Zigzags and central circuits for 3- or 4-valent plane graphs

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I. k-valent two-faced polyhedra

Polyhedra and planar graphs

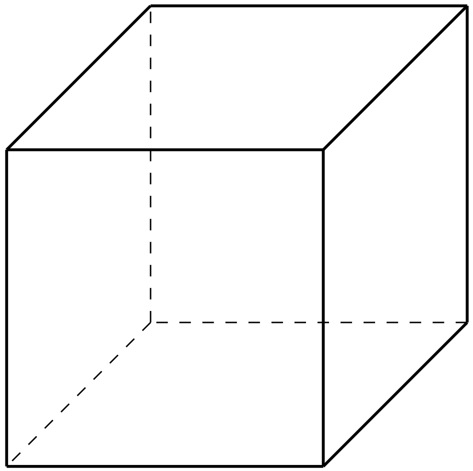
A graph is called k-connected if after removing any set of $k-1$ vertices it remains connected.

The skeleton of a polytope P is the graph $G(P)$ formed by its vertices, with two vertices adjacent if they generate a face of P.

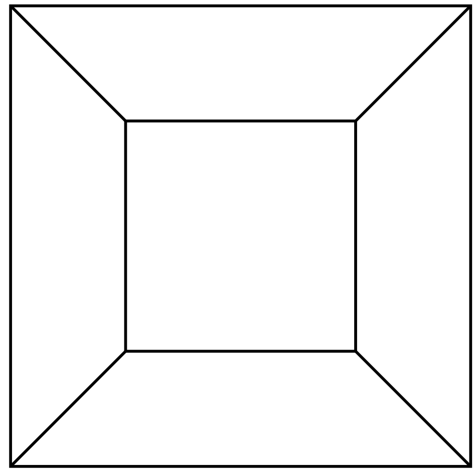
Theorem *(Steinitz)*
(i) A graph G is the skeleton of a 3-polytope if and only if it is planar and 3-connected.
(ii) P and P' are in the same combinatorial type if and only if $G(P)$ is isomorphic to $G(P')$.

The dual graph G^* of a plane graph G is the plane graph formed by the faces of G, with two faces adjacent if they share an edge.

Example

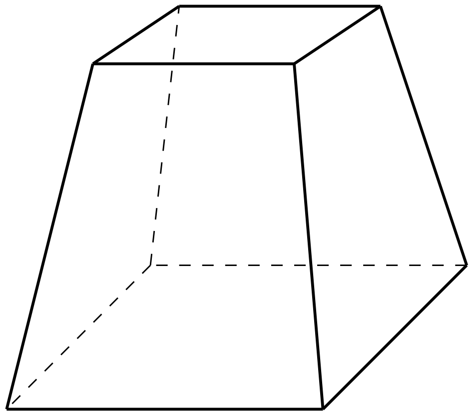


The regular cube

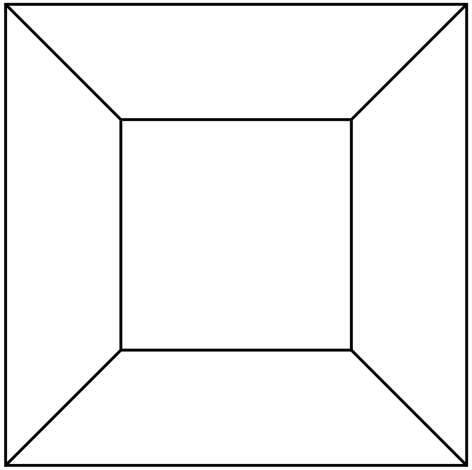


its planar graph

Example



A perturbed cube



its planar graph

A 3-polytope is usually represented by the Schlegel diagram of its skeleton, the program used for this is CaGe by G. Brinkmann, O. Delgado, A. Dress and T. Harmuth.

k-valent two-faced polyhedra

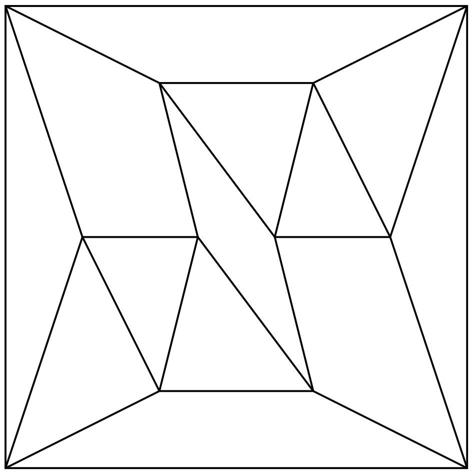
The Euler formula for plane graphs $V - E + F = 2$, take the following form for k-valent graphs:

$$12 = \sum_i (6-i)p_i \quad \text{if} \quad k=3$$
$$\text{and} \quad 8 = \sum_i (4-i)p_i \quad \text{if} \quad k=4$$

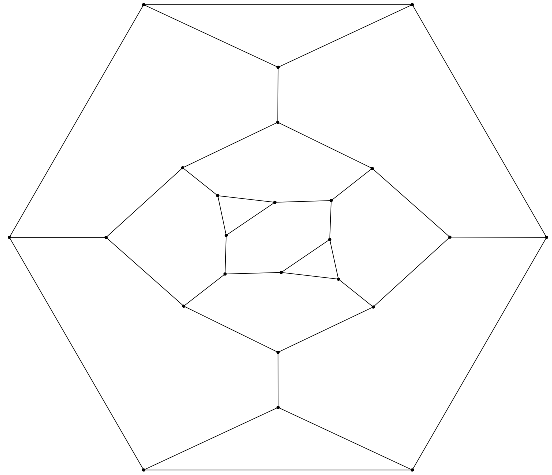
With p_i the number of faces of gonality i.
A k-valent plane graph is called two-faced if the gonality of its faces has only two possible values a and b.

- 3-valent plane graphs with n vertices and faces of gonality q and 6 (classes q_n),
- 4-valent plane graphs with n vertices and faces of gonality 3 or 4 (octahedrites).

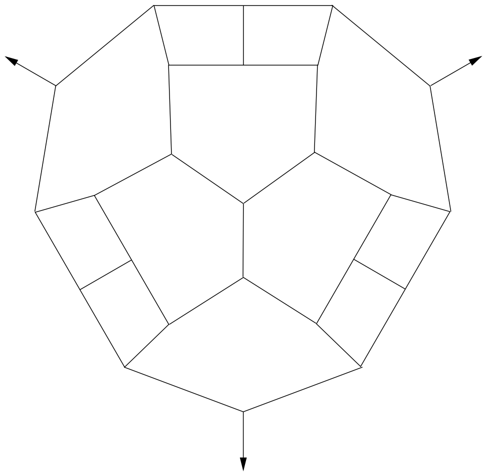
Examples



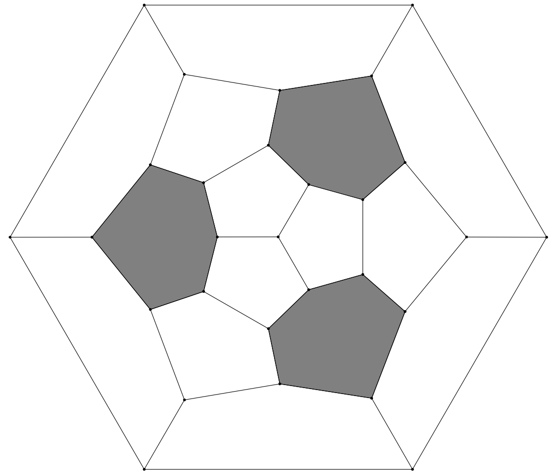
An octahedrite



A 3_{24} plane graph



A 4_{26} plane graph



A 5_{28} plane graph

Classes and their generation

k	(a,b)	Polyhedra	Exist if and only if	p_a	n
3	$(3,6)$	3_n	$\frac{p_6}{2} \in N - \{1\}$	$p_3 = 4$	$4 + 2p_6$
3	$(4,6)$	4_n	$p_6 \in N - \{1\}$	$p_4 = 6$	$8 + 2p_6$
3	$(5,6)$	5_n (fullerenes)	$p_6 \in N - \{1\}$	$p_5 = 12$	$20 + 2p_6$
4	$(3,4)$	octahedrite	$p_4 \in N - \{1\}$	$p_3 = 8$	$6 + p_4$

Generation programs

- 3-valent: CPF for two-faced maps on the sphere by T. Harmuth
  CGF for two-faced maps on surfaces of genus g by T. Harmuth
- 4-valent: ENU by T. Heidemeier
- General: plantri by G. Brinkmann and B. McKay

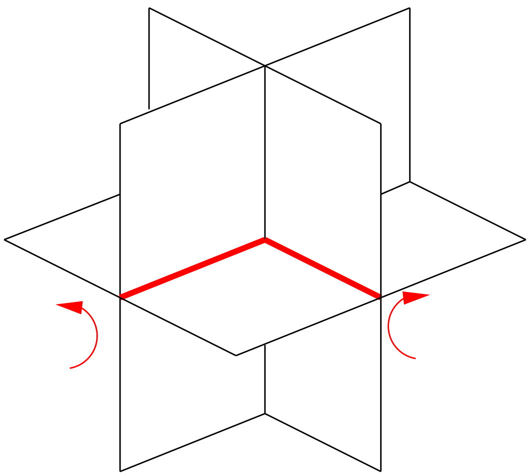
Finite isometry groups

All finite groups of isometries of 3-space are classified. In Schoenflies notations:

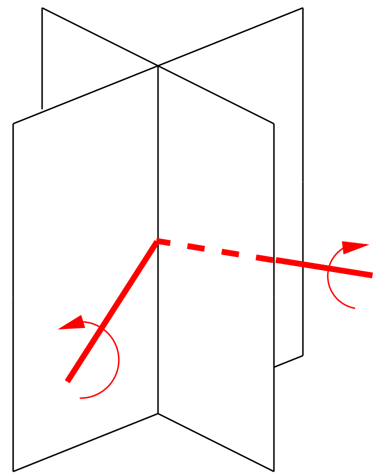
- C_1 is the trivial group
- C_s is the group generated by a plane reflexion
- $C_i = \{I_3, -I_3\}$ is the inversion group
- C_m is the group generated by a rotation of order m of axis Δ
- C_{mv} ($\simeq$ dihedral group) is the group formed by C_m and m reflexion containing Δ
- $C_{mh} = C_m \times C_s$ is the group generated by C_m and the symmetry by the plane orthogonal to Δ
- S_N is the group of order N generated by an antirotation

Finite isometry groups

- D_m ($\simeq$ dihedral group) is the group formed of C_m and m rotations of order 2 with axis orthogonal to Δ
- D_{mh} is the group generated by D_m and a plane symmetry orthogonal to Δ
- D_{md} is the group generated by D_m and m symmetry planes containing Δ and which does not contain axis of order 2



D_{2h}



D_{2d}

Finite isometry groups

- $I_h = H_3 \simeq Alt_5 \times C_2$ is the group of isometries of the regular Dodecahedron
- $I \simeq Alt_5$ is the group of rotations of the regular Dodecahedron
- $O_h = B_3$ is the group of isometries of the regular Cube
- $O \simeq Sym(4)$ is the group of rotations of the regular Cube
- $T_d = A_3 \simeq Sym(4)$ is the group of isometries of the regular Tetrahedron
- $T \simeq Alt(4)$ is the group of rotations of the regular Tetrahedron
- $T_h = T \cup -T$

Point groups

(point group) $Isom(P) \subset Aut(G(P))$ (combinatorial group)

Theorem *(Mani, 1971)*

Given a 3-connected planar graph G, there exist a 3-polytope P, whose group of isometries is isomorphic to $Aut(G)$ and $G(P) = G$.

