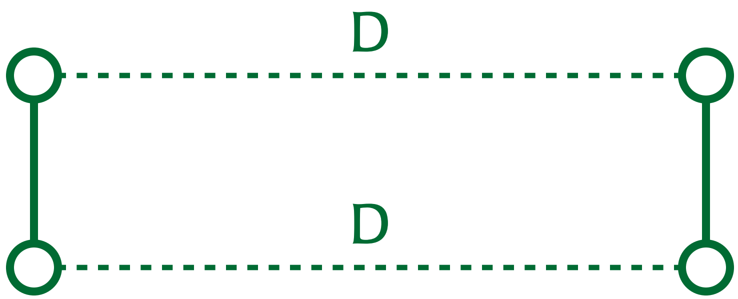
Example: permutahedron
(vertices = permutations of 4 items
edges = flips of adjacent items)



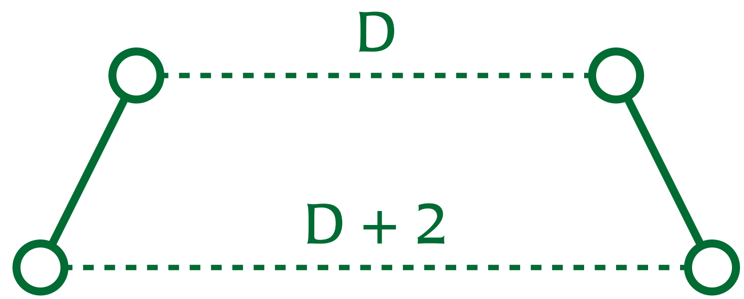
Graph-theoretic characterization

Djokovic-Winkler relation on graph edges [Djokovic 1973, Winkler 1984]:

$$(p,q) \sim (r,s) \text{ iff}$$
$$d(p,r) + d(q,s) \neq d(p,s) + d(q,r)$$



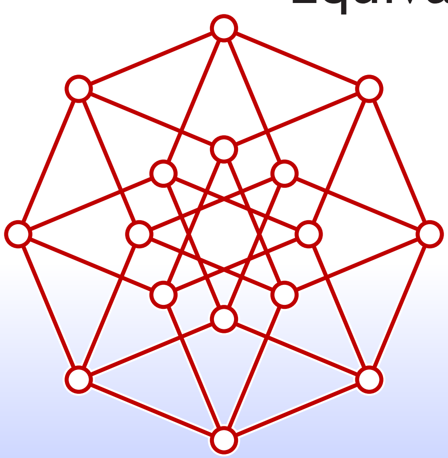
related edges



unrelated edges

G is a partial cube iff it is bipartite and DW-relation is an equivalence relation

Equivalence classes cut graph into two connected subgraphs



0-1 lattice embedding: coordinate per class,
0 in one subgraph, 1 in the other
unique up to hypercube symmetries

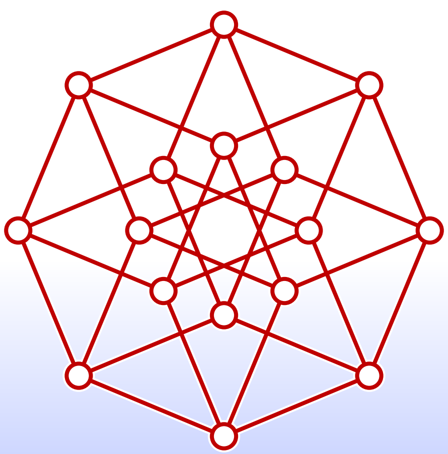
Automaton-theoretic characterization

Medium [e.g. Falmagne and Ovchinnikov 2002]:
System of states and transformations of states ("tokens")
token τ is "effective" on a state S if $S\tau \neq S$

Every token τ has a "reverse" τ^R:
for any two states $S \neq V$, $S\tau = V$ iff $V\tau^R = S$

Any two states can be connected by a "concise message":
sequence of effective tokens
containing at most one from each token-reverse pair

If a sequence of effective tokens returns a state to itself
then its tokens can be matched into token-reverse pairs



States and adjacencies between states
form vertices and edges of a partial cube

Partial cube vertices form medium states
token = "set *i*th coordinate to *b* if possible"

Fundamental components of a partial cube

Vertices and **edges**,
as in any graph, but also:

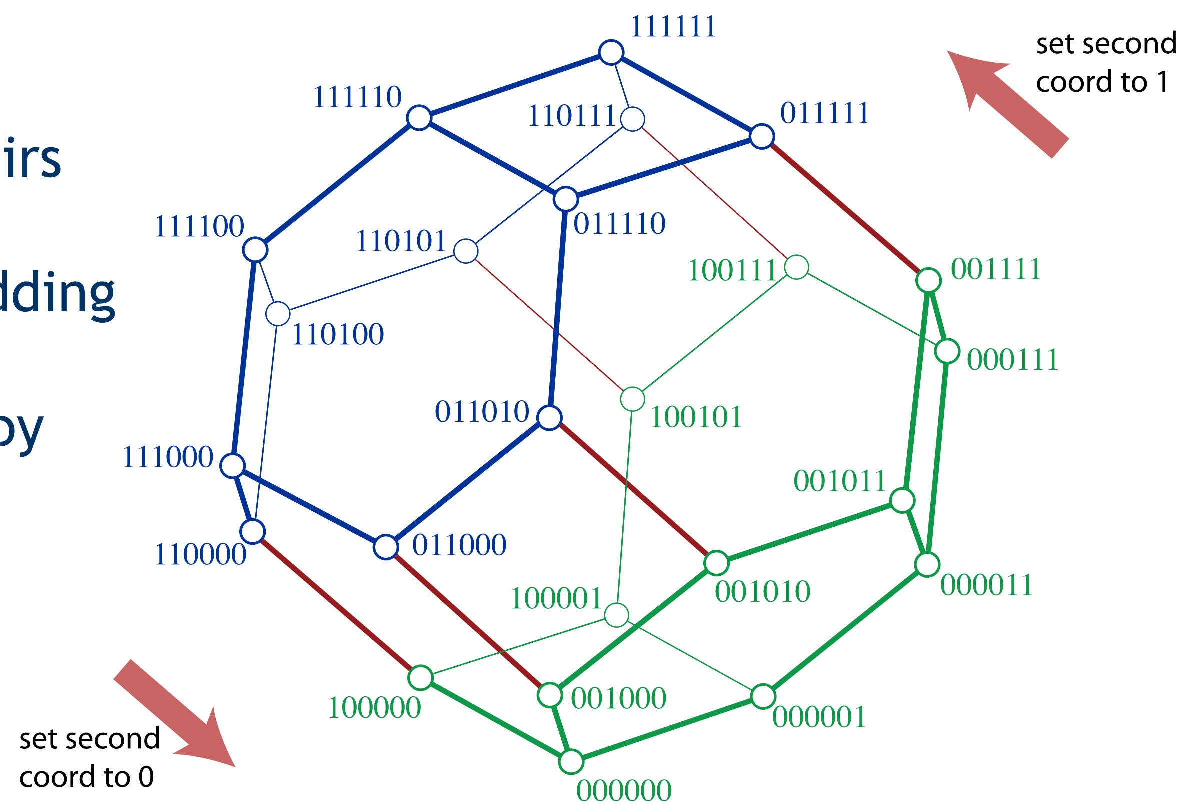
equivalence classes of DW-relation ("zones")

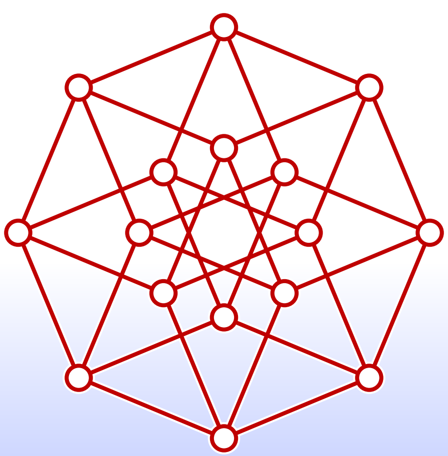
alternatively:

tokens or token-reverse pairs

coordinates of cube embedding

semicubes (subgraphs cut by equivalence classes)





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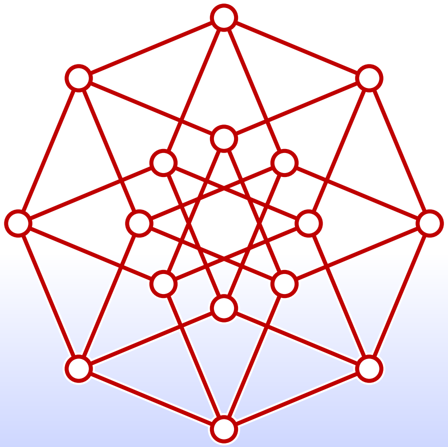
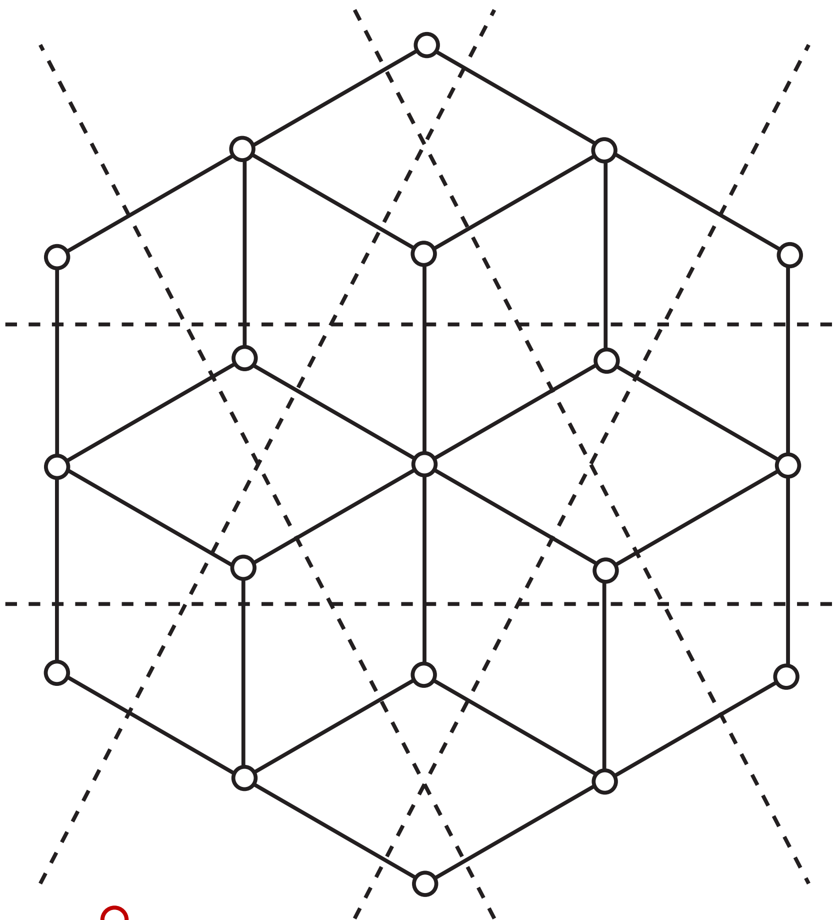
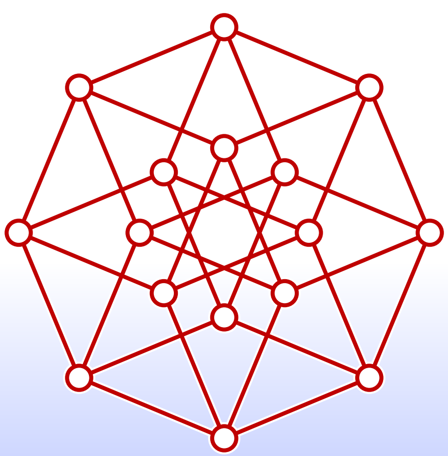


Partial cubes from hyperplane arrangements





Given any arrangement of lines
(or hyperplanes in higher dimensions)

Form dual graph
- vertex for each region
- edge connecting adjacent regions

0-1 labeling:
- coordinate per line
- 0 for regions below line, 1 above

Graph distance ≤ Hamming distance:
- path connecting regions must cross separating lines at least once

Graph distance ≥ Hamming distance:
- line segment connecting regions forms path with ≤ 1 crossing/line

Acyclic orientations of an undirected graph

For any undirected graph $G=(V,E)$:

Number vertices arbitrarily

Form arrangement of hyperplanes

$x_i = x_j$ for each edge v_iv_j in E

Vertices: acyclic orientations

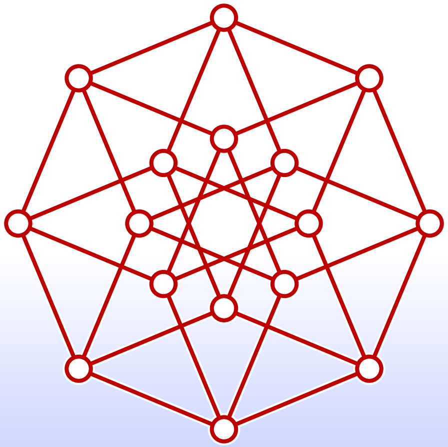
Edges:

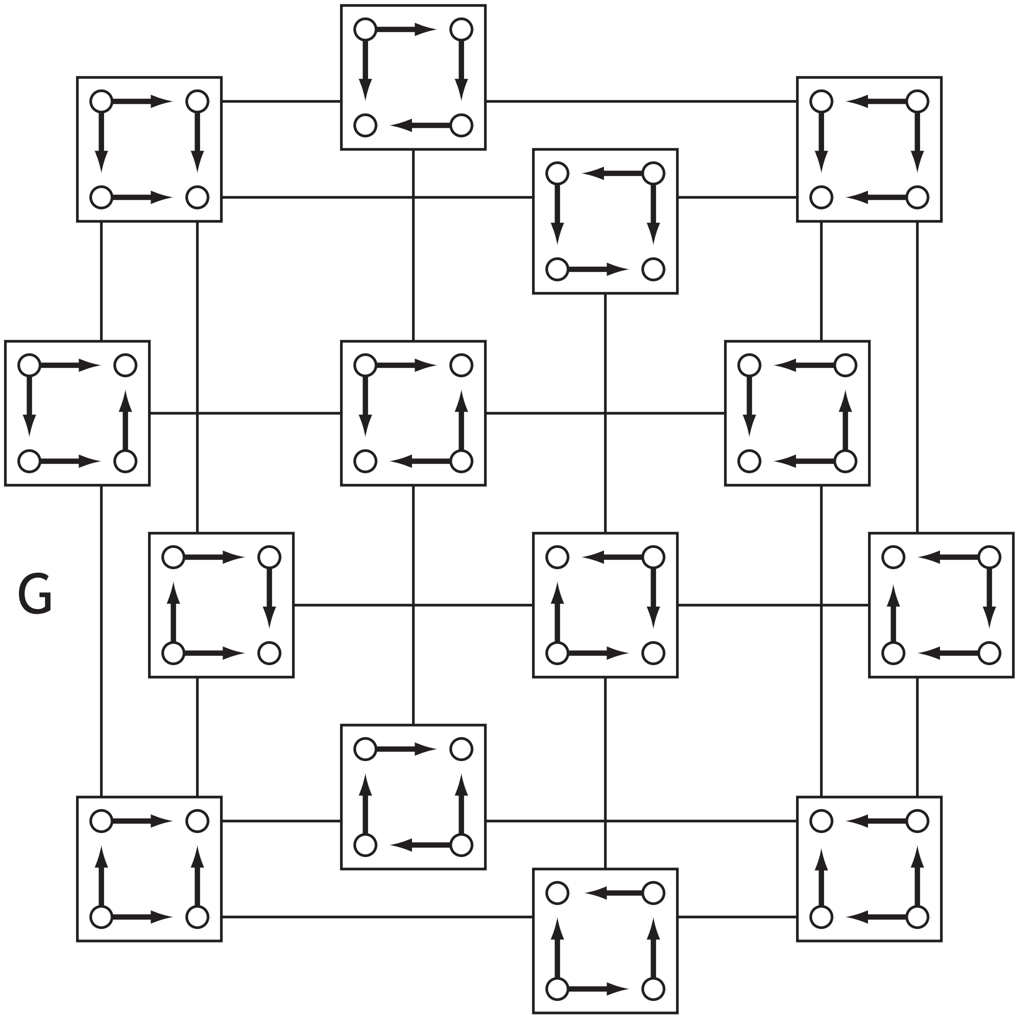
orientations differing by one edge of G

Zones:

flip orientation of a single edge of G

(if the flipped orientation is acyclic)





