On Automorphism Groups of Vertex-transitive Graphs

Ted Dobson

Definition 1 An automorphism of a graph Γ is a bijection $\alpha : V(\Gamma) \to V(\Gamma)$ such that $xy \in E(\Gamma)$ if and only if $\alpha(x)\alpha(y) \in E(\Gamma)$ for every $xy \in E(\Gamma)$. The automorphism group of Γ, denoted by $\mathrm{Aut}(\Gamma)$, is the set of all automorphisms of Γ, which is a group.

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Definition 2 A permutation group G acting on a set X is transitive if for every $x, y \in X$ there exists $g \in G$ such that $g(x) = y$. A graph Γ is said to be vertex-transitive if $\mathrm{Aut}(\Gamma)$ is a transitive group acting of $V(\Gamma)$.

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Definition 3 Let G be a group and $S \subset G - \{1\}$ such that $S^{-1} = S$. Define a graph $\Gamma = \Gamma(G, S)$ by $V(\Gamma) = G$ and $E(\Gamma) = \{(g, gs) : g \in G, s \in S\}$. The graph $\Gamma(G, S)$ is the Cayley graph of G with connection set S. Note that the group G_L of all bijections $g \to hg$ (multiplication by h on the left) is a subgroup of $\mathrm{Aut}(\Gamma)$ and is transitive. Thus a Cayley graph is a vertex-transitive graph.

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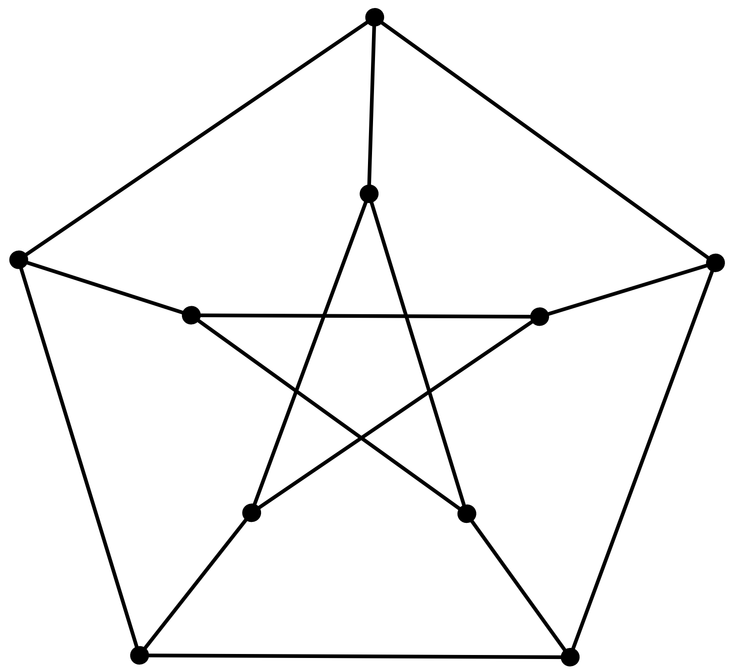
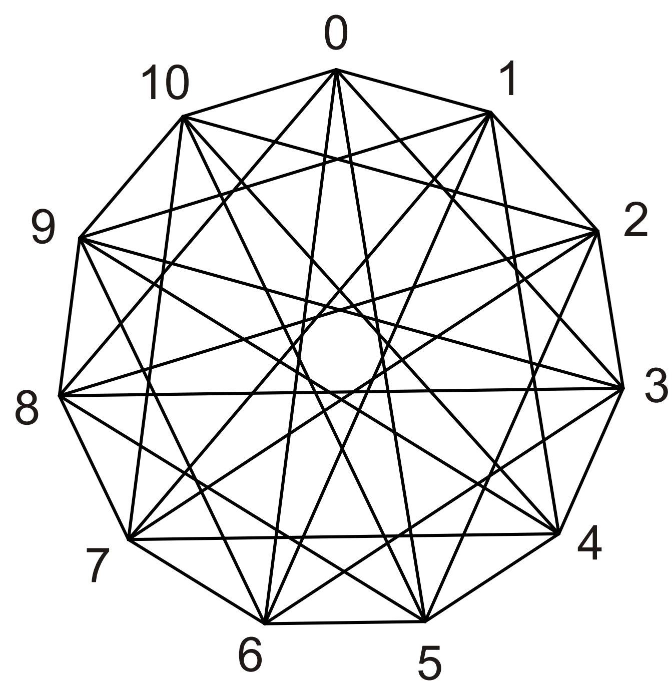
Although we will usually say graph, almost all results hold for digraphs and in fact edge-colored digraphs.

The Petersen graph is vertex-transitive but not isomorphic to a Cayley graph of any group. It's full automorphism group is isomorphic to S_5.



A *circulant graph of order n* is simply a Cayley graph of $\mathbb{Z}_n$. The following graph is $\Gamma(\mathbb{Z}_{11}, \{1, 3, 5, 6, 8, 10\})$. Notice that there is an edge between any two vertices if and only if the difference of the vertices (modulo 11) is an element of $\{1, 3, 5, 6, 8, 10\}$.