- For octahedrites: (C_1, C_s, C_i), (C_2, C_{2v}, C_{2h}, S_4), (D_2, D_{2d}, D_{2h}), (D_3, D_{3d}, D_{3h}), (D_4, D_{4d}, D_{4h}), (O, O_h). (Deza and al.)
- For 3_n: (D_2, D_{2h}, D_{2d}), (T, T_d) (Fowler and al.)
- For 4_n: (C_1, C_s, C_i), (C_2, C_{2v}, C_{2h}), (D_2, D_{2d}, D_{2h}), (D_3, D_{3d}, D_{3h}), (D_6, D_{6h}), (O, O_h) (Deza and al.)
- For 5_n: (C_1, C_s, C_i), (C_2, C_{2v}, C_{2h}, S_4), (C_3, C_{3v}, C_{3h}, S_6), (D_2, D_{2h}, D_{2d}), (D_3, D_{3h}, D_{3d}), (D_5, D_{5h}, D_{5d}), (D_6, D_{6h}, D_{6d}), (T, T_d, T_h), (I, I_h) (Fowler and al.)

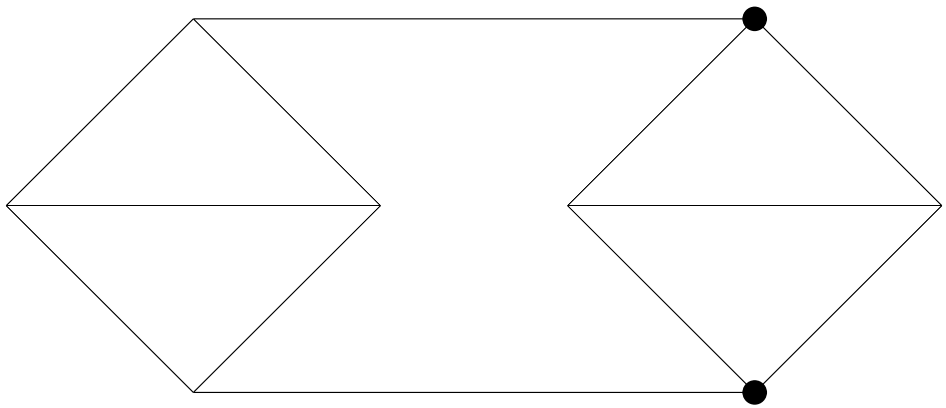
k-connectedness

Theorem

(i) Any octahedrite is 3-connected.

(ii) Any 3-valent plane graph without (> 6)-gonal faces is 2-connected.

(iii) Moreover, any 3-valent plane graph without (> 6)-gonal faces is 3-connected except of the following serie G_n:



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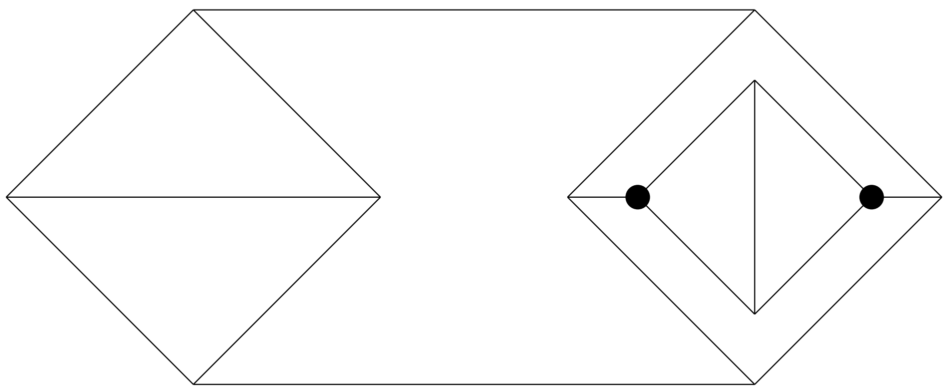

k-connectedness

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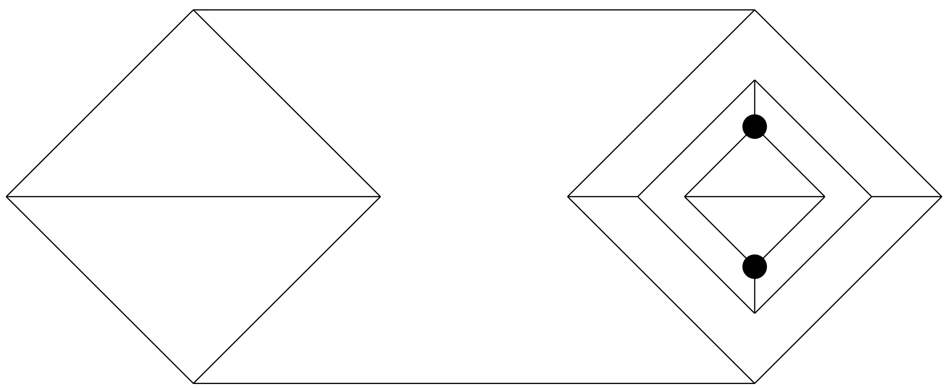
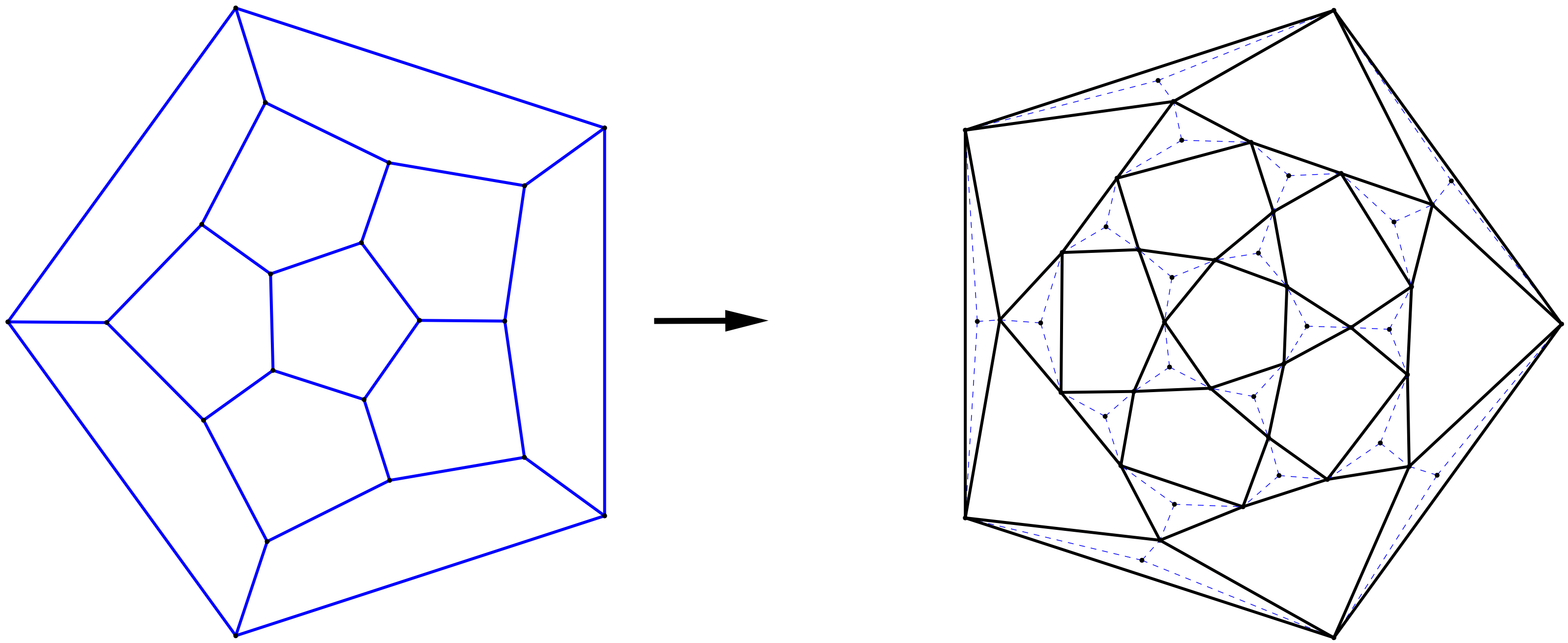


Medial graph

Given a plane graph G, the 4-valent plane graph $Med(G)$ is defined as the graph having as vertices the edges of G with two vertices adjacent if and only if they share a vertex and belong to a common face.