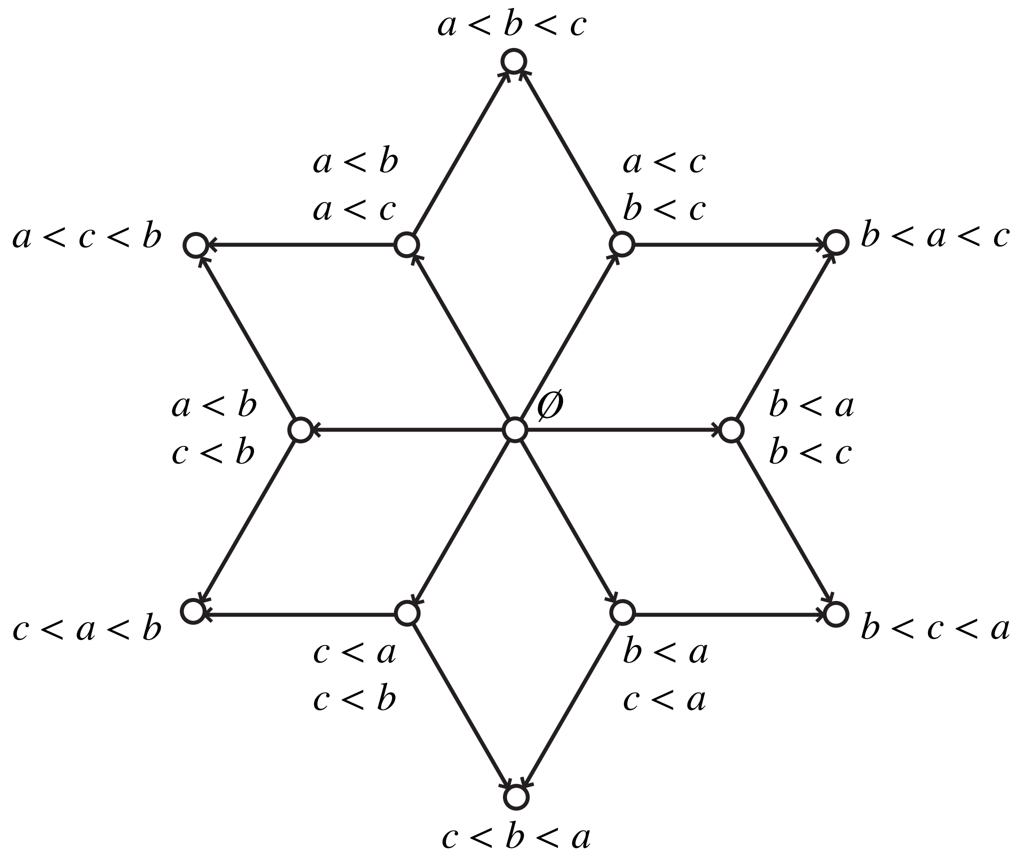
The acyclic orientations of a 4-cycle

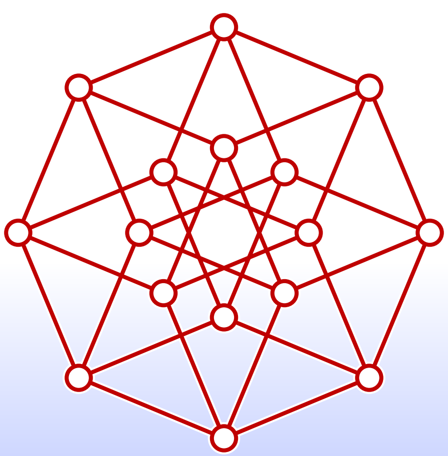
Weak orders

Weak order =
equivalence relation
+ total order on equivalence classes

Model geometrically as
face lattice of arrangement
of hyperplanes $x_i = x_j$
[Ovchinnikov 2006]

Applications to voter modeling
in social choice theory
[Hsu, Falmagne, Regenwetter 2005]





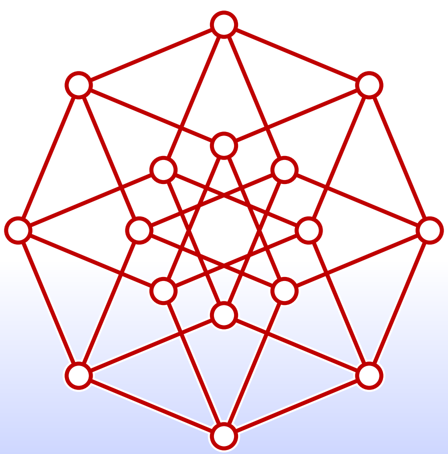
Antimatroids

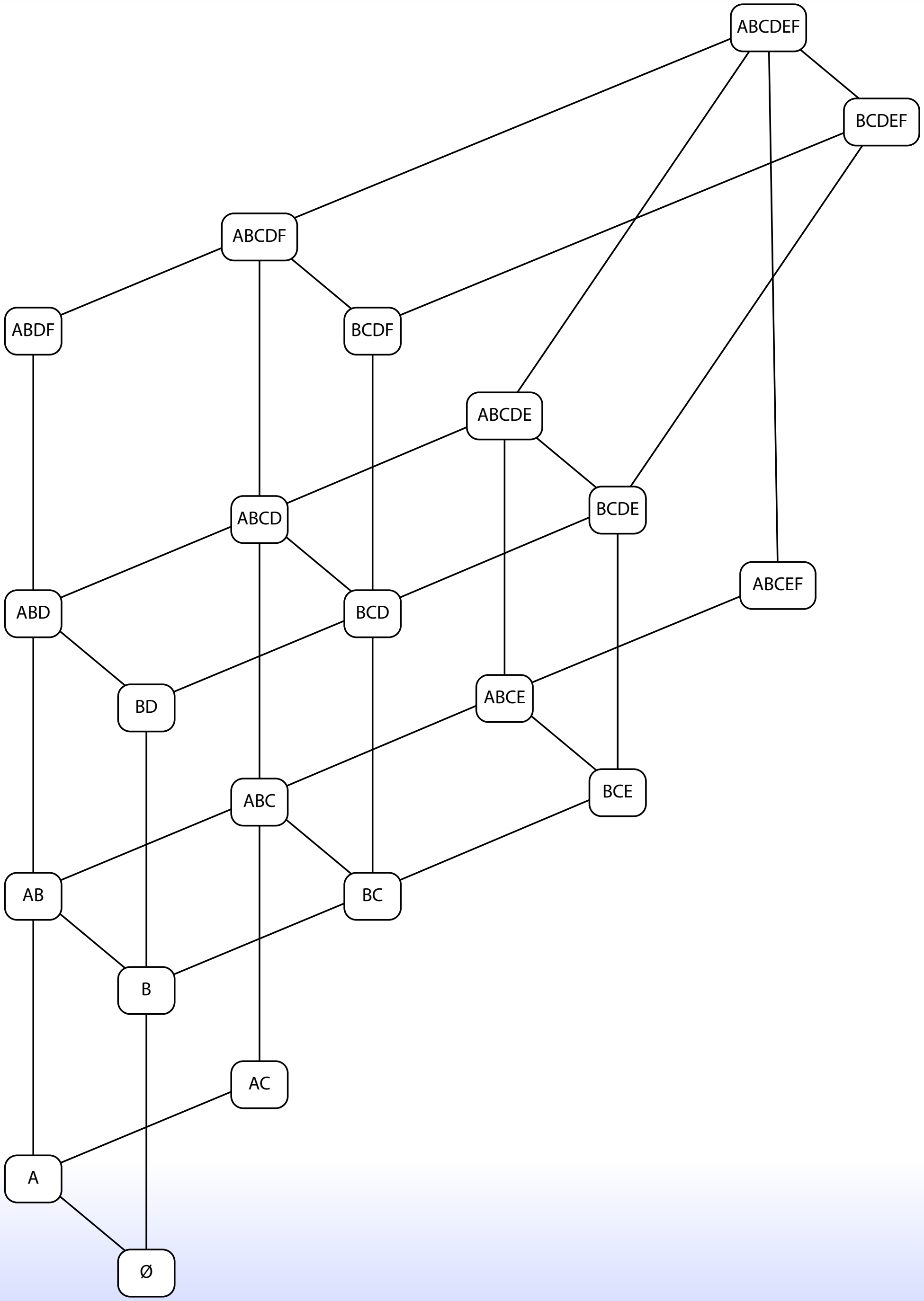
Antimatroid = family of sets,
closed under unions, s.t.
each nonempty set has removable item

Models processes of adding items one by one; once available for inclusion, an item never becomes unavailable

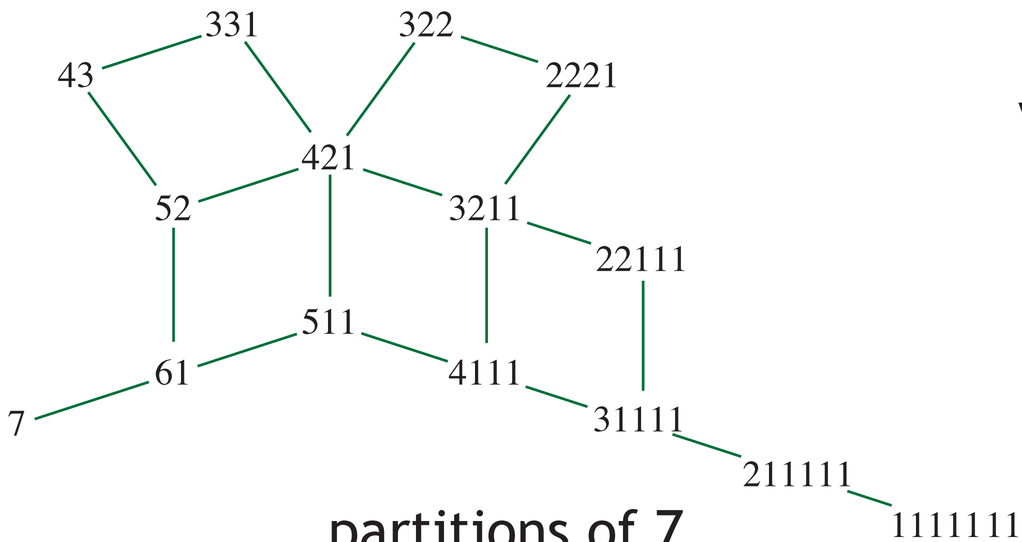
Applications in discrete geometry...
(shelling sequences of point sets)

...and mathematical psychology
(states of knowledge of human learner)
[Doignon and Falmagne 1999]





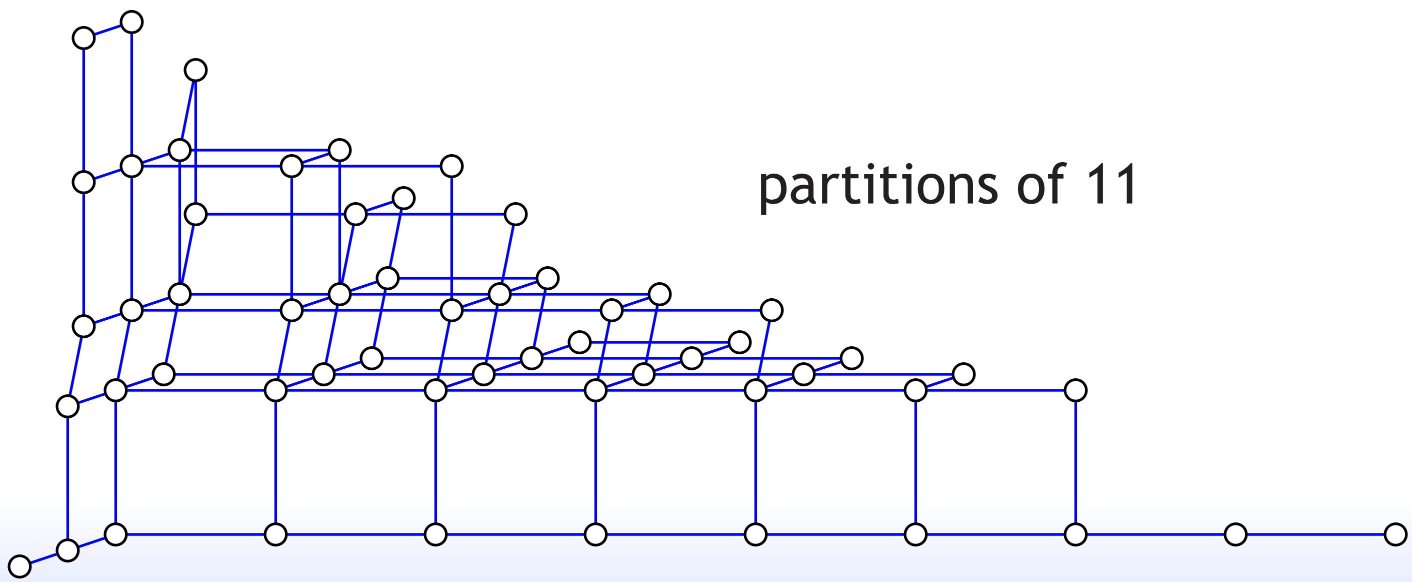
Integer partitions



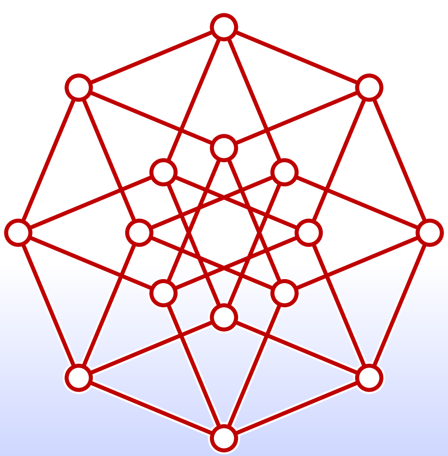
partitions of 7

Vertex = partition

Edge = increment largest value
and decrement some other value
(or vice versa)



partitions of 11



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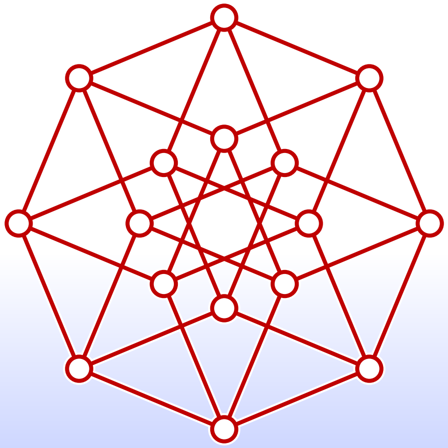


Concepts of dimension for partial cubes

Isometric dimension:

dimension of (unique) hypercube embedding
maximum dimension of lattice embedding
number of zones
polynomial time

Lattice dimension:

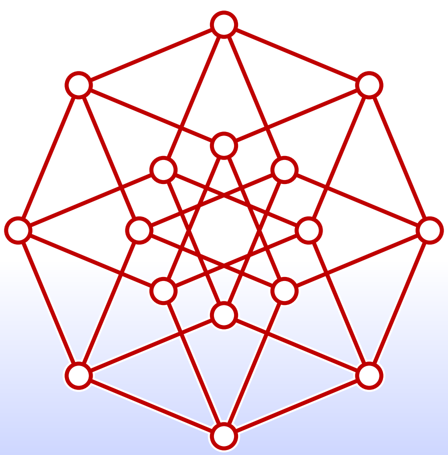
minimum dimension of lattice embedding
isometric embedding into product of minimum number of paths
polynomial time

Tree dimension:

isometric embedding into product of min number of trees
polynomial for $d = 2$, NP-complete for $d \geq 3$

Arrangement dimension:

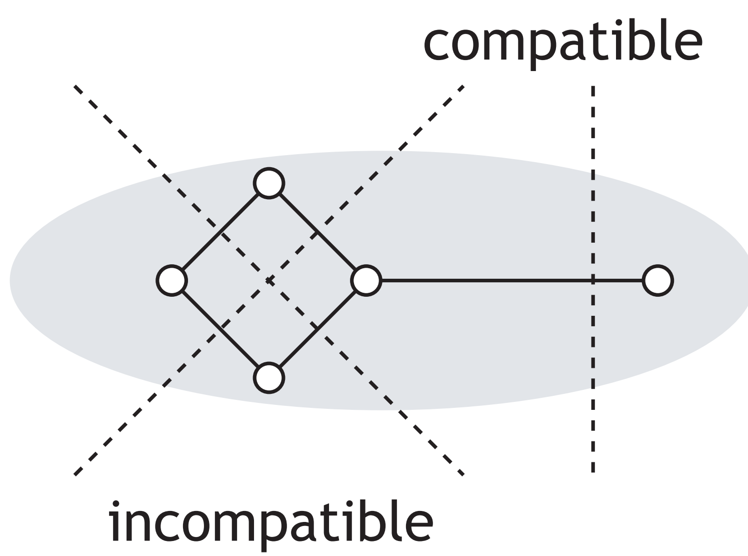
min dim of representation as arrangement (if possible)
NP-complete even for $d = 2$
via stretchability of pseudolines [Shor 1991]