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2. Normal Cayley graphs - determine Cayley graphs Γ of G such that G_L is normal in $\mathrm{Aut}(\Gamma)$.

3. Is it true that almost all Cayley graphs have automorphism group as small as possible? are normal? These are conjectures of B. Alspach and M. Y. Xu.

4. Determine arc-transitive and half arc-transitive graphs.

   Arc-transitive graphs are vertex-transitive graphs that transitive on the set of directed edges while half arc-transitive are vertex-transitive and edge-transitive but not arc-transitive. Every automorphism group of a vertex-transitive graph can be written as the intersection of the automorphism groups of arc or half arc-transitive graphs.

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Observe that every vertex-transitive graph Γ of prime order is isomorphic to a circulant graph of prime order as $\mathrm{Aut}(\Gamma)$ contains a subgroup of order p, which, after a relabeling, we may assume is $\langle(0,1,\ldots,p-1)\rangle = \langle x \to x+1\rangle = (\mathbb{Z}_p)_L$. So circulant!

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Theorem 1 (Burnside, 1901) *Let $G \leq S_p$, p a prime, contain $(\mathbb{Z}_p)_L$. Then $G \leq \mathrm{AGL}(1,p) = \{x \to ax+b : a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$ or G is doubly-transitive.*

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Note that if $G \leq \mathrm{Aut}(\Gamma)$ is doubly-transitive, then $\mathrm{Aut}(\Gamma) = S_X$ and Γ is complete or has no edges.

This gives an algorithm for determining the full automorphism group of a circulant graph $\Gamma = \Gamma(\mathbb{Z}_p, S)$. Note that $x \to x + b$ is always contained in $\mathrm{Aut}(\Gamma)$, so we need only check which $a \in \mathbb{Z}_p^*$ satisfy $a \cdot S = \{as : s \in S\} = S$ (we observe that $\mathrm{AGL}(1, p)$ is itself doubly-transitive, so if all such $x \to ax$ are in $\mathrm{Aut}(\Gamma)$, then $\mathrm{Aut}(\Gamma) = S_p$). Doing this with our example Γ on 11 vertices, we see that $\mathrm{Aut}(\Gamma) = D_{11}$.

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Burnside's Theorem is equivalent to

Theorem 2 *Let $G \leq S_p$, p a prime, be transitive. Then either G contains a normal Sylow p-subgroup or G is doubly-transitive.*

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Theorem 3 (D., D. Witte, 2002) *There are exactly* $2p-1$ *transitive* p*-subgroups* P *of* S_{p^2} *up to conjugation, and all but three have the property that if* $G \leq S_{p^2}$ *with Sylow* p*-subgroup* P*, then either* $P \triangleleft G$ *or* G *is doubly-transitive.*

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Theorem 4 (D., 2005) *Let* P *be a transitive* p*-subgroup of* S_{p^k}*,* p *an odd prime,* $k \geq 1$*, such that every minimal transitive subgroup of* P *is cyclic. If* $G \leq S_{p^k}$ *with Sylow* p*-subgroup* P*, then either* $P \triangleleft G$ *or* G *is doubly-transitive.*

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It is probable that many more generalizations are true, but it also seems likely that "most" automorphism groups of graphs cannot be obtained in this way. We will see an indication of why this may be true later.

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We employ the above strategy to determine the automorphism groups of Cayley graphs of $\mathbb{Z}_p \times \mathbb{Z}_p$, p a prime.

Some more terminology

Definition 5 Let G be a transitive group acting on X and $B \subseteq X$. If $g(B) = B$ or $g(B) \cap B = \emptyset$ for every $g \in G$, then B is a block of G. Singletons and the set X are always blocks and are called trivial. If B is a block, then $\mathcal{B} = \{g(B) : g \in G\}$ is called a complete block system of G - each element of $\mathcal{B}$ is a block, and $\mathcal{B}$ partitions X. We remark that in this case G also acts on $\mathcal{B}$. If G has a nontrivial block, then G is imprimitive, otherwise primitive.

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In general, primitive groups can, under many circumstances, be explicitly determined using the O'Nan-Scott Theorem (which gives the structure of primitive groups, usually in terms of a normal direct product of simple groups) and the Classification of the Finite Simple Groups. We say no more about this ...

We remark that there exist vertex-transitive graphs whose automorphism group is imprimitive, but the induced action on the complete block system is not the automorphism group of a graph. There are also vertex-transitive graphs whose automorphism group is imprimitive, but the induced action on a block (the set-wise stabilizer) is not the automorphism group of a graph. This basically says that induction on the number of prime divisors of the order of a graph will not work easily ...

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Definition 6 Let G act on X and H act on Y. The wreath product of G and H, written $G \wr H$, is the set of all permutations of $X \times Y$ of the form $(x, y) \to (g(x), h_x(y))$, $g \in G$, and each $h_x \in H$. We remark that the fibers $\{x\} \times Y$ form a complete block system of $G \wr H$.