Medial graph

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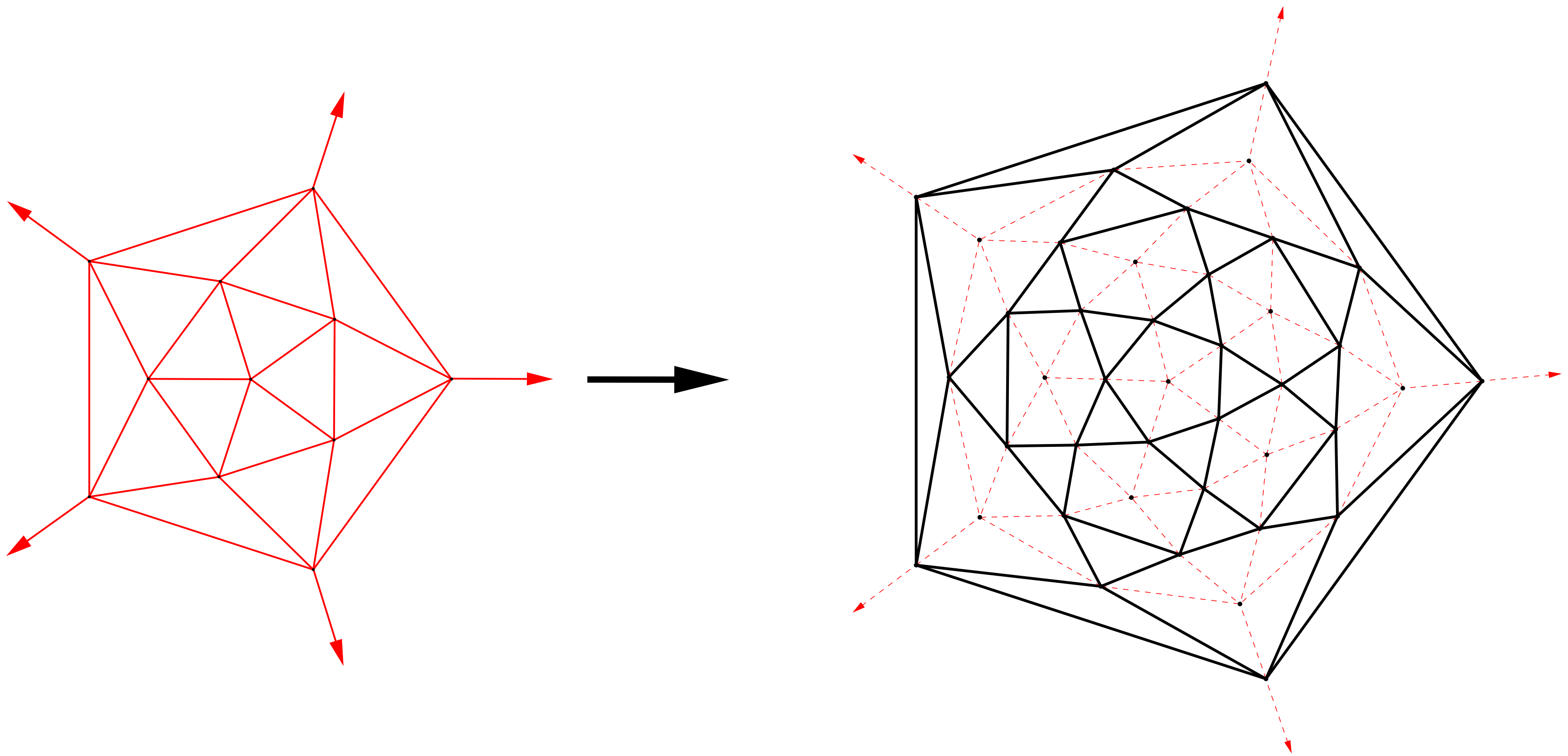
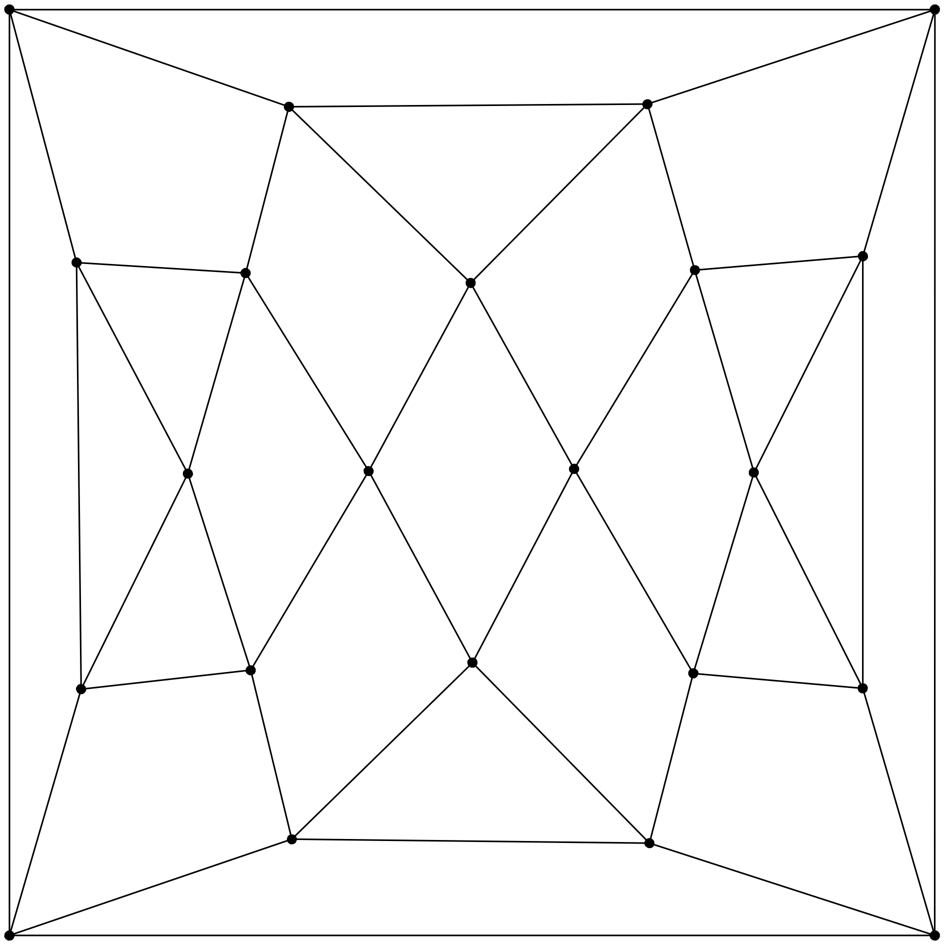


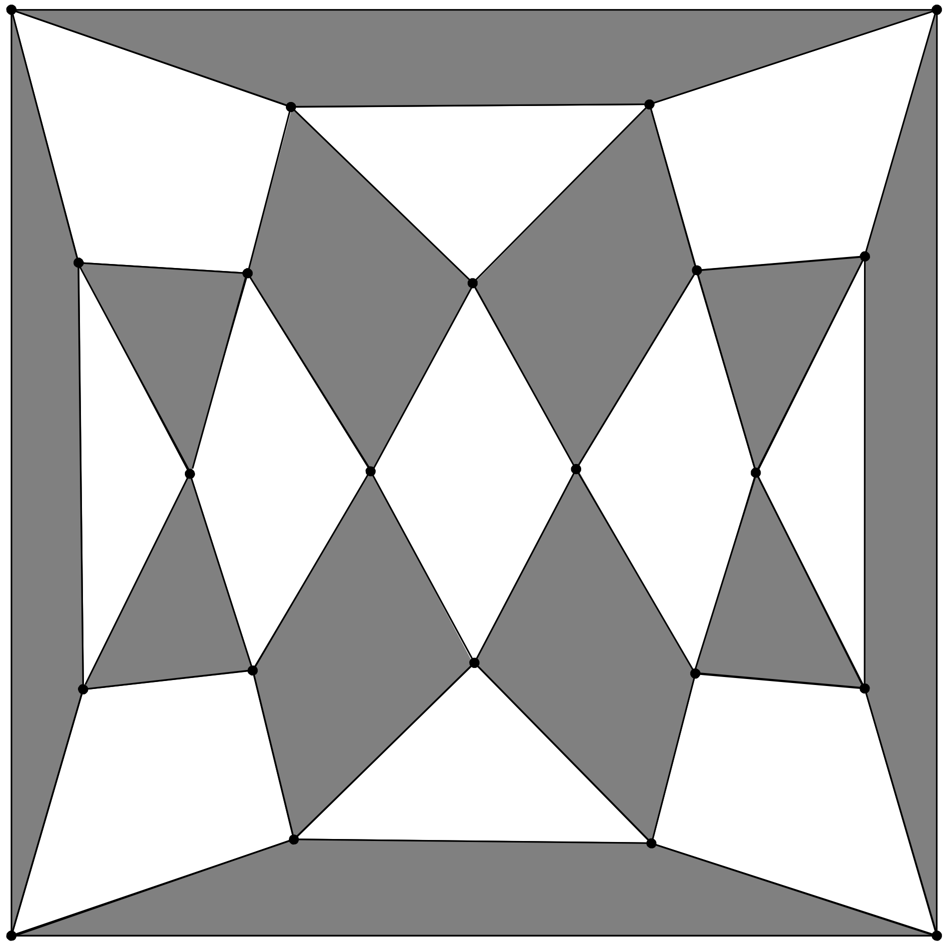
Inverse medial graph

If G is a 4-valent plane graph, we want to find the graphs H such that $G = Med(H)$.



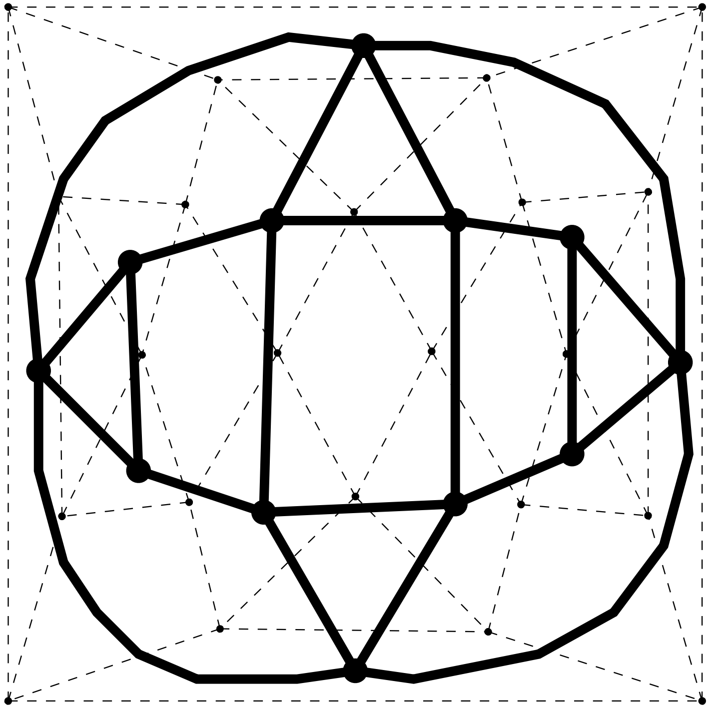
Inverse medial graph

Take $\mathcal{C}_1$( ), $\mathcal{C}_2$( ) a bipartition of the face-set of G or in other word a chess coloring.

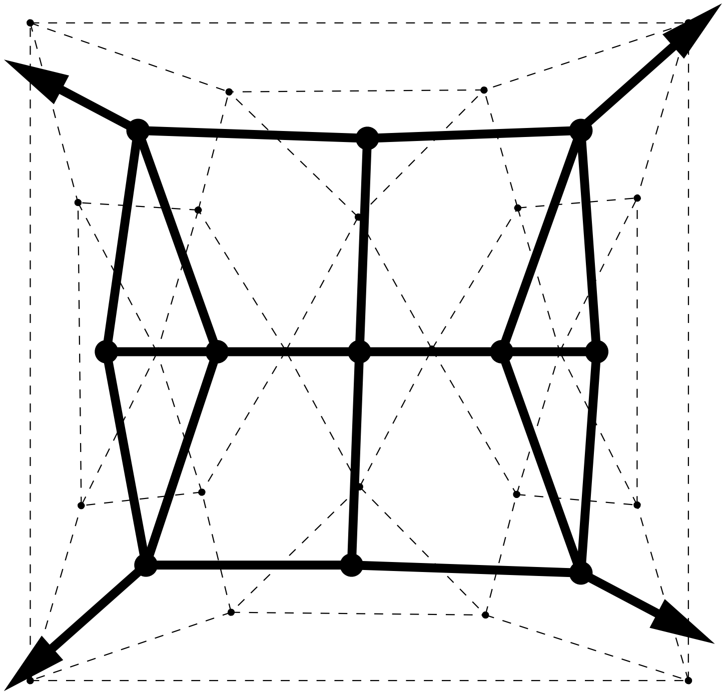


Inverse medial graph

We form two plane graphs H_{black} and H_{white} with the black and white faces.