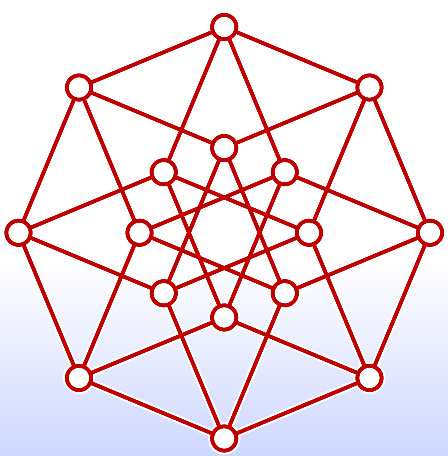


Tree dimension

Representation as an isometric subset of a product of trees
= partition of zones into compatible subsets

"compatible": for each pair of zones,
not all four combinations of semicubes are occupied



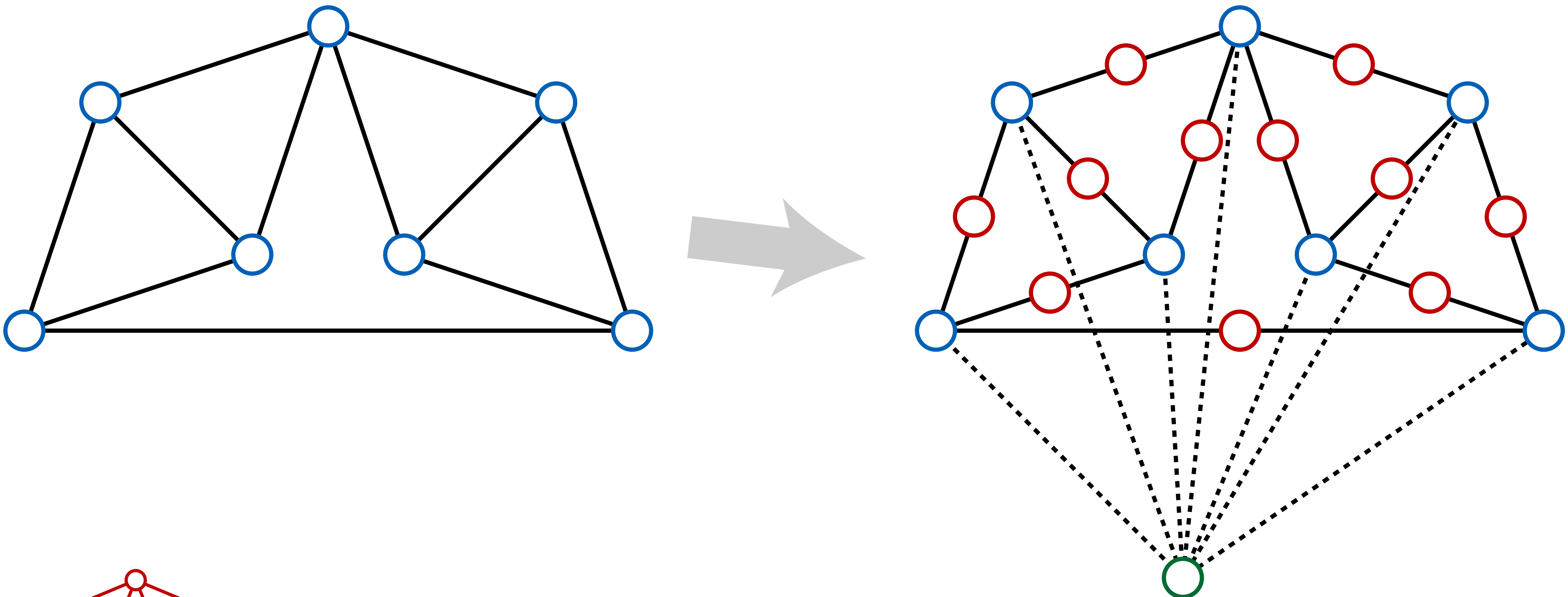


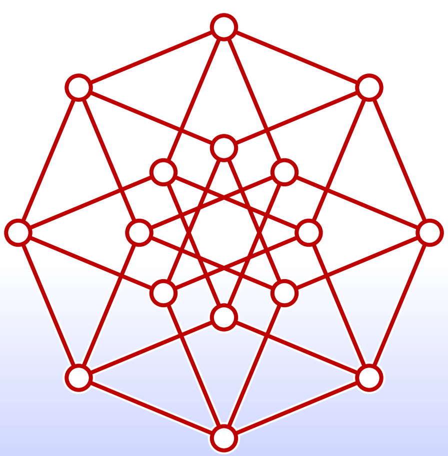
So dim ≤ k iff compatibility graph k-colorable

Polynomial time for k = 2 [Dress 1992]

Tree dimension as graph coloring

Any graph is the compatibility graph of some partial cube
(more specifically, of some median graph: Klavzar and Mulder 2002)



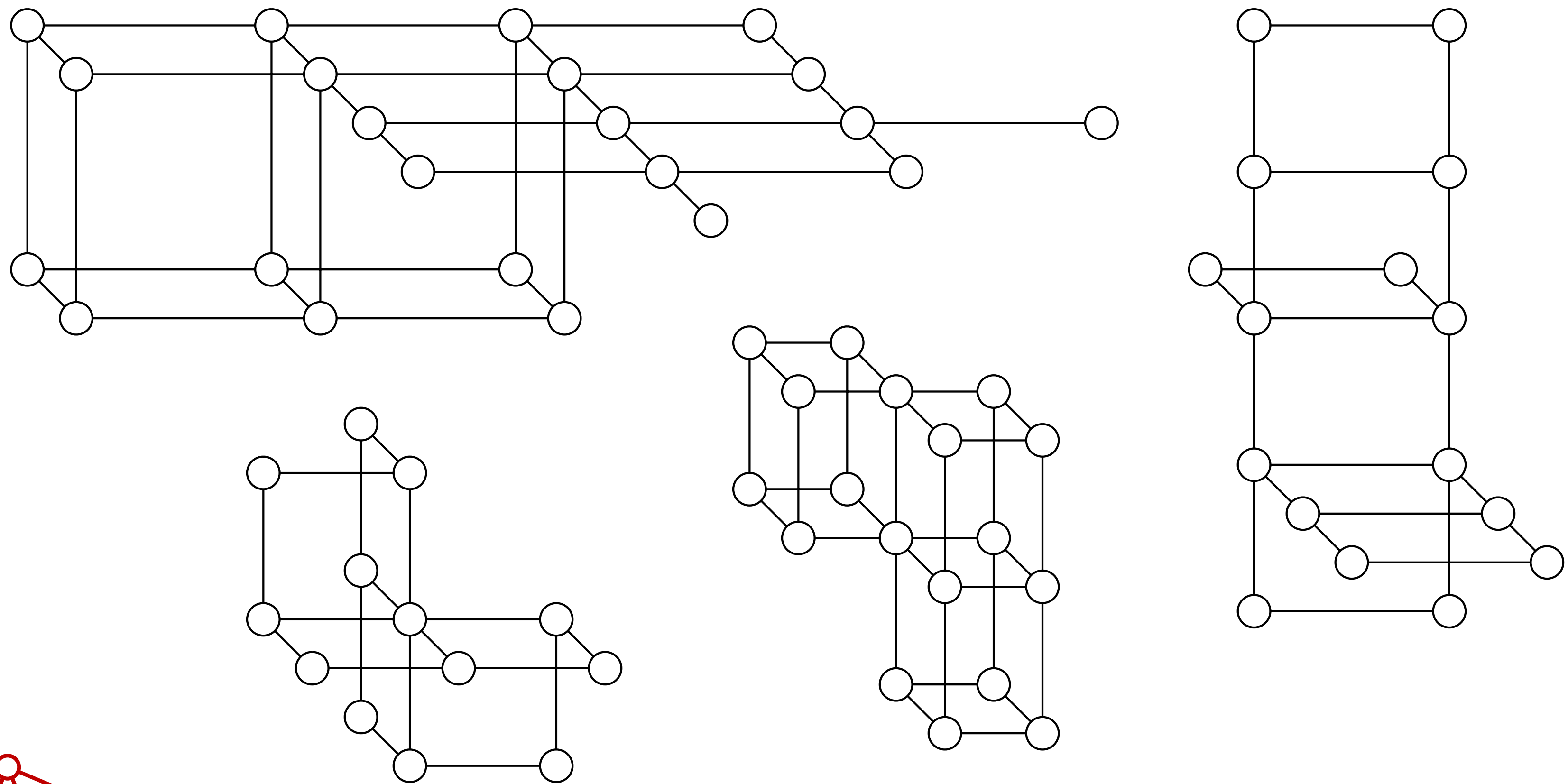


So finding tree dimensions ≥ 3
is as hard as graph coloring

[Bandelt and van de Vel 1989]

Lattice dimension

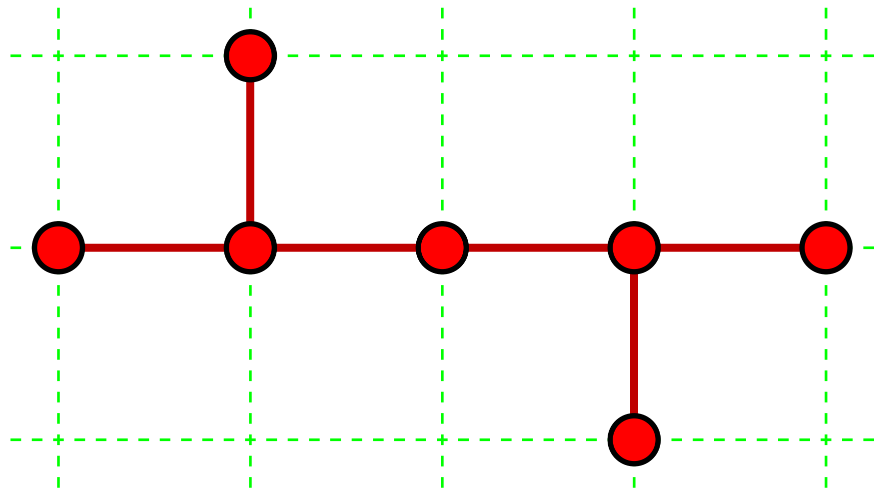
Embed partial cube isometrically into integer lattice
with as low a dimension as possible



Note: embedding into lattices non-isometrically
(but with all vertex positions distinct)
is NP-complete even for trees
[Bhatt and Cosmodakis 1987]

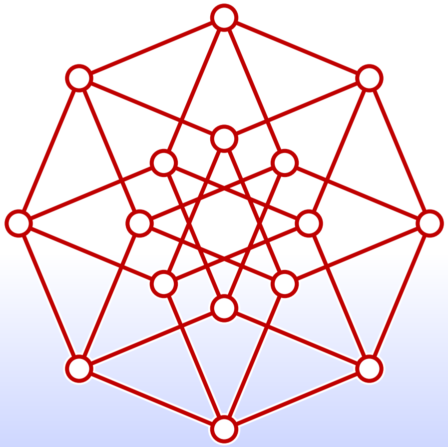
Lattice dimension of trees

Lattice dimension of a tree = ceiling(leaves/2)
[Hadlock and Hoffman 1978; Ovchinnikov 2004]



Lower bound: at most two extreme points in each direction

Upper bound: remove two leaf paths, embed rest, add leaves back



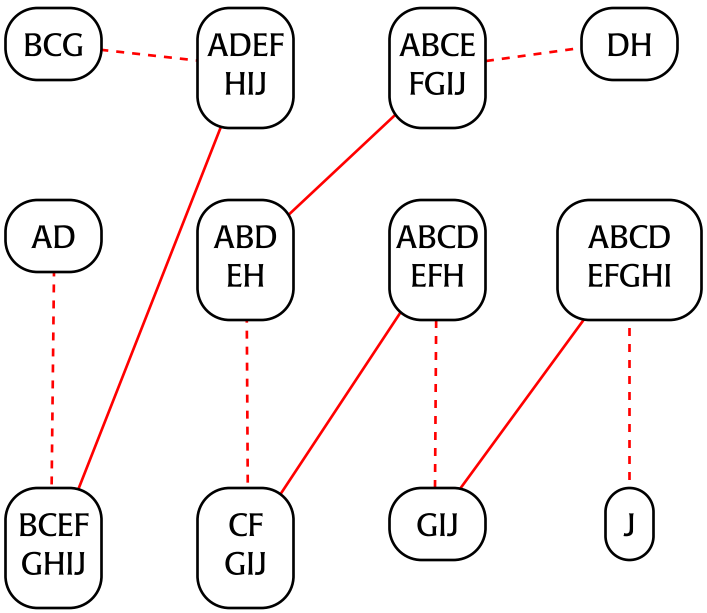
Corollary: dimension ≥ degree/2

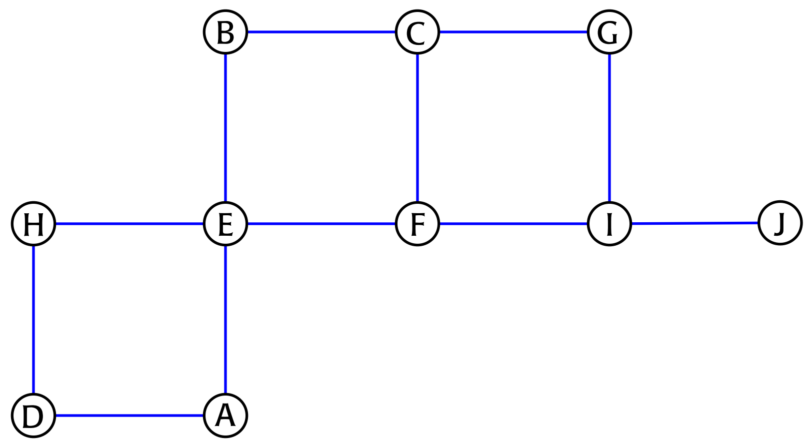
e.g. for integer partitions of n, dim = Θ(sqrt n)

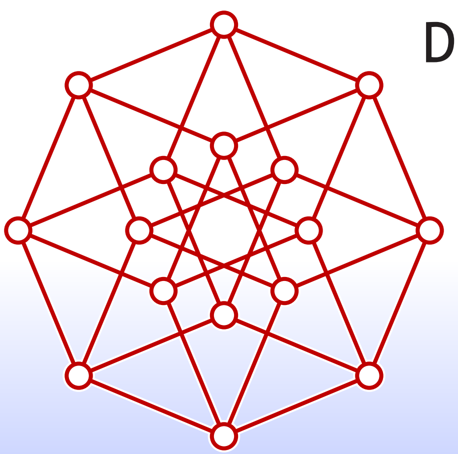
Lattice embeddings as paths of semicubes

Semicubes = sets of points with *i*th coordinate ≤ or ≥ some constant

For each coordinate of embedding, form path of semicubes, alternating ≤/≥





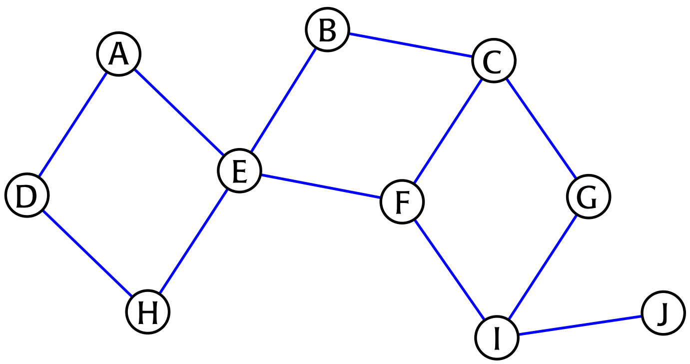


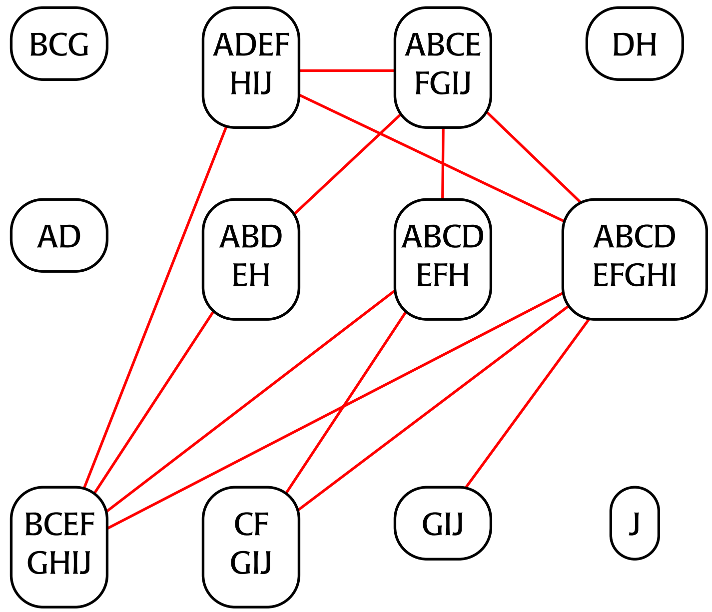
Dashed edges: complementary pairs of semicubes