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The Sylow p-subgroups of automorphism groups of Cayley graphs of $\mathbb{Z}_p \times \mathbb{Z}_p$ were determined by C. Godsil (1983). They are $\mathbb{Z}_p \times \mathbb{Z}_p$ and $\mathbb{Z}_p \wr \mathbb{Z}_p$.

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Note that if $p \geq 5$, then $S_p \times S_p$ and $S_p \wr S_p$ are both transitive but do not have a normal Sylow p-subgroup, and their Sylow p-subgroups are $\mathbb{Z}_p \times \mathbb{Z}_p$ and $\mathbb{Z}_p \wr \mathbb{Z}_p$, so both of these transitive p-groups are among the three exceptions given above ... In general, we cannot obtain a generalization of Burnside's Theorem for groups with Sylow p-subgroup a direct product or a wreath product.

If a Sylow p-subgroup of $\mathrm{Aut}(\Gamma)$ is $\mathbb{Z}_p \wr \mathbb{Z}_p$, then $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2)$, where Γ_1 and Γ_2 are prime order circulants or $\mathrm{Aut}(\Gamma) = S_{p^2}$ by a classical result of Sabidussi (1959). Otherwise, we need the following result of G. Jones (1979)

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Determining which such H correspond to automorphism groups of graphs and calculating their normalizers (all straightforward) we have:

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2. *If G is simply primitive and solvable, then $G \leq \mathsf{AGL}(2, p)$.*

3. *If G is simply primitive and nonsolvable, then $G \leq \mathsf{AGL}(2, p)$ or $G = S_2 \wr S_p$ in its product action.*

4. *If G is imprimitive, solvable, and has elementary abelian Sylow p-subgroup, then either $G < \mathsf{AGL}(1,p) \times \mathsf{AGL}(1,p)$ or $G = S_3 \times S_3$ (and $p = 3$).*

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6. *If G is imprimitive with Sylow p-subgroup of order at least p^3, then $G = G_1 \wr G_2$, where G_1 and G_2 are automorphism groups of circulant graphs of order p.*

Sylow p-subgroups of automorphism groups of Cayley graphs of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ are known (D, 2000) and for $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ (D, 1995, and M.Y. Xu, unpublished). Corresponding subgroups for $\mathbb{Z}_p \times \mathbb{Z}_q^2$ are known (D, and independently, I. Kovacs and M. Muzychuk). The same strategy can be implemented (D.) for $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ - there are 8 families of p-groups to consider which lead to 8 families of automorphism groups. In a similar fashion, the full automorphism groups of Cayley graphs of $\mathbb{Z}_p^3$ and $\mathbb{Z}_p \times \mathbb{Z}_q^2$ should be obtainable for p and q primes.

The following result generalizes Jones' result mentioned above:

Theorem 7 *Let* $G \leq S_{p^k}$, p *a prime and* $k \geq 2$ *be transitive with Sylow* p*-subgroup* $P = \mathbb{Z}_{p^{\ell_1}} \times \mathbb{Z}_{p^{\ell_2}}$, $\ell_1 + \ell_2 = k$. *Then* G *contains a normal subgroup* $H = H_1 \times H_2$ *where each* H_i *is a doubly-transitive simple group or* $H_i \cong \mathbb{Z}_{p^{\ell_i}}$.

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Note that both this result and Jones' result can be viewed as "Burnside type" results, with the obvious exceptions of direct products.

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4. algorithms to produce automorphism groups without a listing of the automorphism groups

Automorphism Groups of Circulants

The following result is a group-theoretic translation of several papers on Schur rings by Leung and Man (1996 and 1998), and Evdomikov and Ponomarenko (2002)

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1. $\mathrm{Aut}(\Gamma) = G_1 \times G_2 \times \cdots \times G_r$, *where* $r \geq 1$, *each* $G_i \leq S_{n_i}$ *and* $G_i \cong S_{n_i}$, *or contains a normal regular cyclic group, such that* $\gcd(n_i, n_j) = 1$ *and* $n = n_1 n_2 \cdots n_r$, *or*

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2. $\mathrm{Aut}(\Gamma) = G_1 \cap G_2$, *where* $G_1 = S_r \wr H_1$ *and* $G_2 = H_2 \wr S_k$, H_1 *is an automorphism group of a circulant graph of order* mk/r, *and* H_2 *is an automorphism group of order* m, $r|m$.

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D. and J. Morris in 2005 independently proved the an equivalent form of the above result when n is square-free.

Let $\mathcal{C}G_n$ be the set of all circulant digraphs of order n, and $\mathcal{D}RR_n$ be the set of all circulant digraphs Γ of order n such that $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_n$. Thus $\mathcal{D}RR_n$ is the set of all circulant digraphs that are digraphical regular representations (DRR's) of $\mathbb{Z}_n$.

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Theorem 9 (J. Araújo, D., J. Konieczny, J. Morris) *Almost all circulant digraphs are DRR's. That is,*

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The only other result on whether almost all Cayley digraphs are DRR's is by Godsil (1981) for certain p-groups.

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