From the black faces



From the white faces

Note that H_{black} can be isomorph to H_{white};
$Med(Tetrahedron) = Octahedron$.

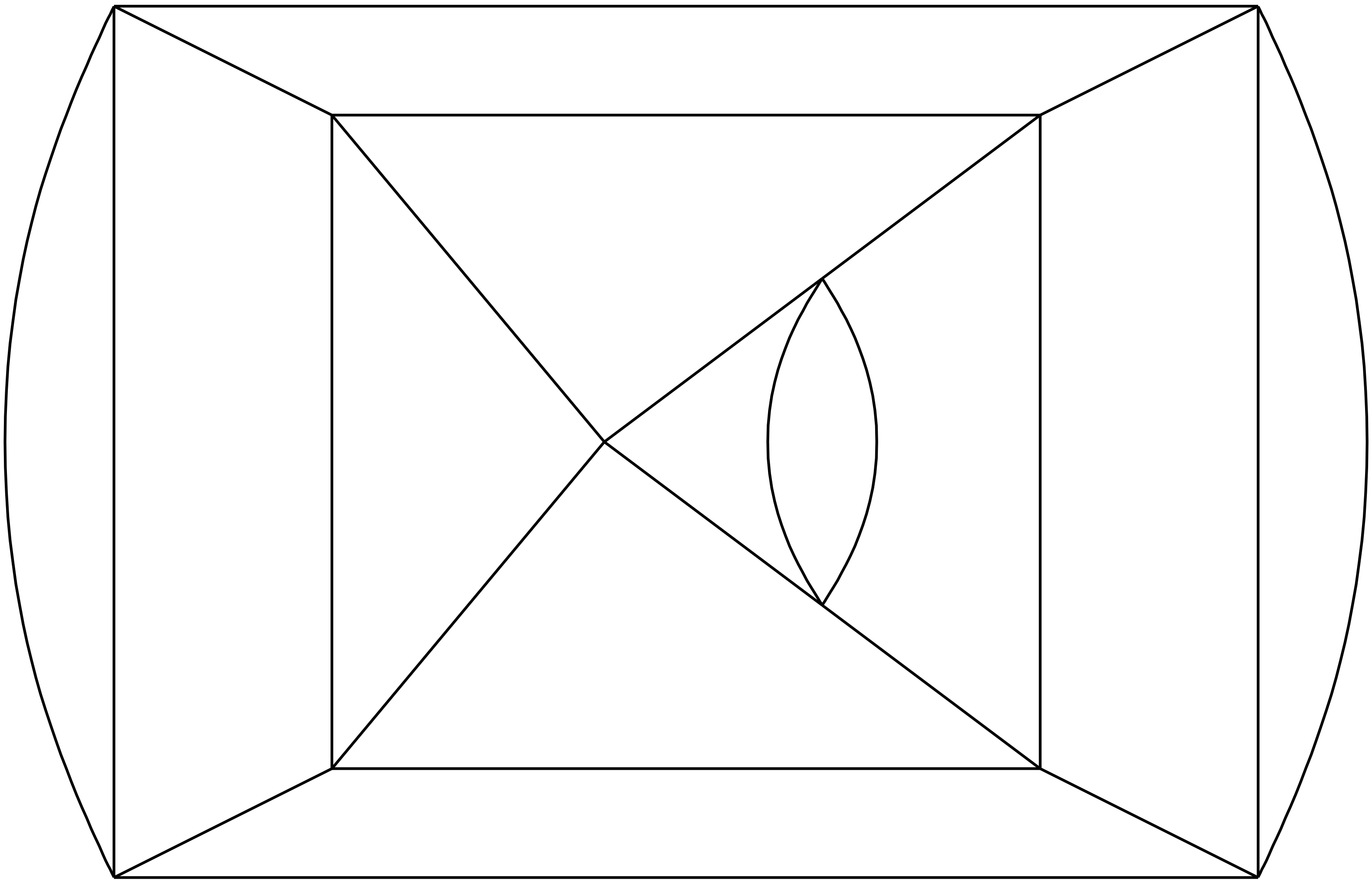
II. Zigzags

and

central circuits

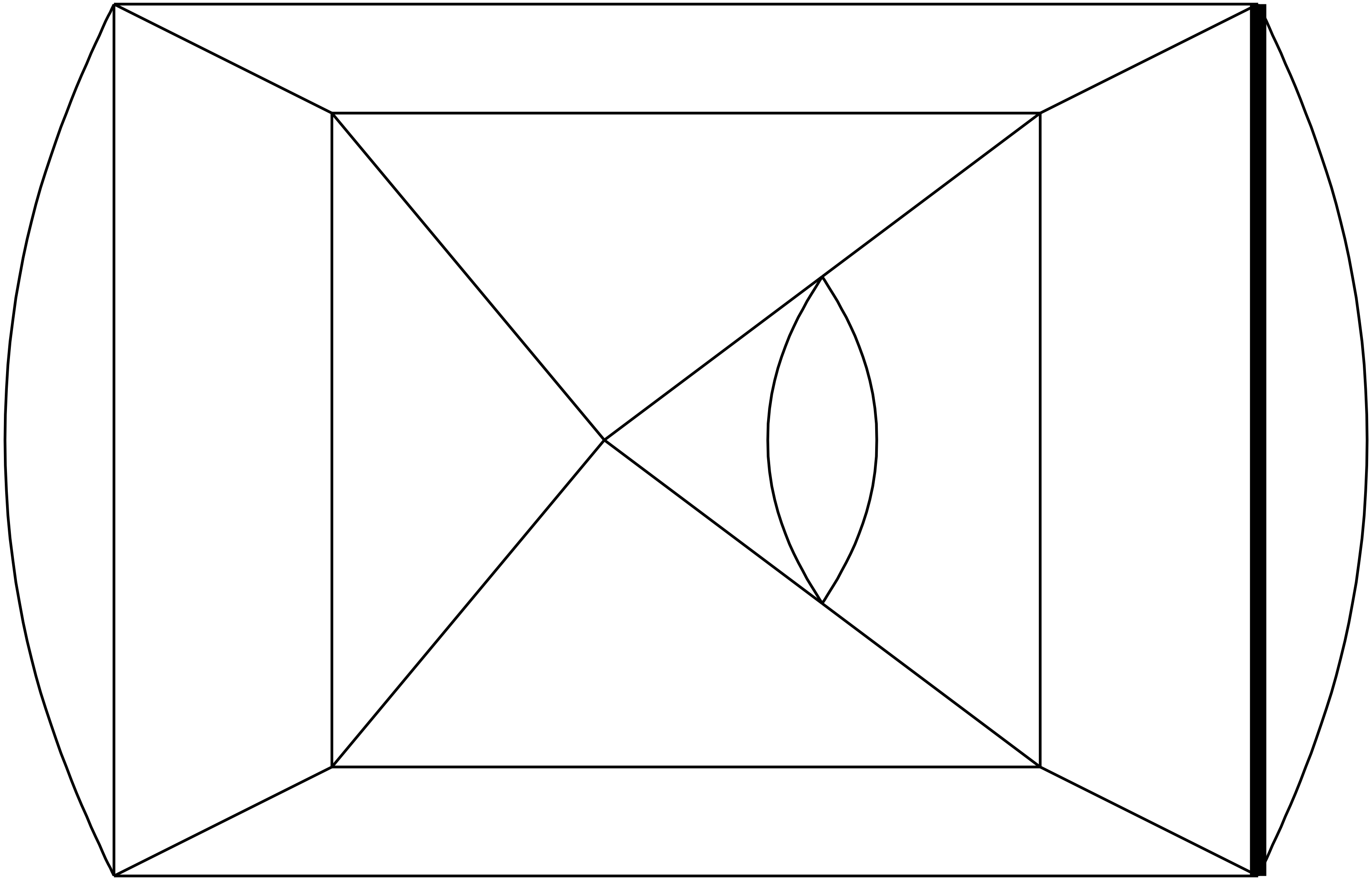
Central circuits

A 4−valent plane graph G



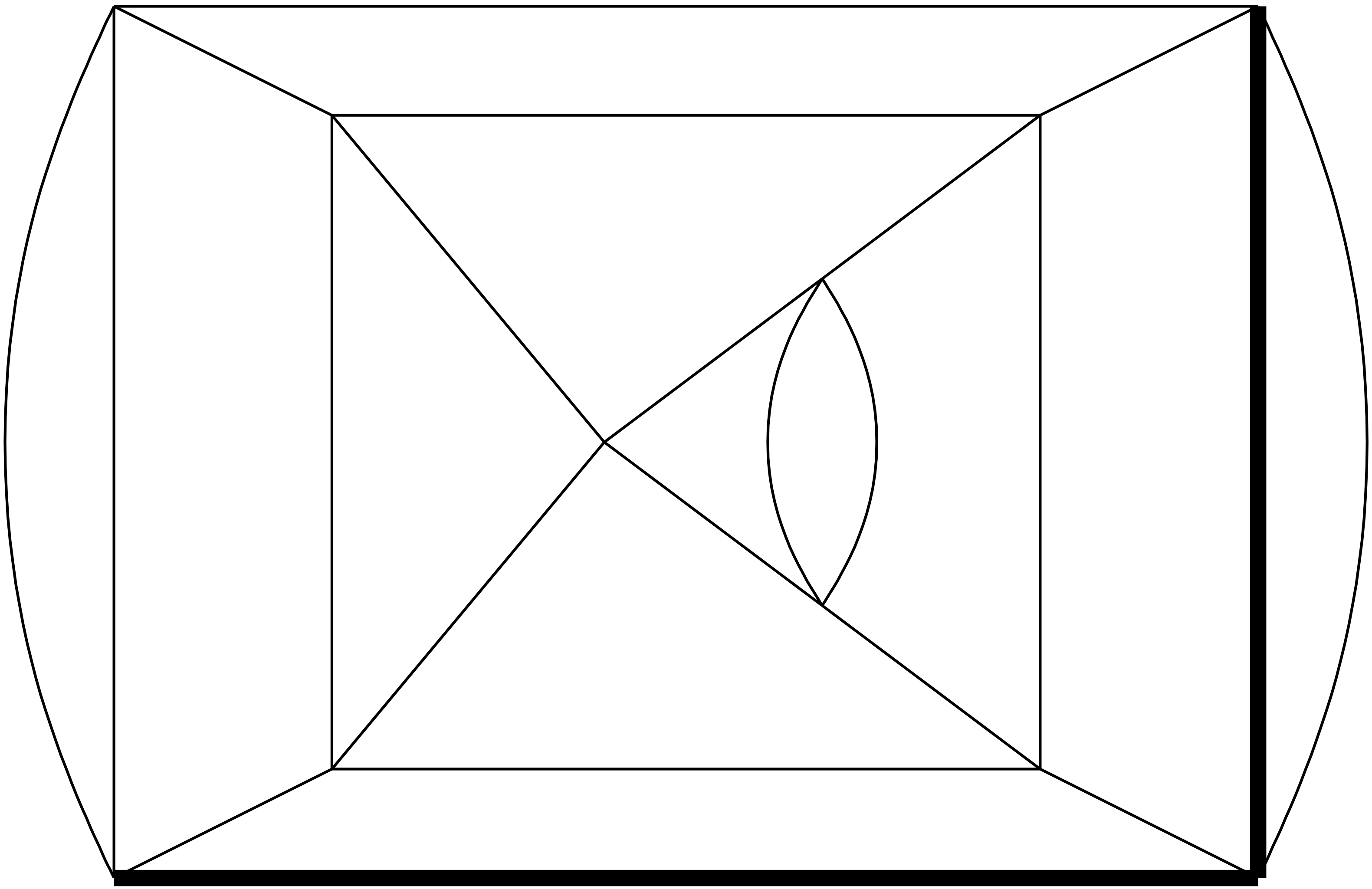
Central circuits

Take an edge of G



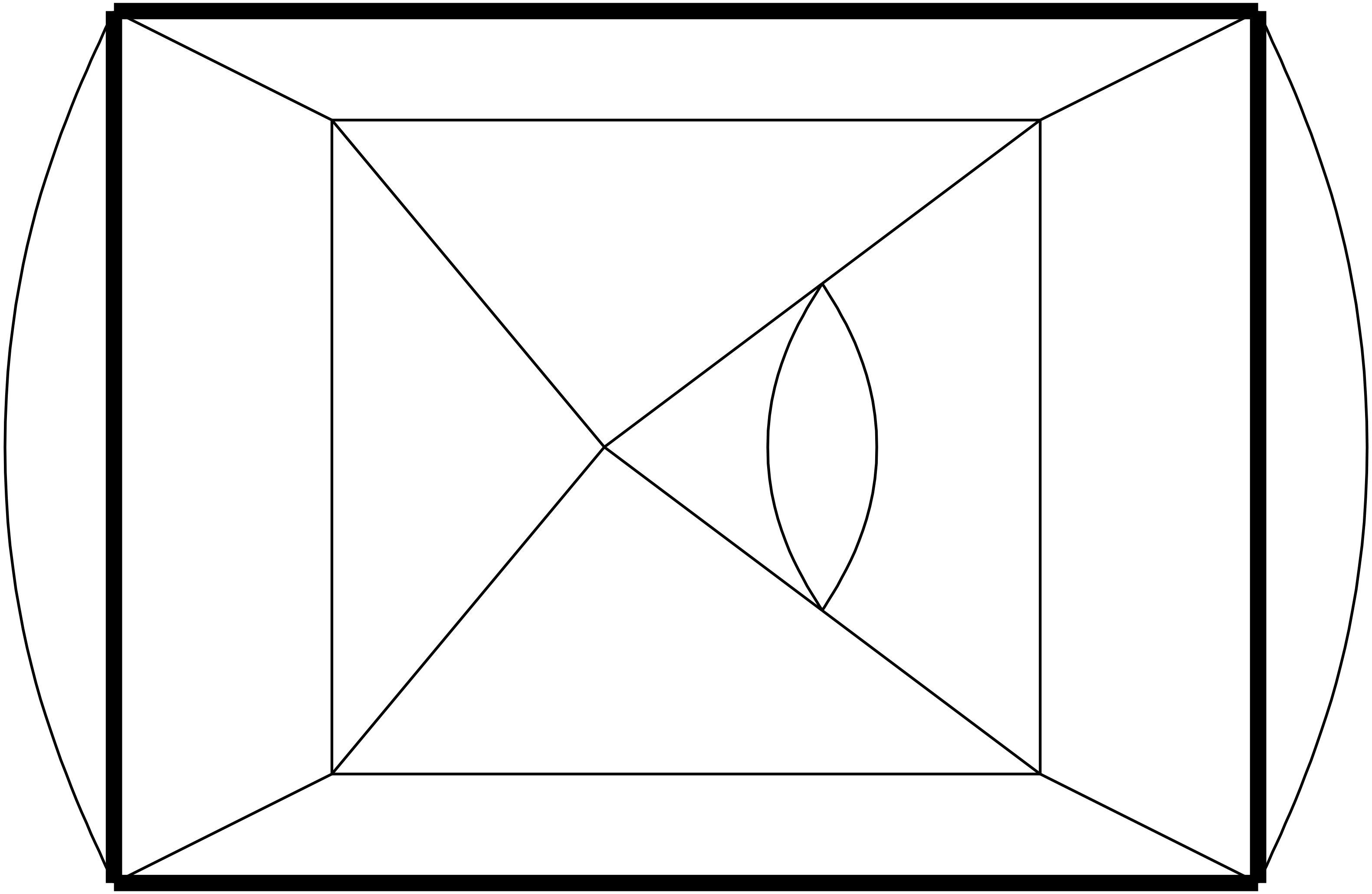
Central circuits

Continue it straight ahead ...