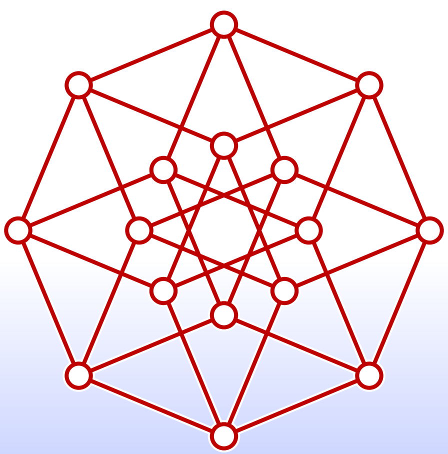
Solid edges: noncomplementary semicubes
that cover the entire partial cube

Characterizing lattice dimension

Theorem: lattice dimension = isometric dimension – M
where M = cardinality of maximum matching in semicube graph
[E. 2005]







Leads to polynomial time algorithm
for finding minimum dimension embedding

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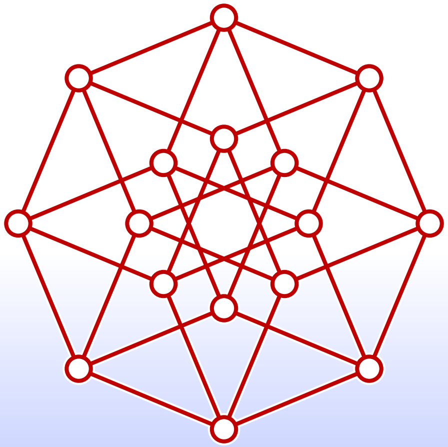


Graph drawing desiderata

Visualize graphs by placing points (or small disks) at vertices, drawing edges as line segments as more complex curves

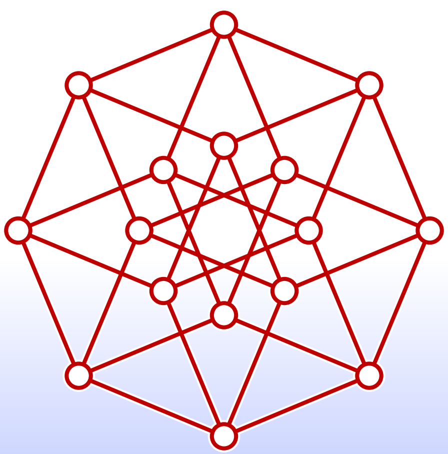
Keep unrelated vertices and edges well separated from each other

Minimize crossings
(planar graph: no crossings)

Use small area relative to vertex separation

Make important graph structure visually apparent: drawing technique for special class of graphs should produce a drawing specific to that class

(e.g. draw planar graphs without crossings)



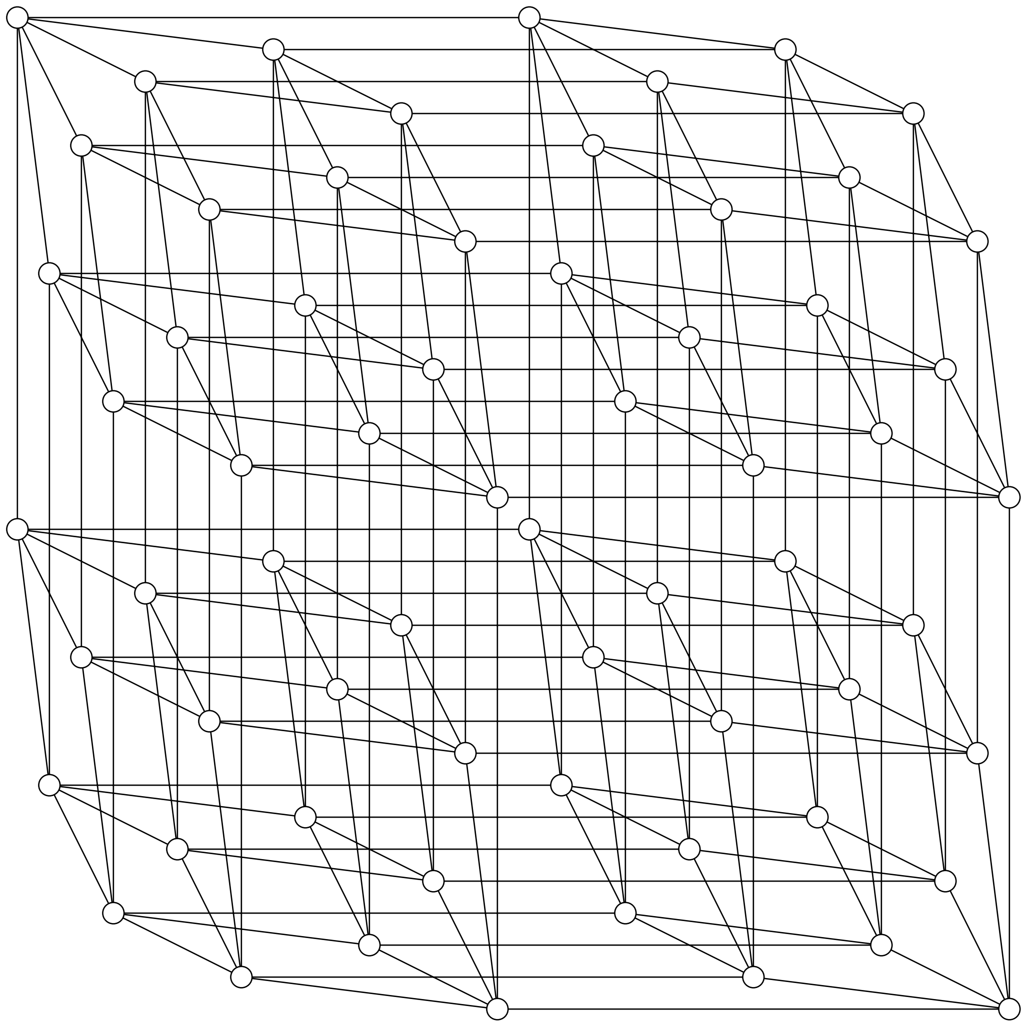
Visualize partial cube by projecting lattice embedding

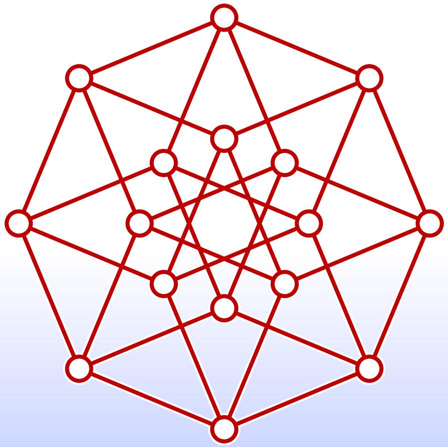
Works for any partial cube

Parallel edges in lattice
shown parallel in drawing
(embedding structure is visible)

Space-efficient for hypercube

Can be less closely packed
for other partial cubes





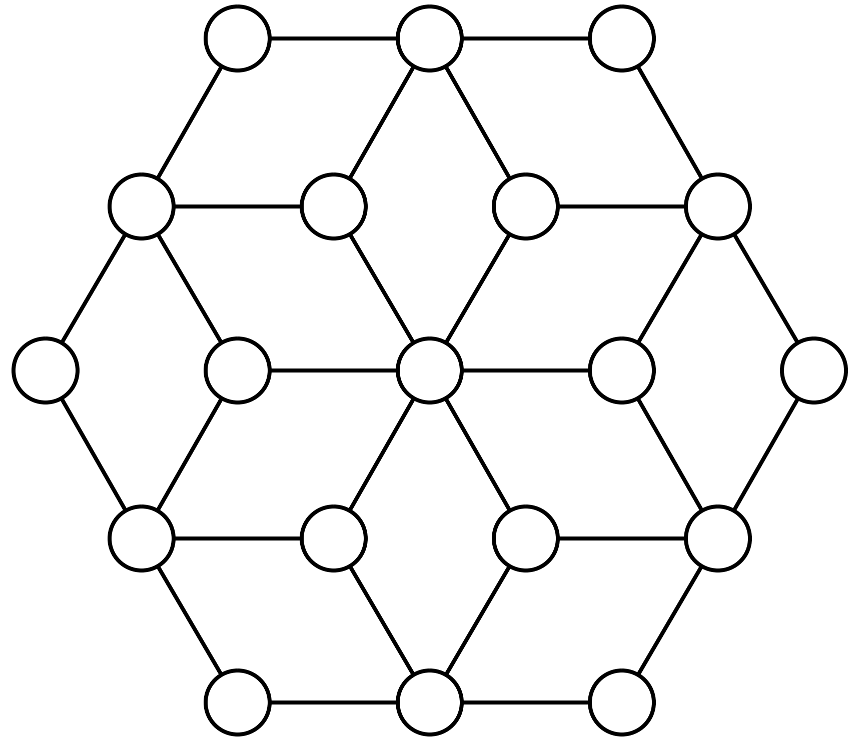
Planar projection of three-dimensional lattices

Easy to find good projection
of fixed 3d embedding:

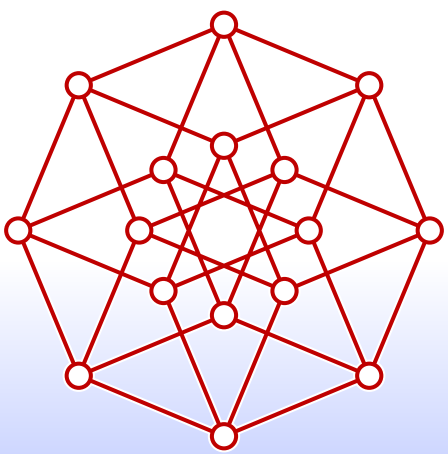
Always along main diagonal

Projects onto hex lattice

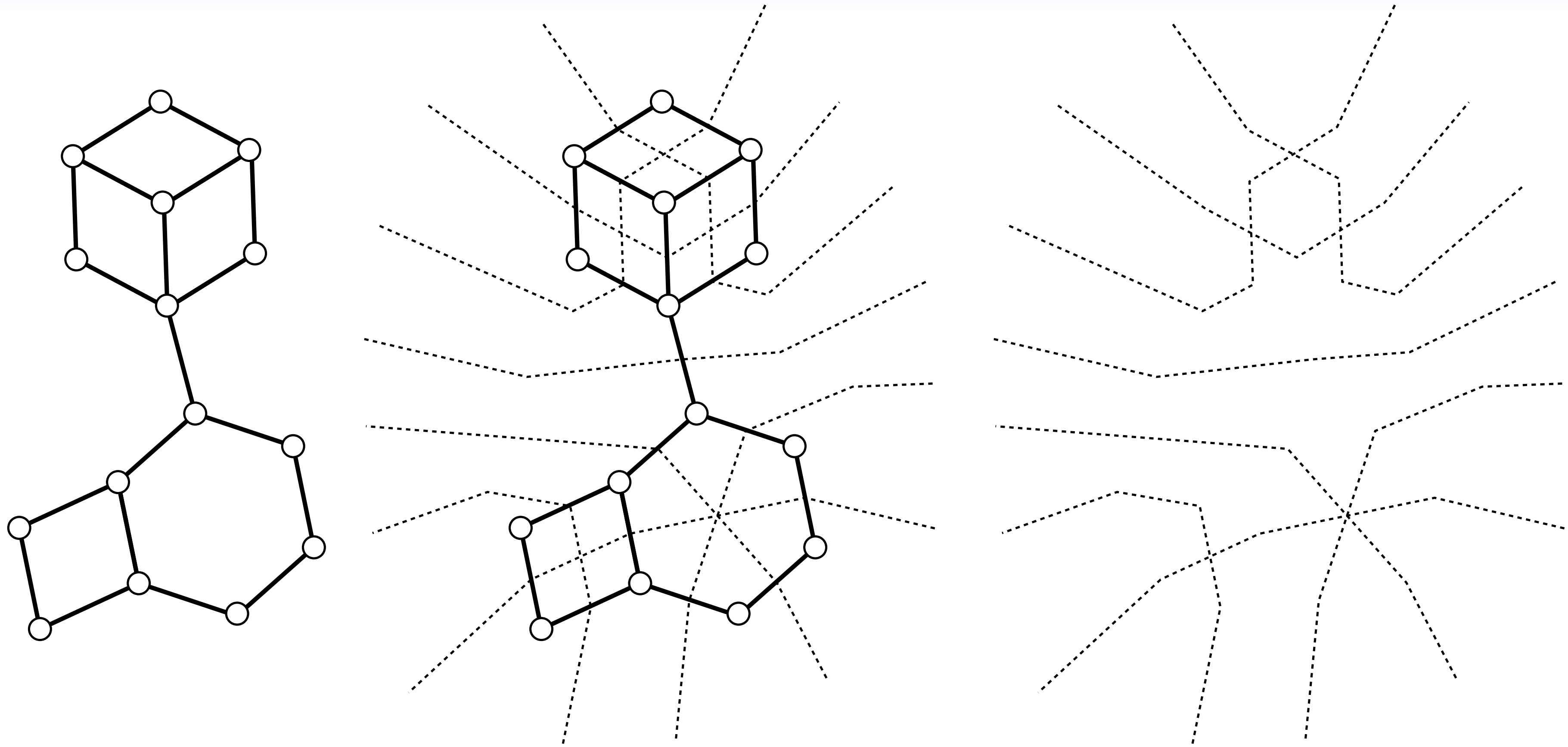
Open: how to find
projectable embedding?



partial orders on 3 elements

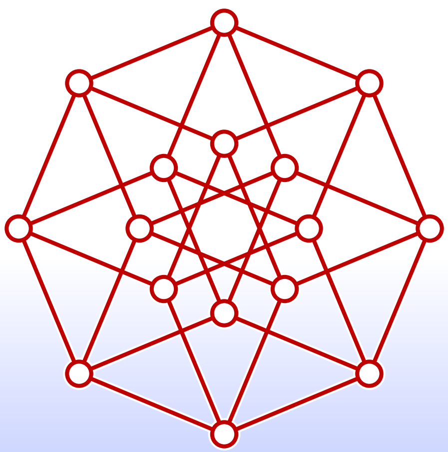


Planar graphs with centrally-symmetric faces



= region graphs of (weak) pseudoline arrangements

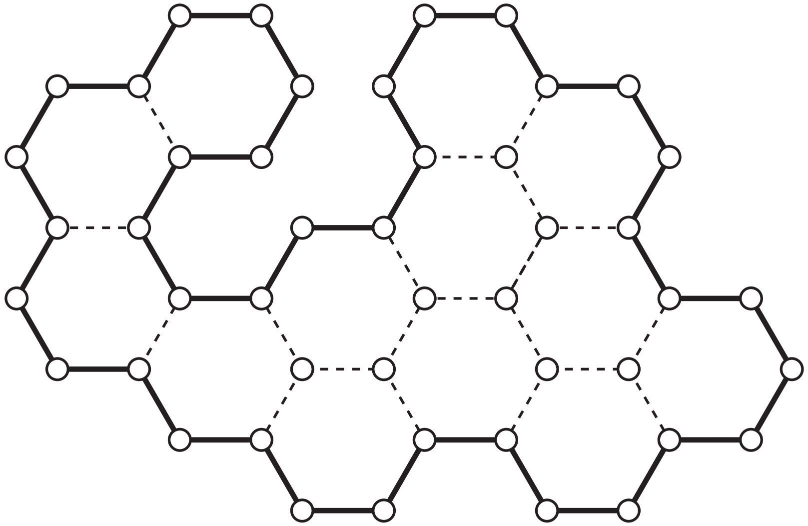
[E. 2004]