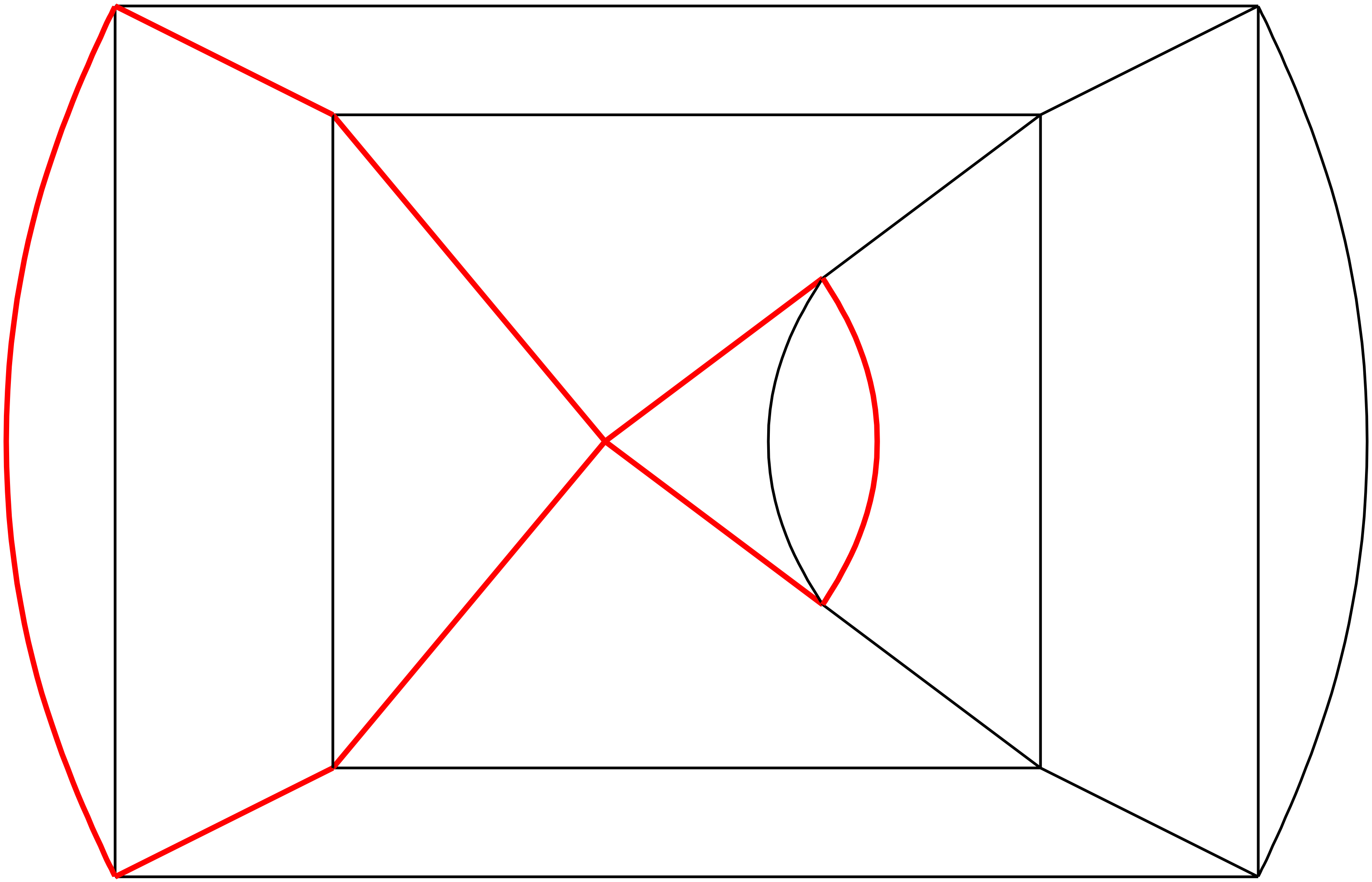
Central circuits

... until the end



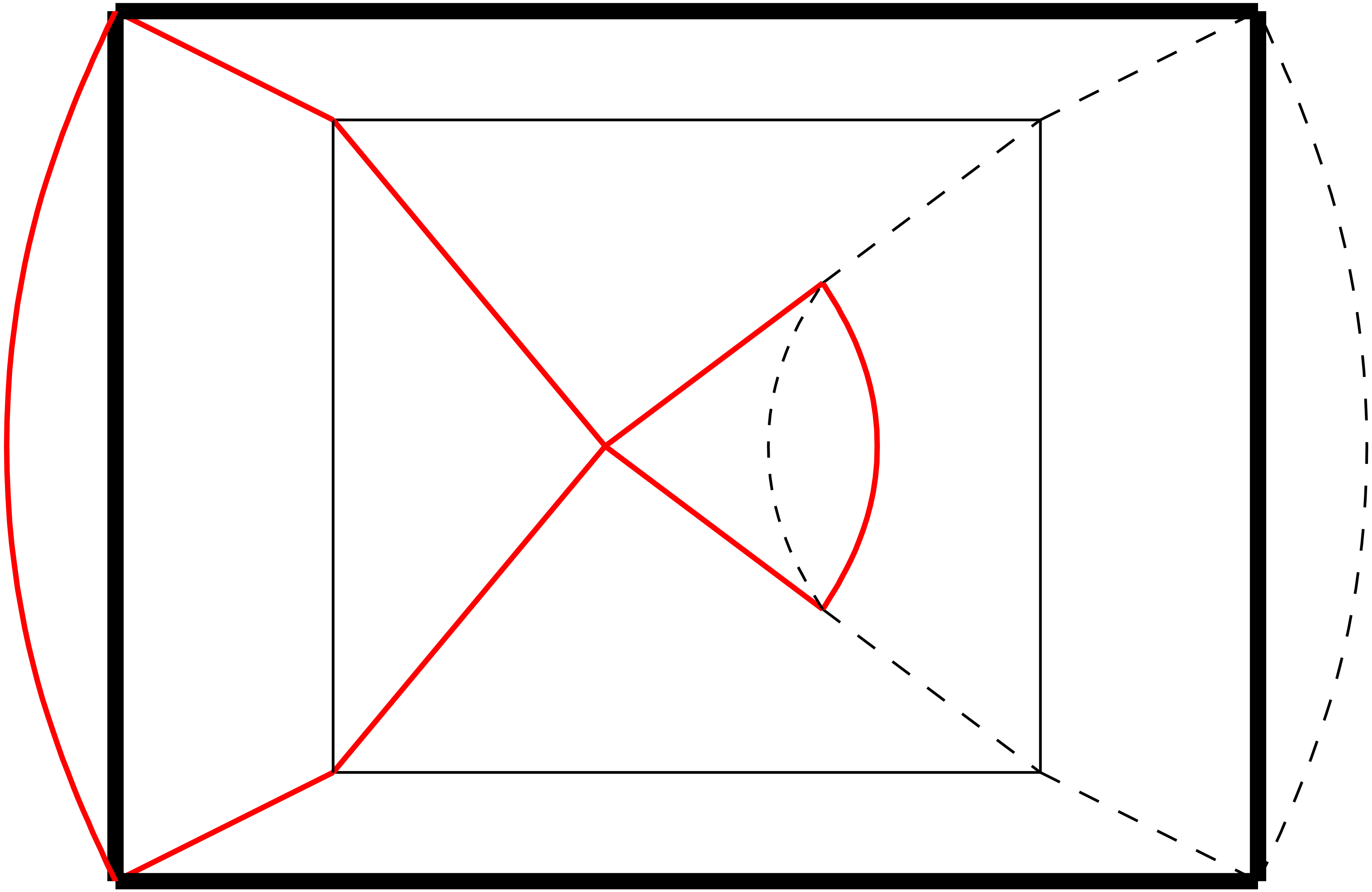
Central circuits

A self−intersecting central circuit



Central circuits