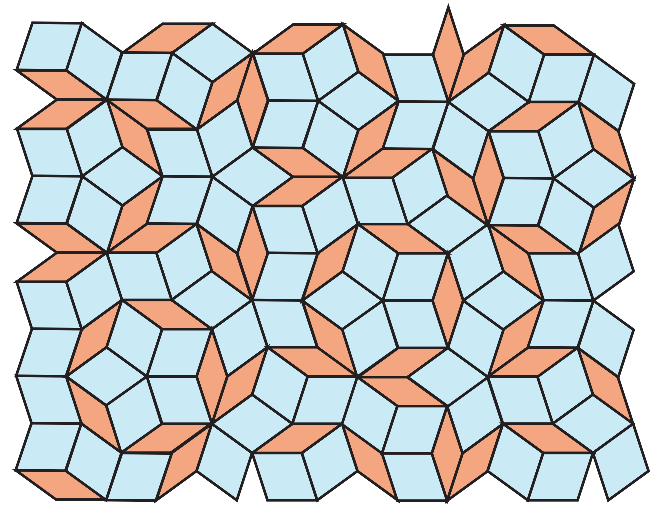
pseudolines: curve topologically equivalent to lines
(unbounded, partition plane, cross at most once)

weak: some pairs of pseudolines may not cross

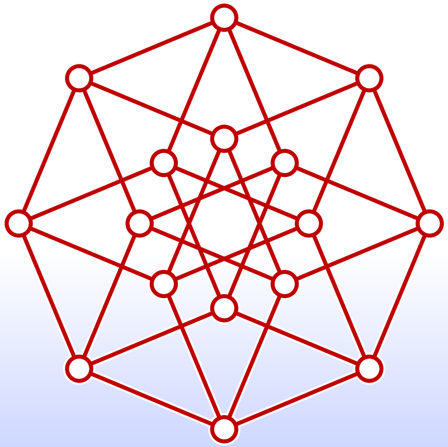
Examples of planar graphs with centrally-symmetric faces

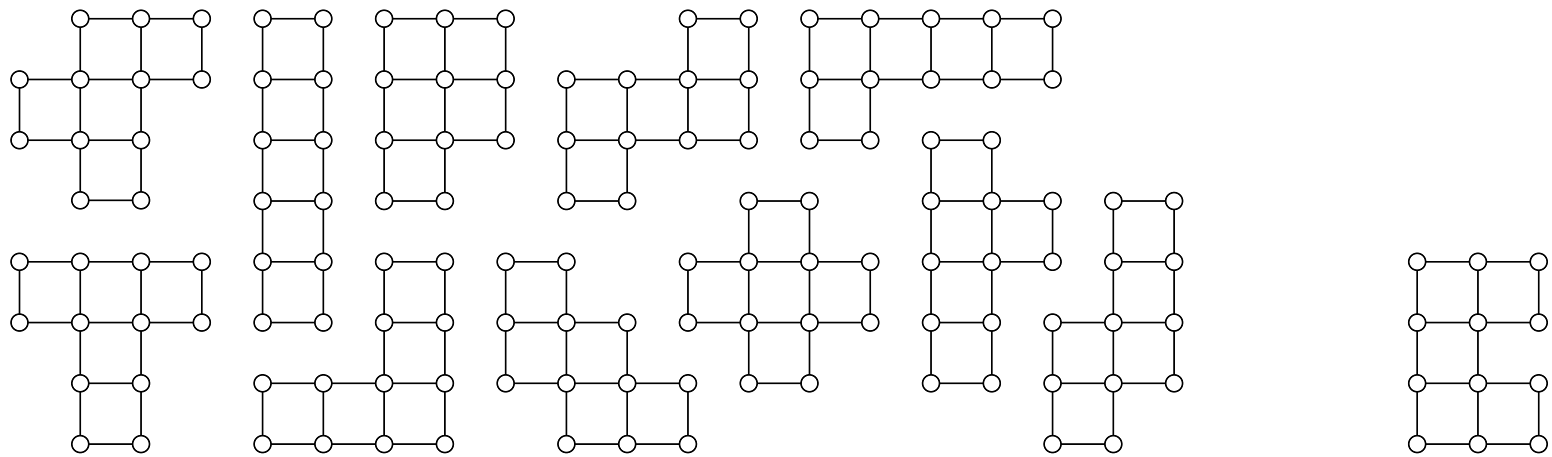


benzenoid system (chemistry)



Penrose rhombic aperiodic tiling





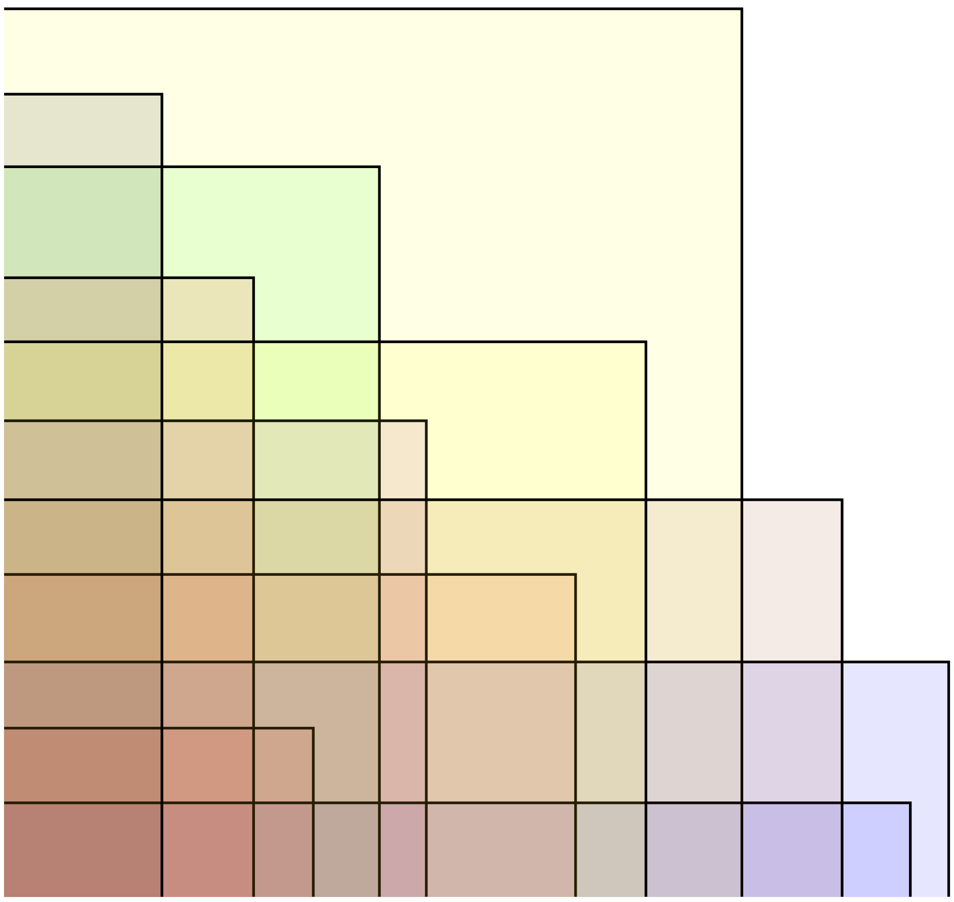
polyominoes

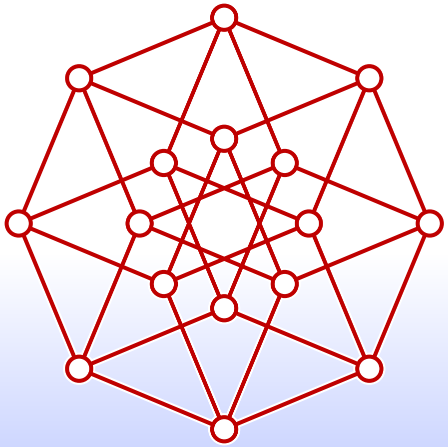
Arrangement of translates of a wedge

Wedge translates are pseudolines
Region graph forms partial cube

Theorem: wedge-representable
partial cubes are exactly
antimatroids with order dim = 2

[E. 2006]



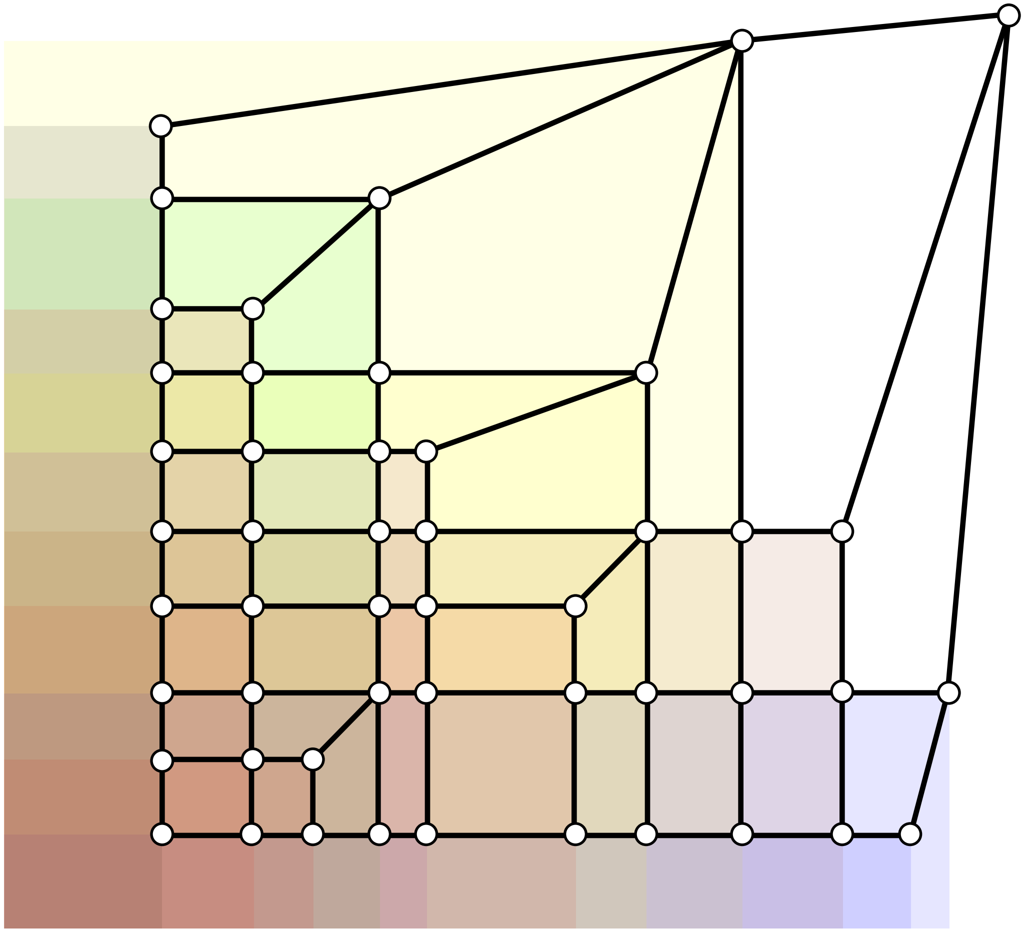


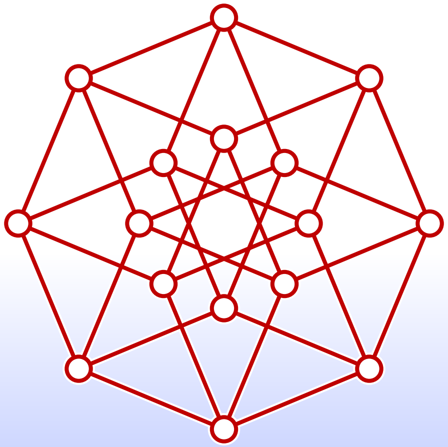
Upright-quad drawing

Place vertex at upper left corner of each region

Region graph of arrangement has all faces convex quads with bottom and left sides axis-parallel

All drawings of this type come from order-dim-2 antimatroids





Outline

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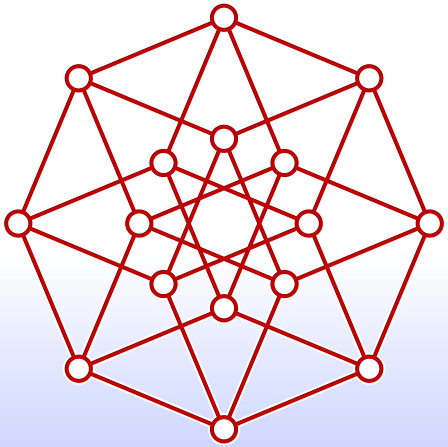
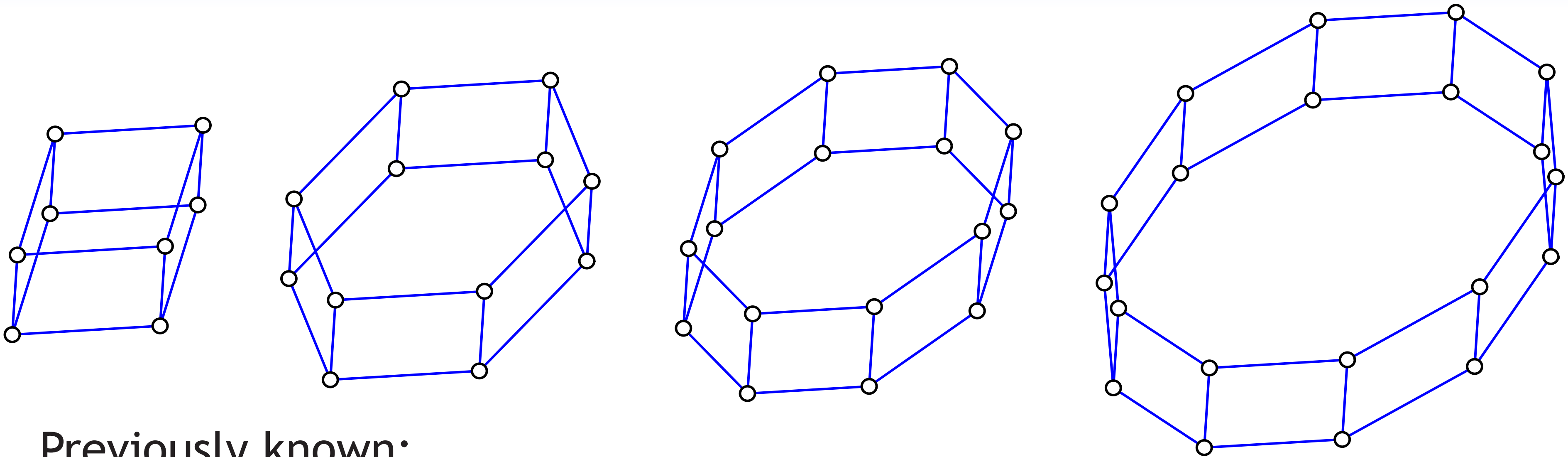
Graph Drawing

Cubic Partial Cubes

Flip Distance



Cubic (3-regular) partial cubes

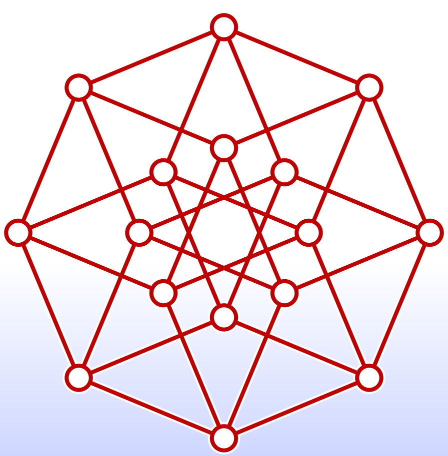


Previously known:

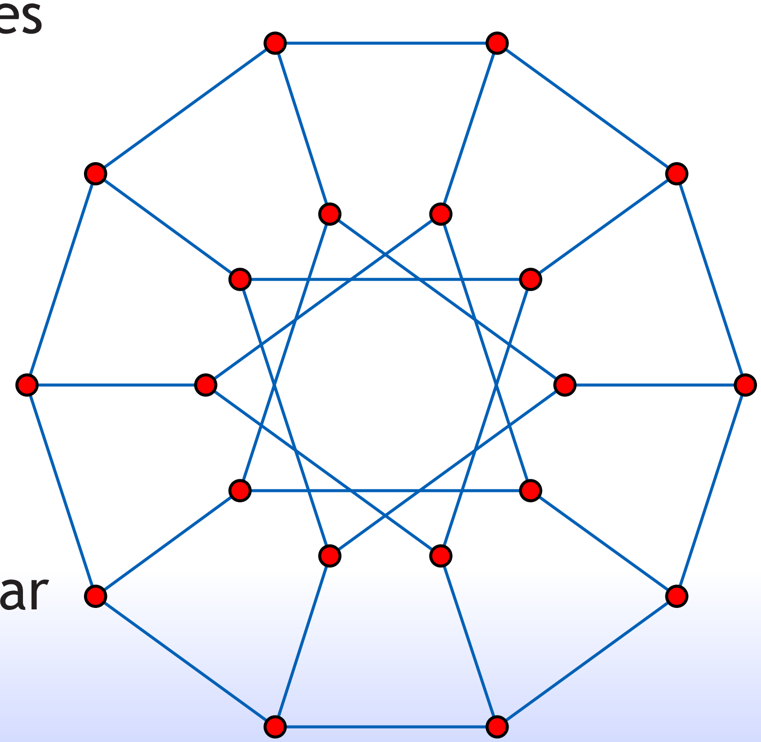
one infinite family (prisms over even polygons)

a small number of additional sporadic examples

[Bonnington, Klavzar, Lipovec 2003]
[Bresar, Klavzar, Lipovec, Mohar 2004]
[Deza, Dutour-Sikiric, Shpectorov 2004]
[Klavzar, Lipovec 2003]

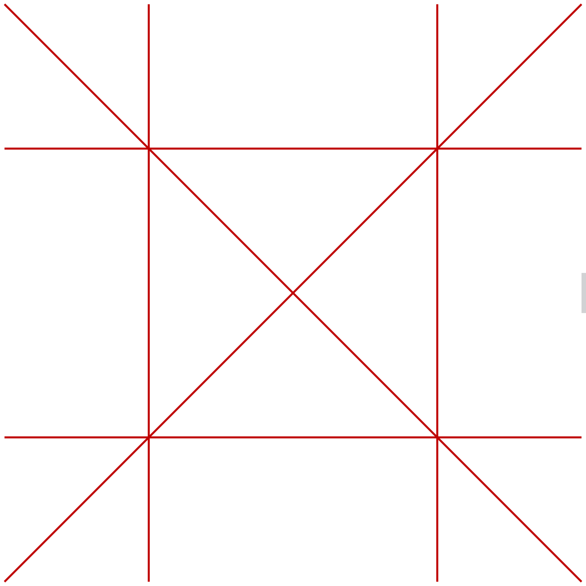


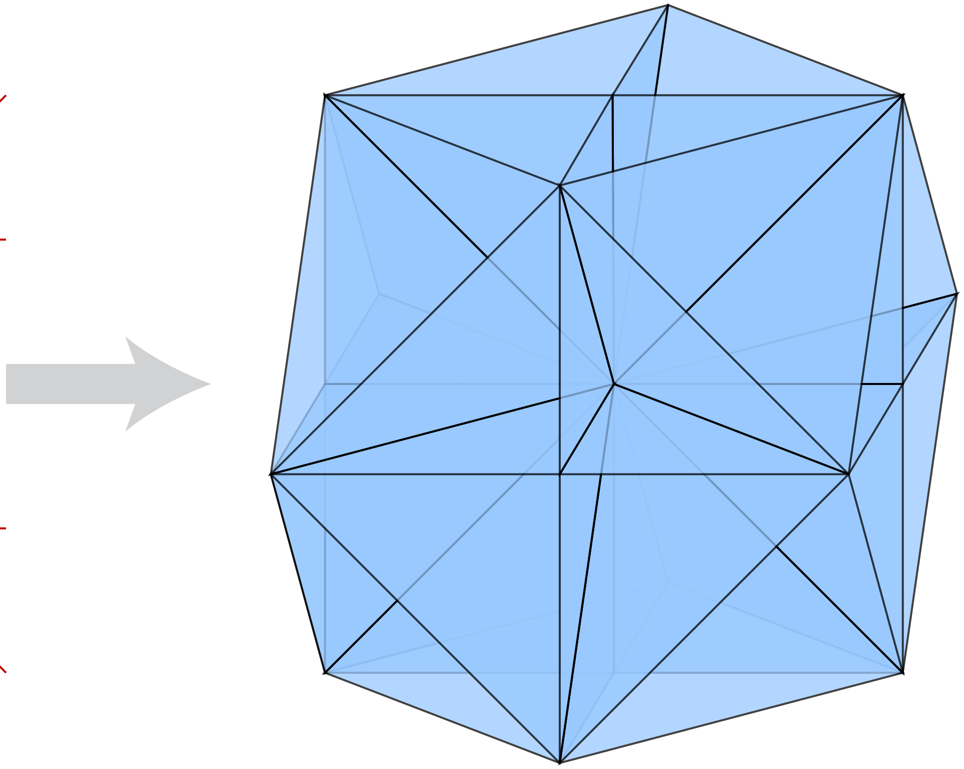
Desargues graph,
only known nonplanar
cubic partial cube

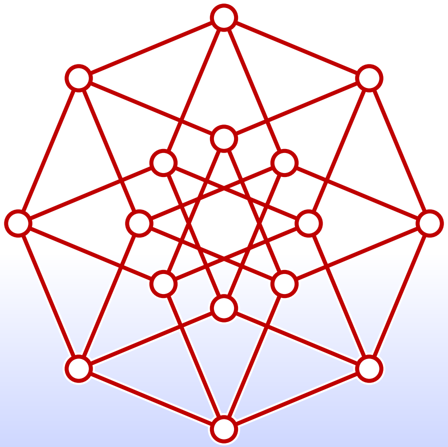


Cubic partial cubes from simplicial arrangements

[E., 2006]







Central projection lifts lines in plane
to planes through origin in 3d

Cell of line arrangement forms two cells of 3d arrangement

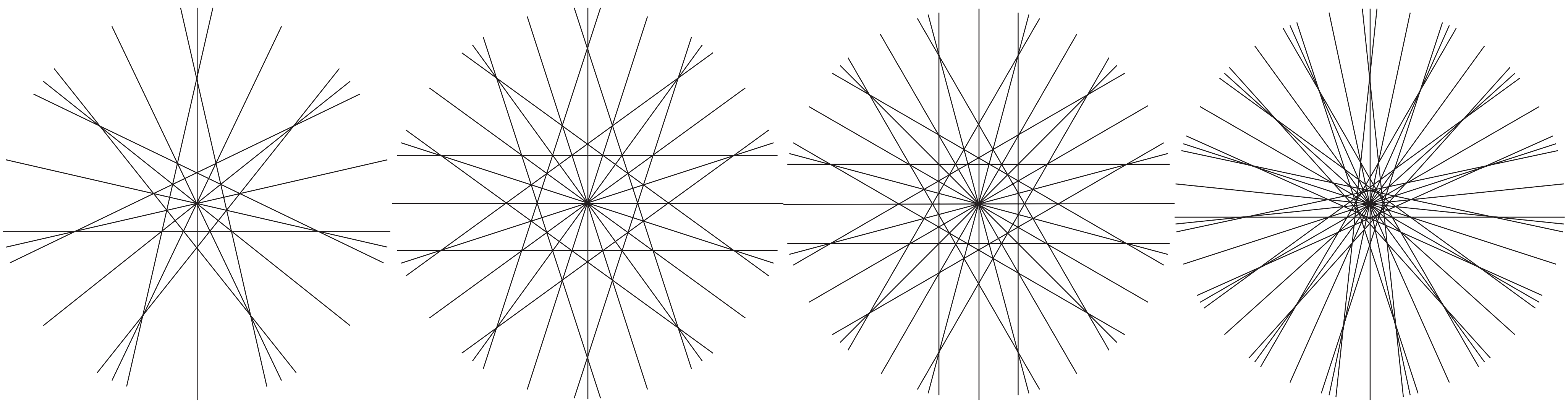
Simplicial line arrangement gives cubic region graph

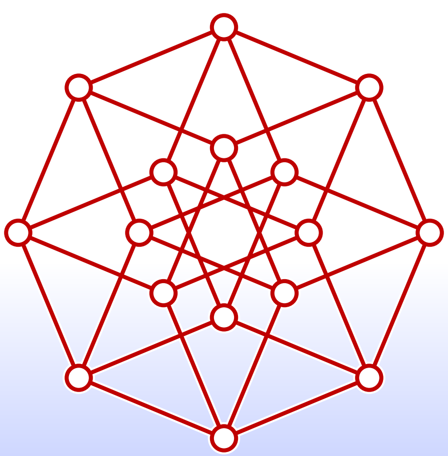
Infinite families of cubic partial cubes

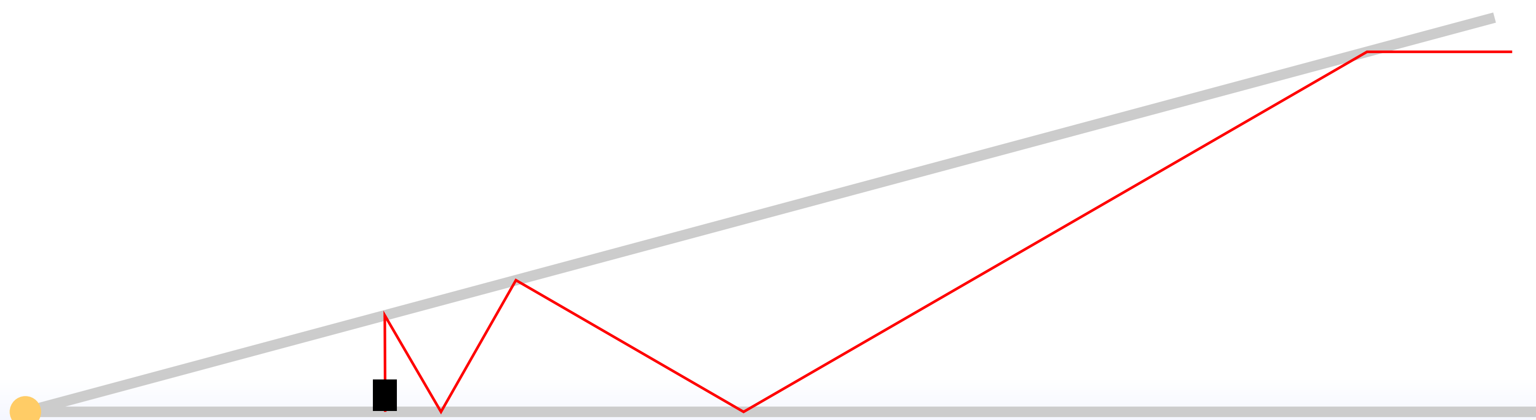
Several infinite families of simplicial arrangements known [Grünbaum 1972]

Simplest: pencil of lines (leading to even prism partial cubes)

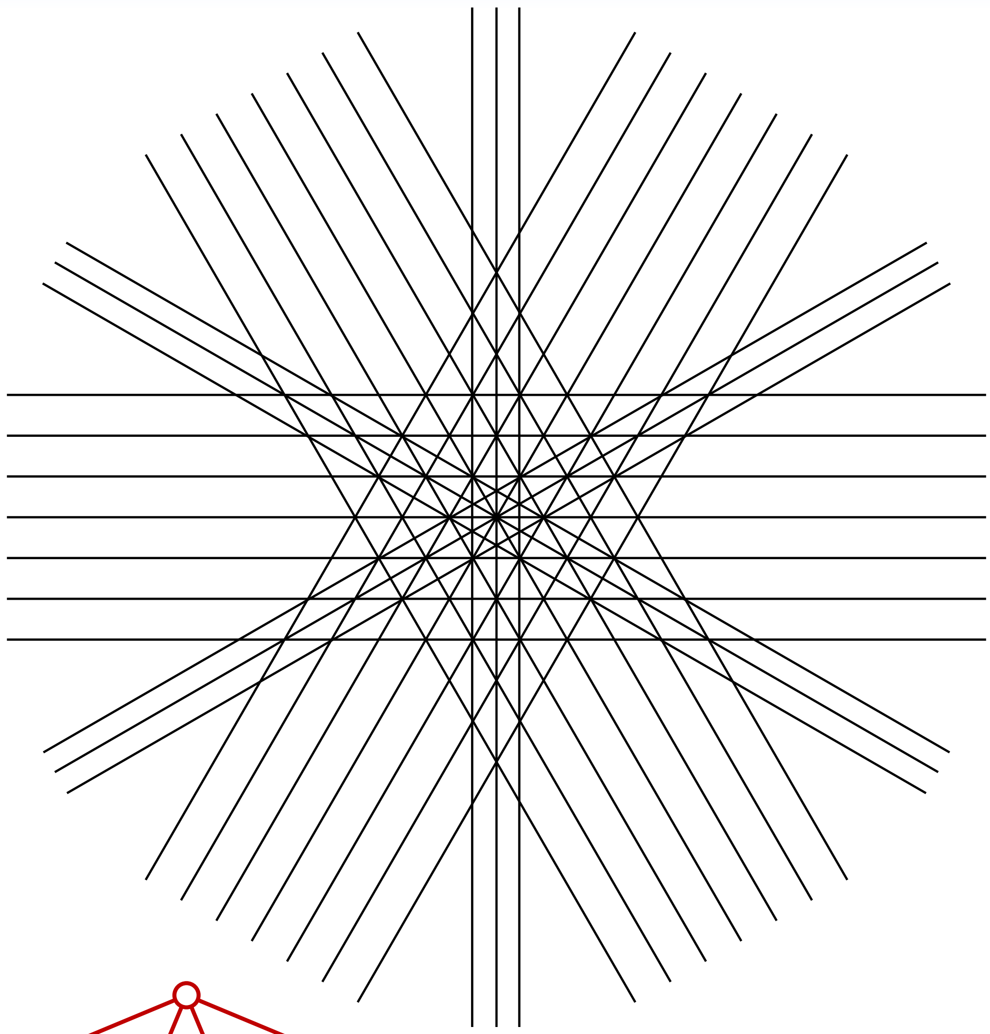
Second simplest: two mirrors and a laser

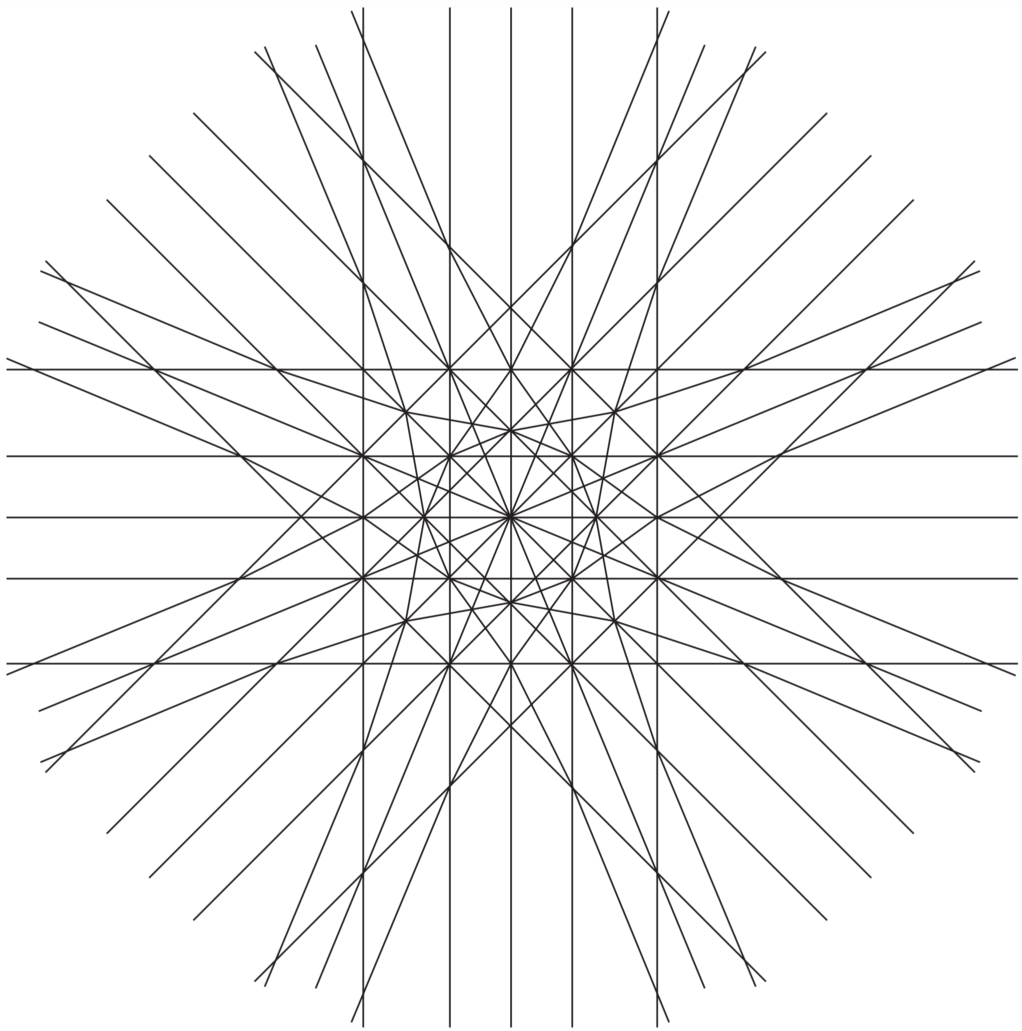


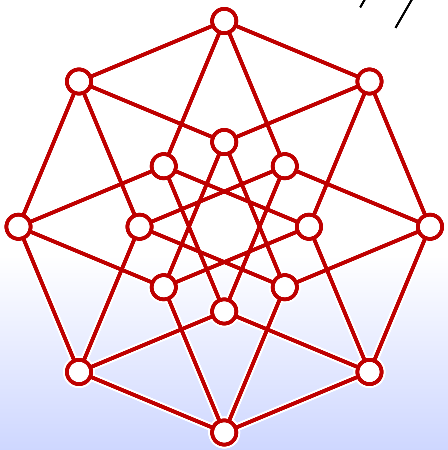




Additional sporadic cubic partial cubes



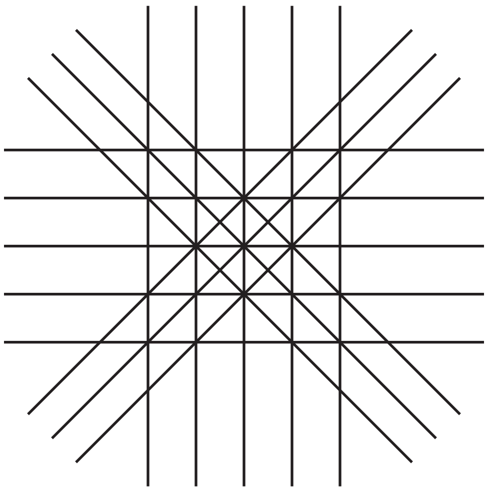




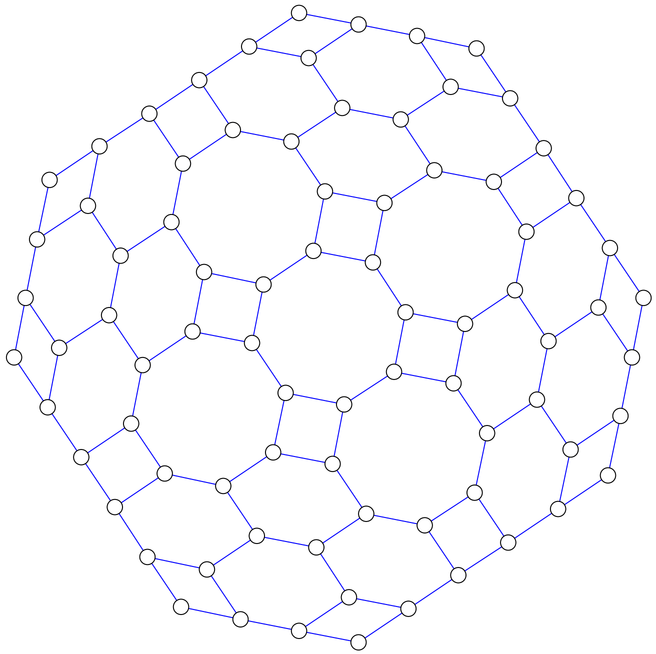
Many additional sporadic examples

Works also for pseudoline arrangements

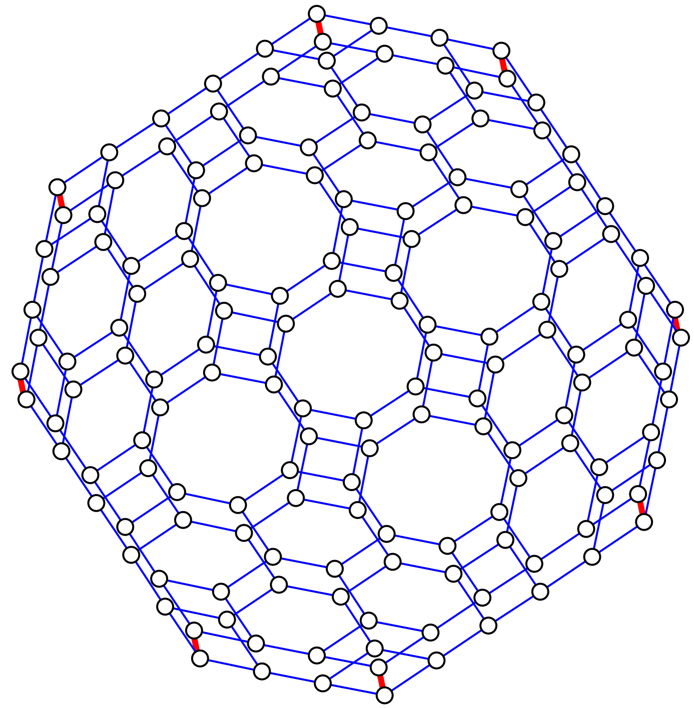
Connection to symmetric-face planar drawing



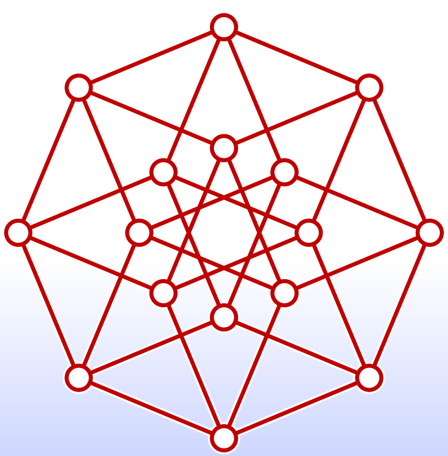
simplicial arrangement



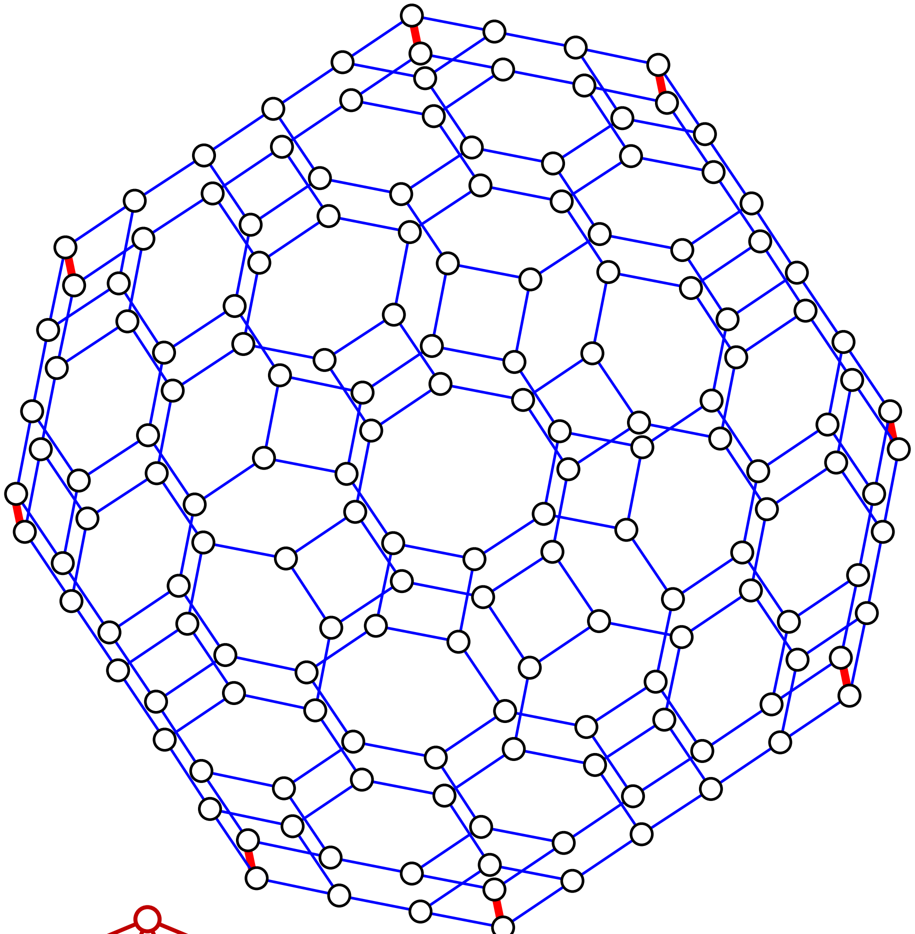
planar dual: symmetric-faced drawing



connect to inverted copy of drawing: cubic partial cube

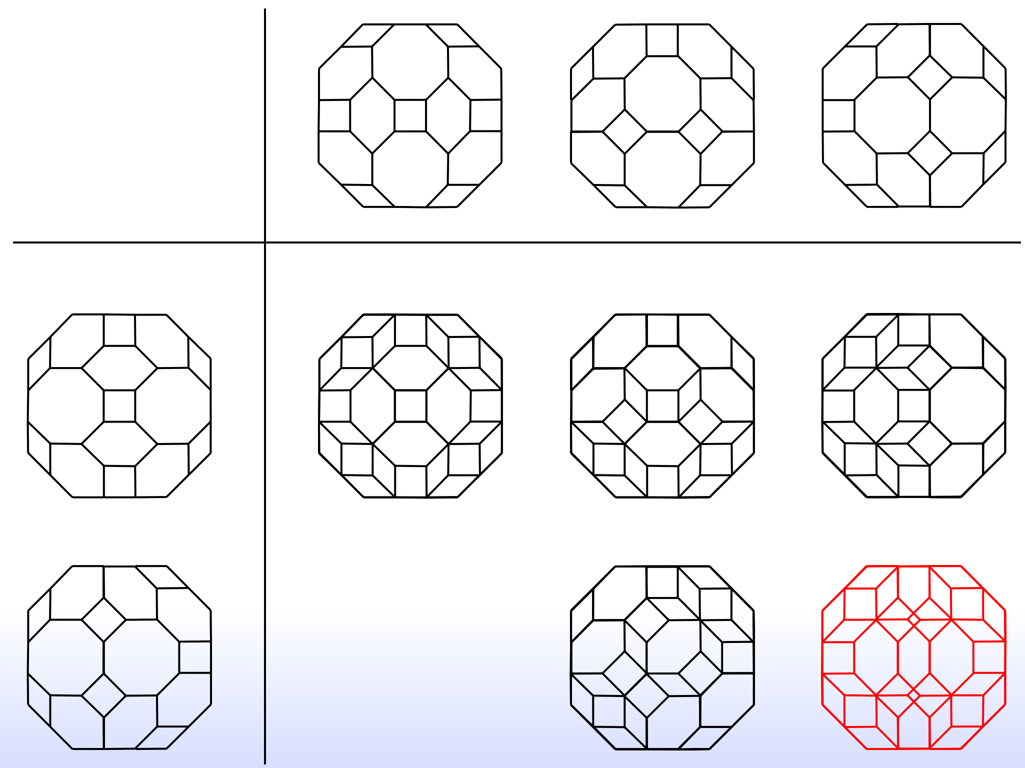


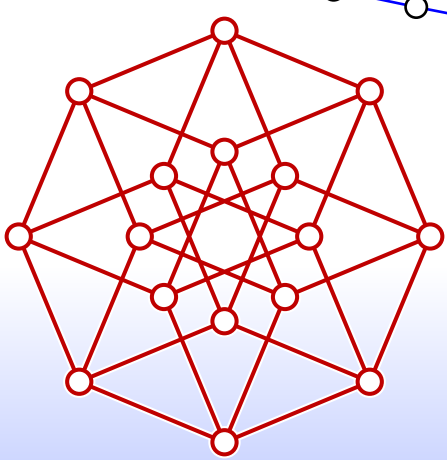
Sometimes, connecting two different drawings works



Sufficient condition:

overlaying the two drawings
produces another
symmetric-faced planar drawing





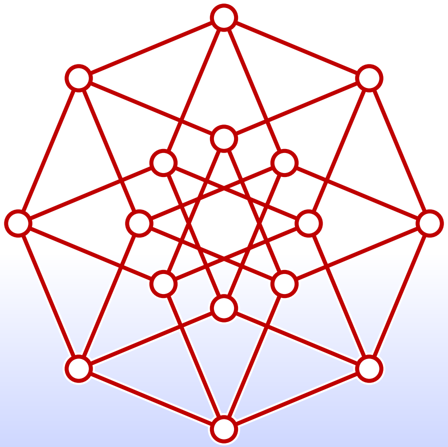
Summary of progress on cubic partial cubes

Several new infinite families
(both lines and pseudolines)

Very large number of sporadic examples

Small number of new non-arrangement examples

Still open: are there infinitely many cubic partial cubes
not formed from a line or pseudoline arrangement?



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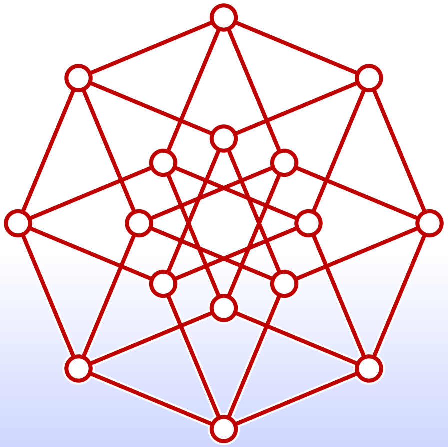


Flip graphs of point sets

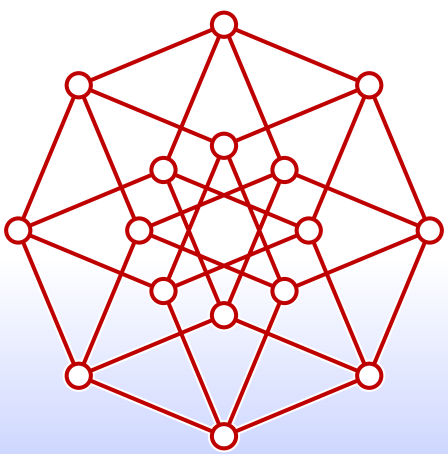
Vertices = triangulations
(here, of 3x3 grid)