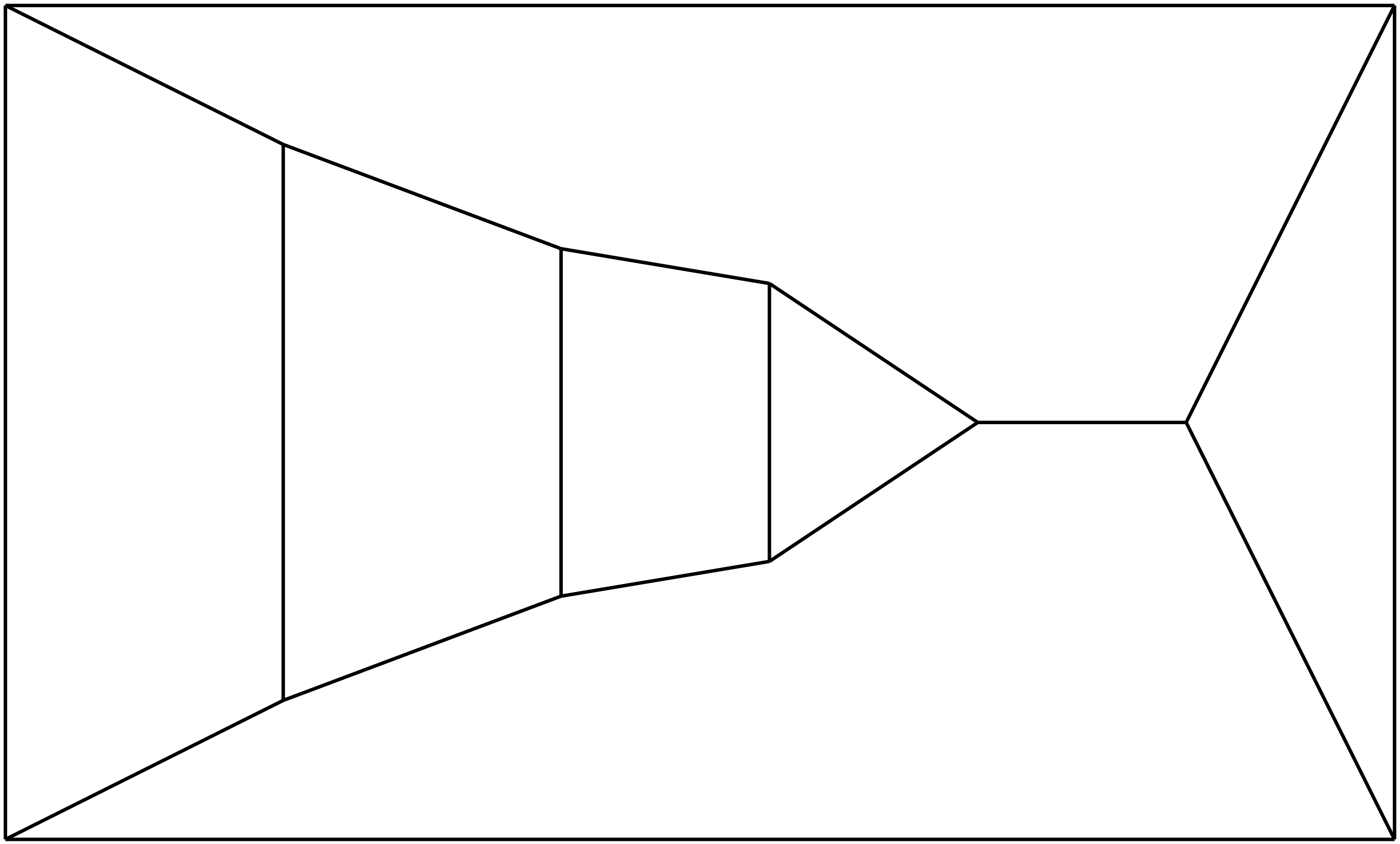
A partition of edges of G



CC=4^2, 6, 8

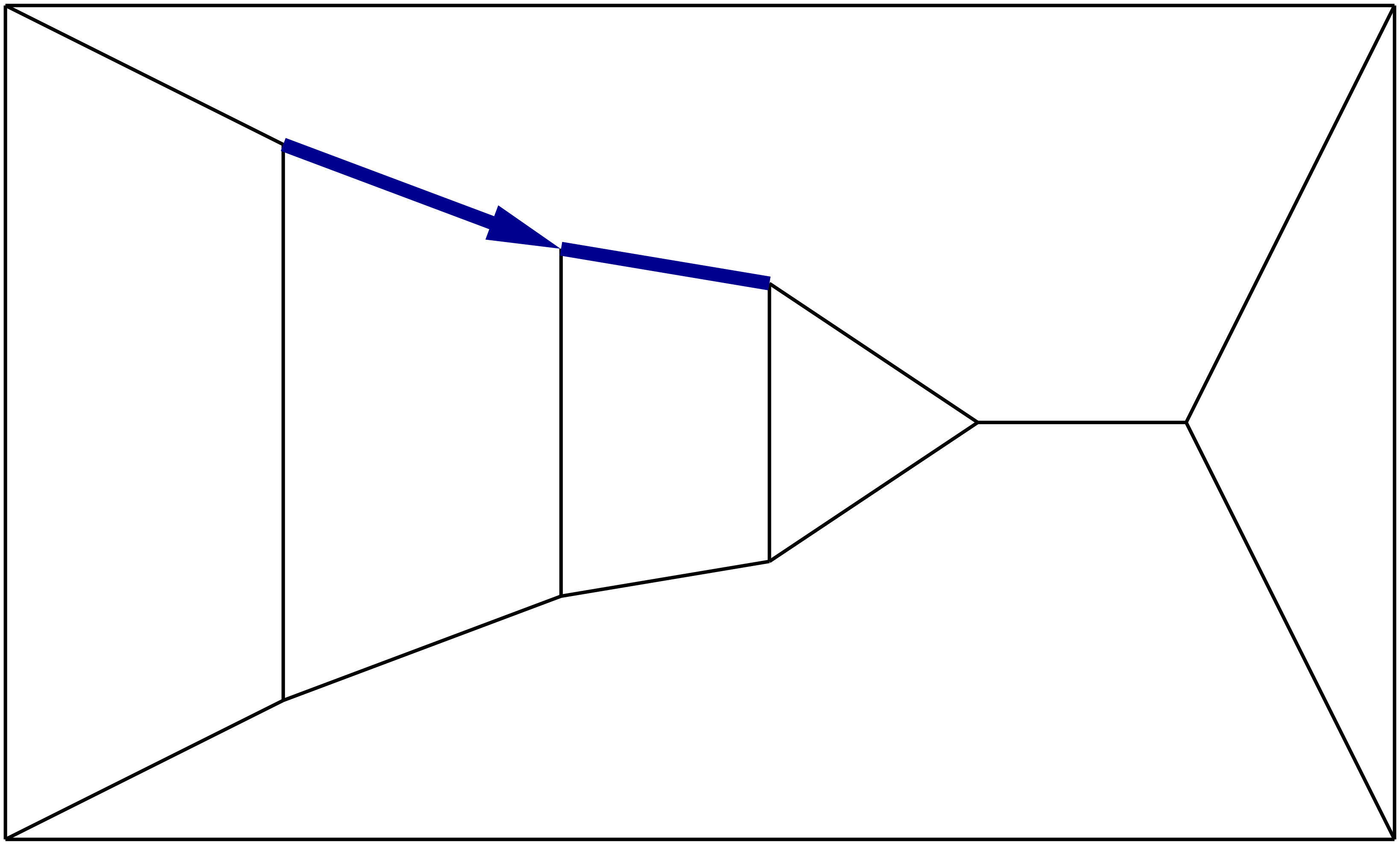
Zigzags

A plane graph G



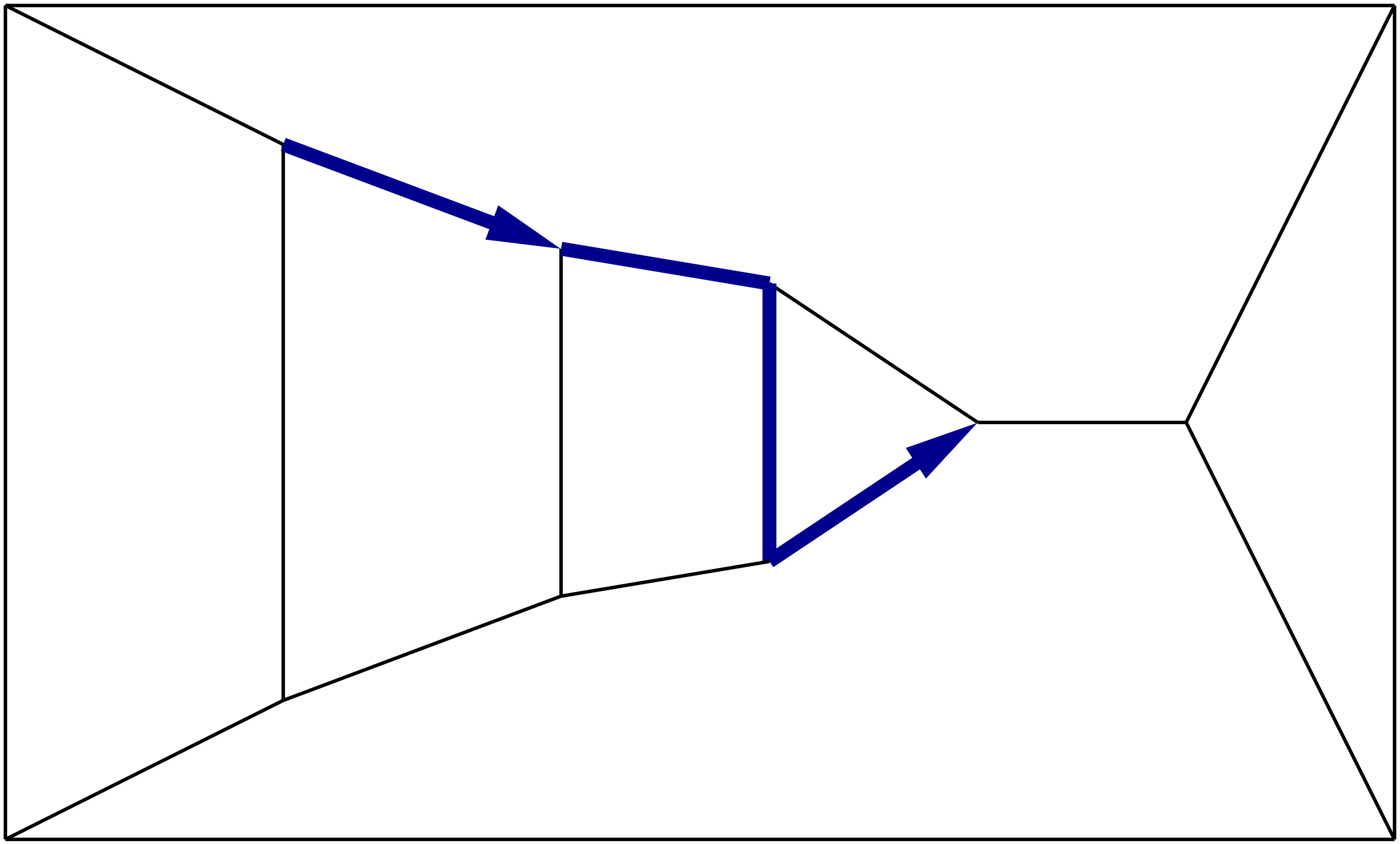