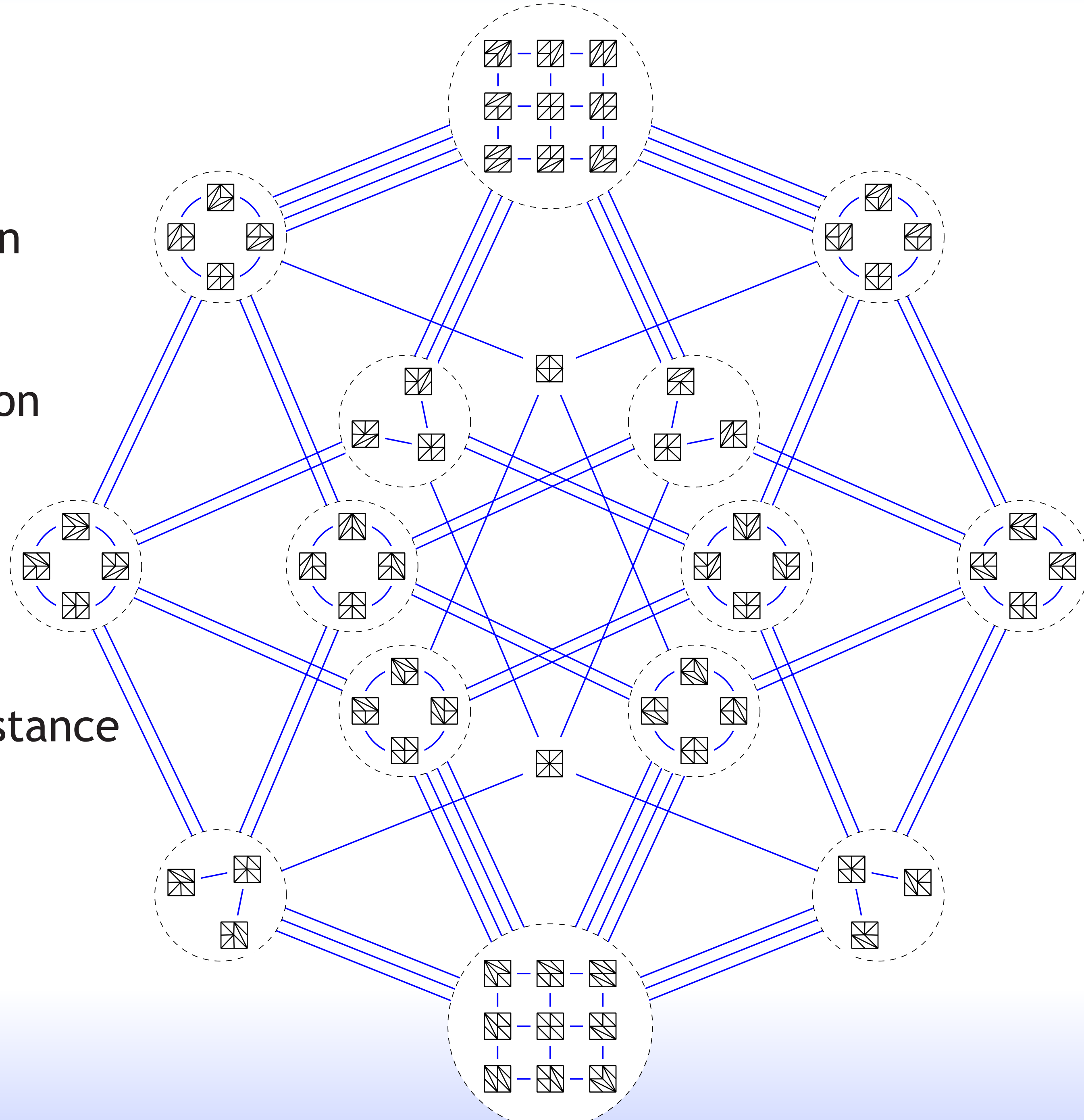
Edges = change triangulation
by one edge ("flip")

Important in mesh generation

Generalizes rotation
in binary trees

Important open problem in
algorithms: compute flip distance



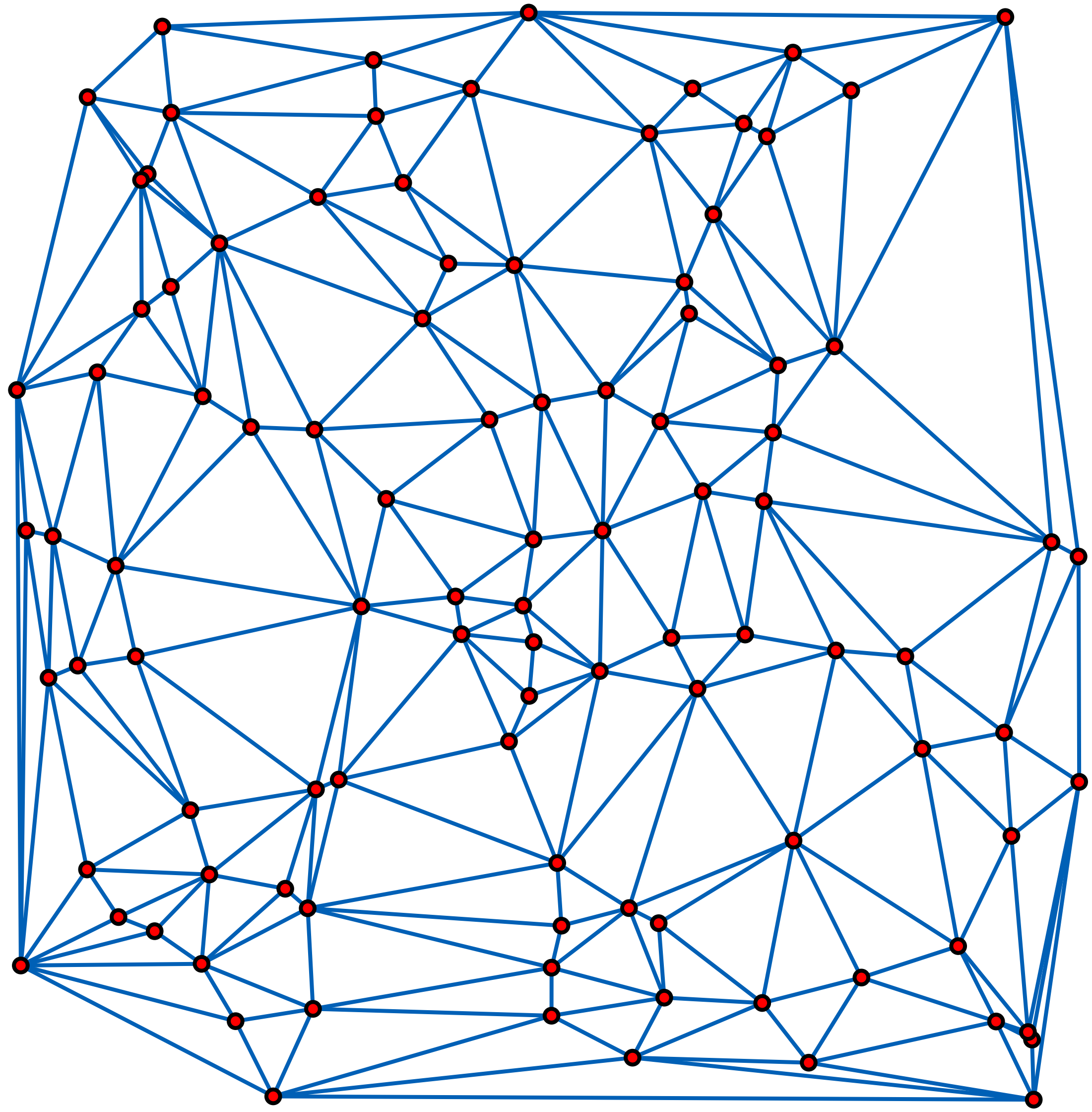


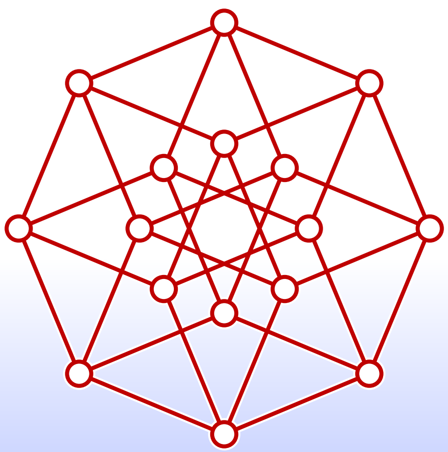
Delaunay triangulation

Connect two points if there exists a circle containing them and no others

Flip towards DT of the four flipped vertices:
always reaches DT of whole set in $O(n^2)$ flips

Any two triangulations have flip distance $O(n^2)$:
flip both to Delaunay

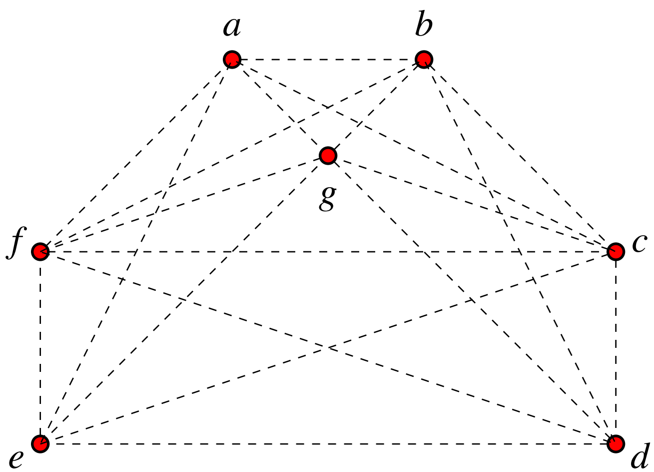


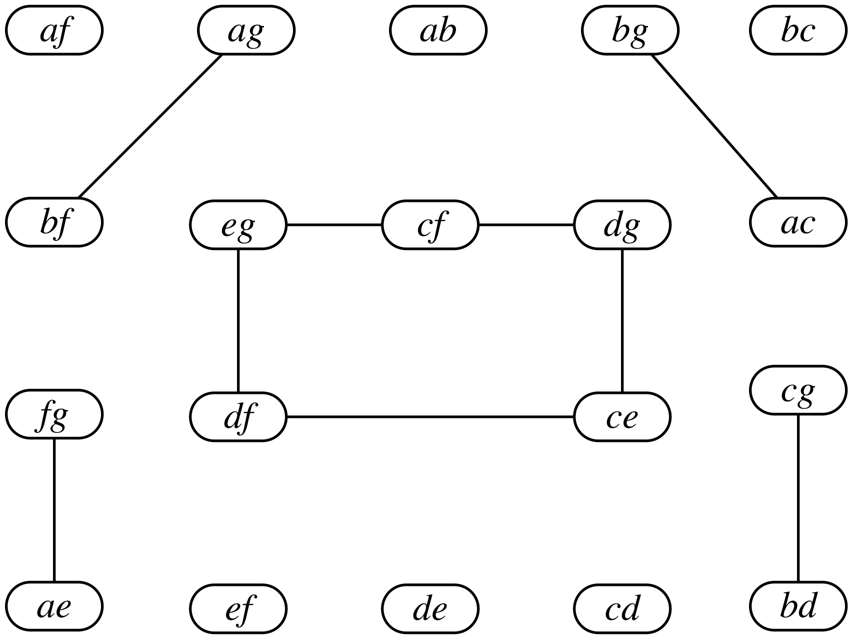


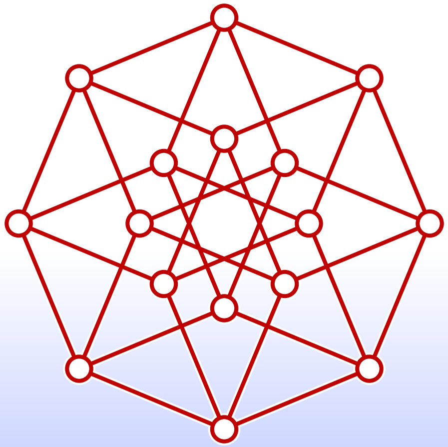
The quadrilateral graph of a point set

Vertices:
Unobstructed line segments
= Possible edges of triangulations

Edges:
Pairs of segments that cross
= Possible flips of triangulations
= Empty quadrilaterals







(if no empty quad, only one triangulation)

Flip graphs and partial cubes

Theorem [E., 2007]. The following are equivalent:

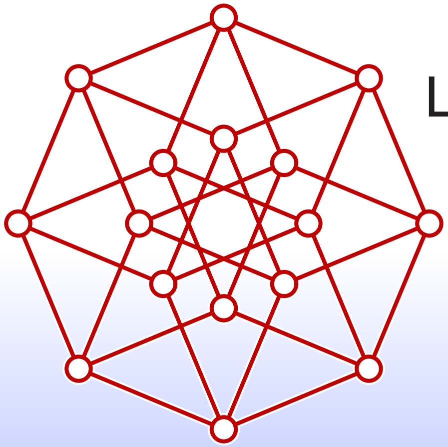
the flip graph is a partial cube
iff the flip graph is bipartite
iff the quadrilateral graph is a forest
iff the quadrilateral graph is bipartite
iff the points have no empty convex pentagon

Proof ideas:

If empty pentagon exists, other statements all false (easy)

else...

Orient quad graph by flipping towards Delaunay triangulation
Each node has a unique parent, so must be a forest
Embed flip graph into product of its trees
Use Constrained DT to show that distance-reducing flip exists

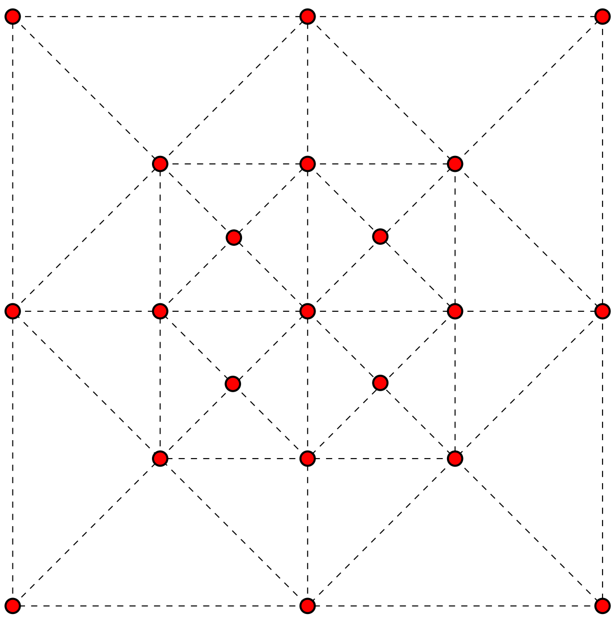


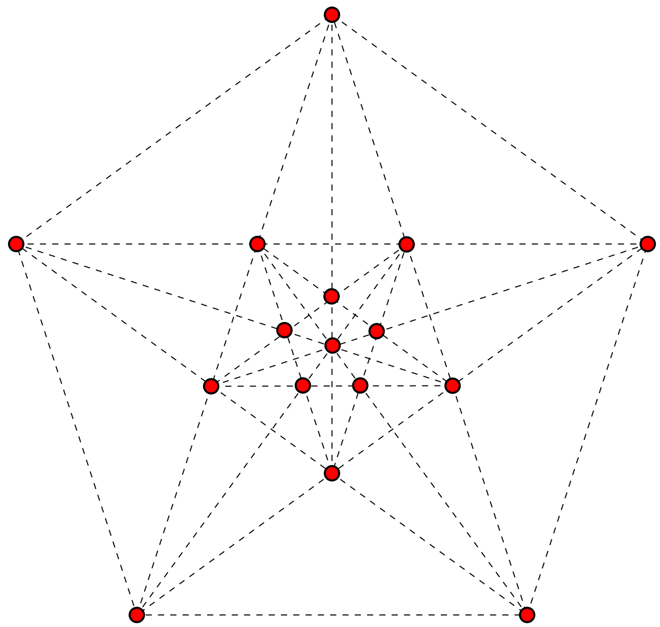
Leads to **quadratic-time flip distance algorithm**:
flip both triangulations towards Delaunay
count edges occurring in only one flip sequence

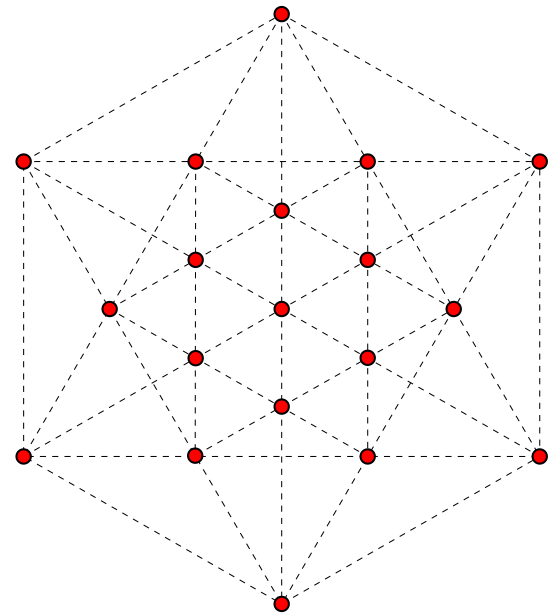
Point sets with no empty pentagon

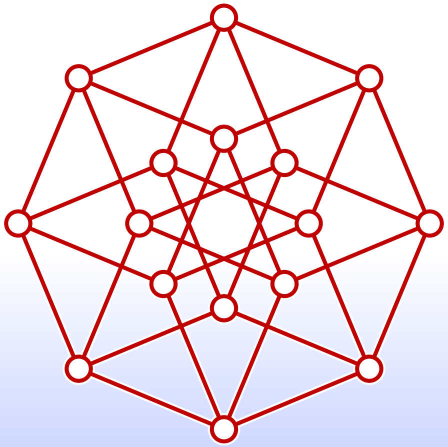
Cannot be in general position and have 10 or more points [Harborth 1978]

But many non-general-position examples possible...









any convex subset of a lattice

any set of points on two lines

many additional sporadic examples

Conclusions: Geometry and partial cubes

Geometric constructions can help us understand the structure of partial cubes

(e.g. cubic partial cube examples)

Geometric information about partial cubes is interesting and useful to study

(e.g. dimension, graph drawing techniques)

Understanding partial cubes can help us solve geometry problems

(e.g. flip distance)