Zigzags

take two edges



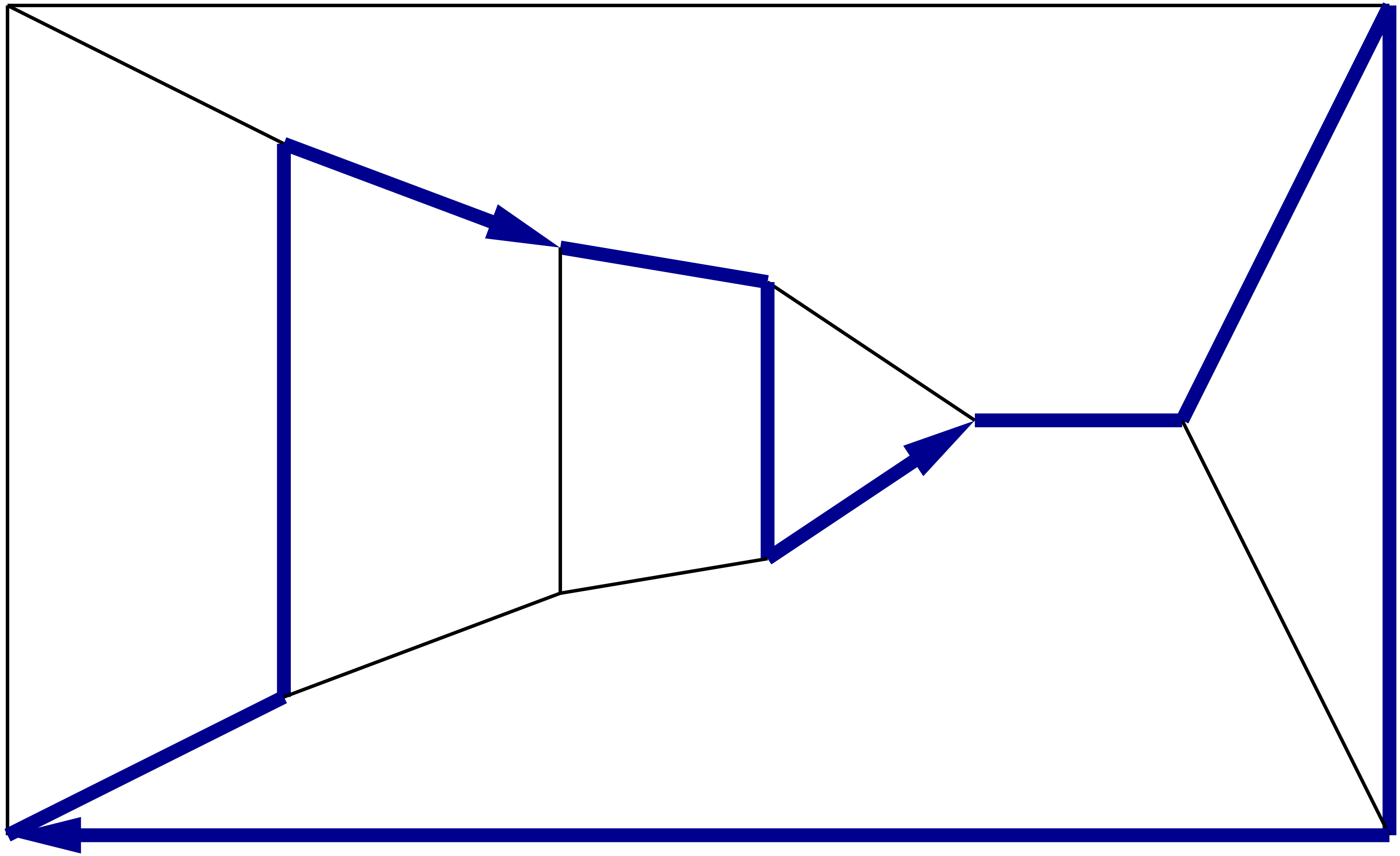
Zigzags

Continue it left−right alternatively ....

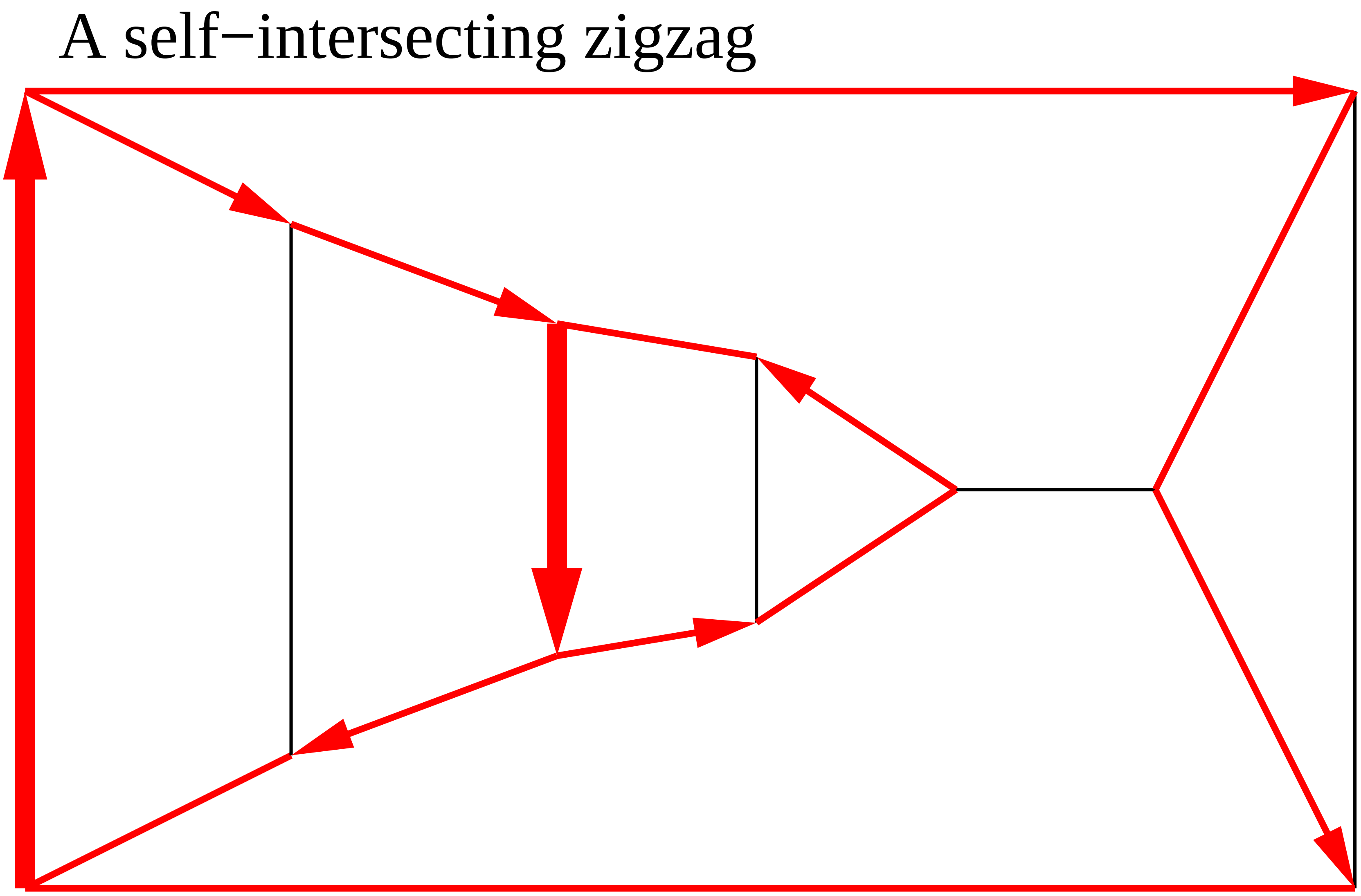


Zigzags

... until we come back.

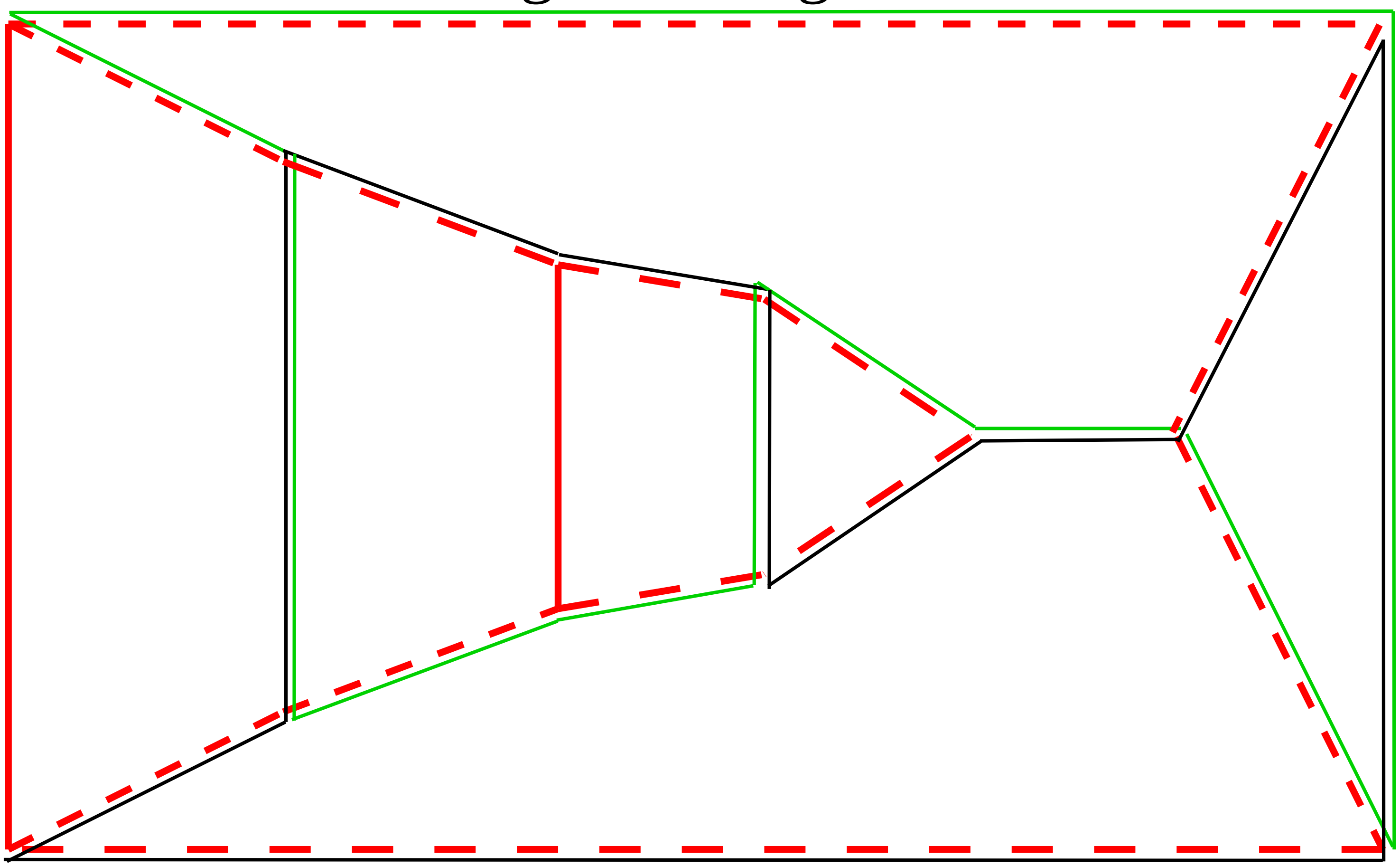


Zigzags



Zigzags

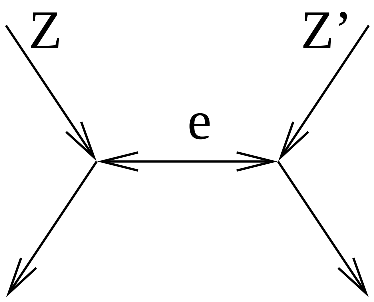
A double covering of 18 edges: 10+10+16



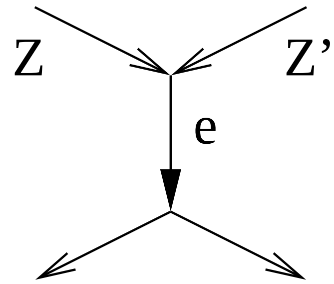
z−vector $z = 10^2, 16_{2,0}$

Intersection types for zigzags

Let Z and Z' be (possibly, $Z = Z'$) zigzags of a plane graph G and let an orientation be selected on them. An edge of intersection $Z \cap Z'$ is called of type I or type II, if Z and Z' traverse e in opposite or same direction, respectively



type I



type II

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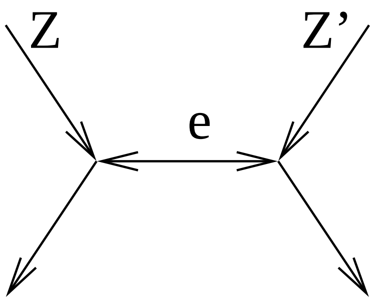
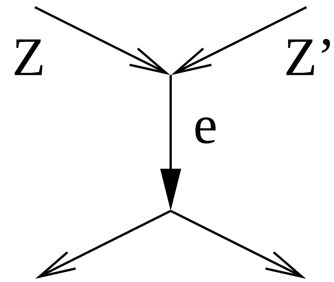
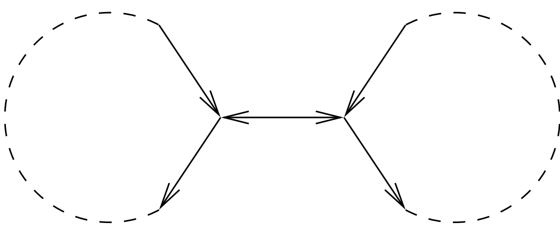


type I

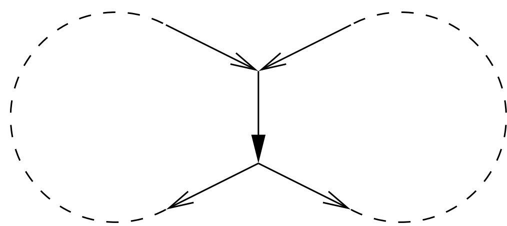


type II

The types of self-intersection depends on orientation chosen on zigzags except if $Z = Z'$:



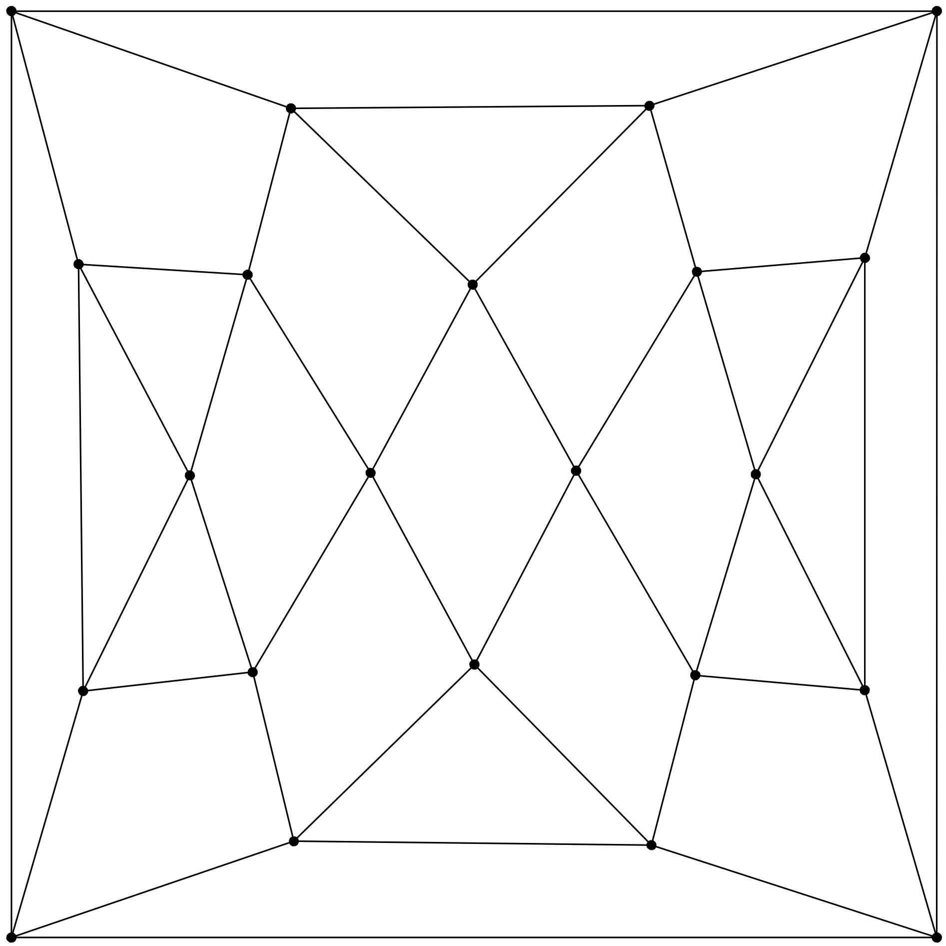
type I



type II

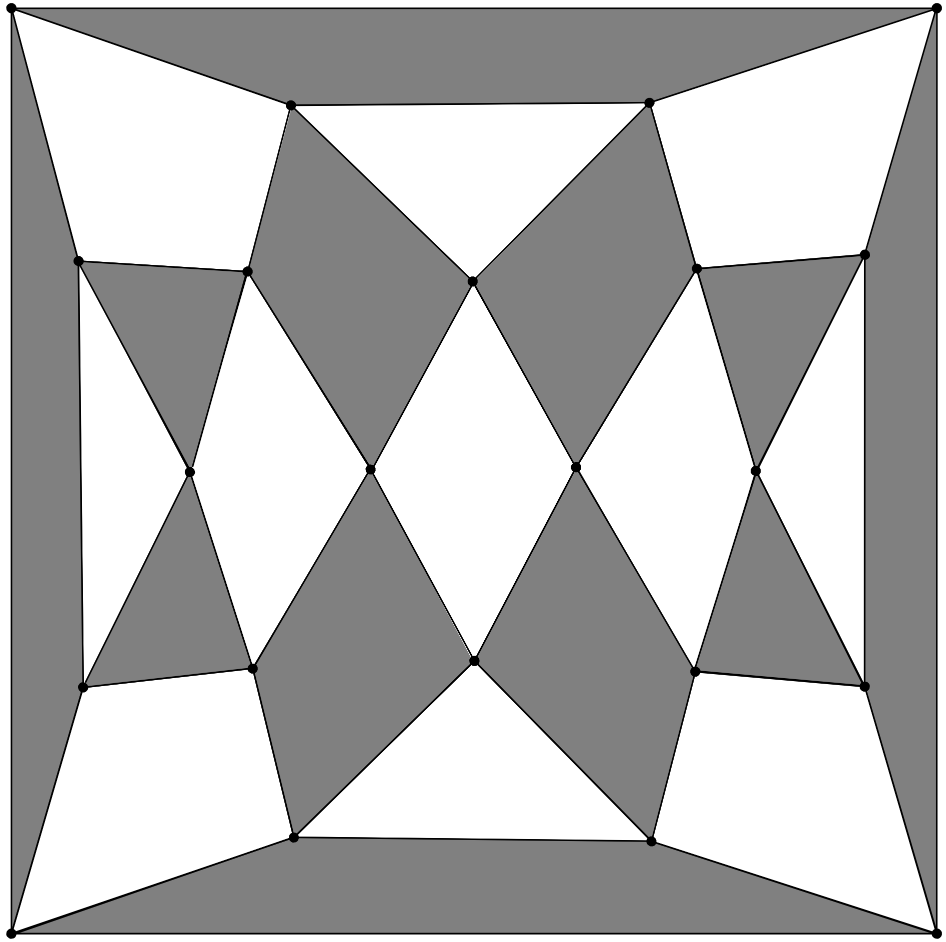
Intersection types for central circuits

Let G be a 4-valent plane graph



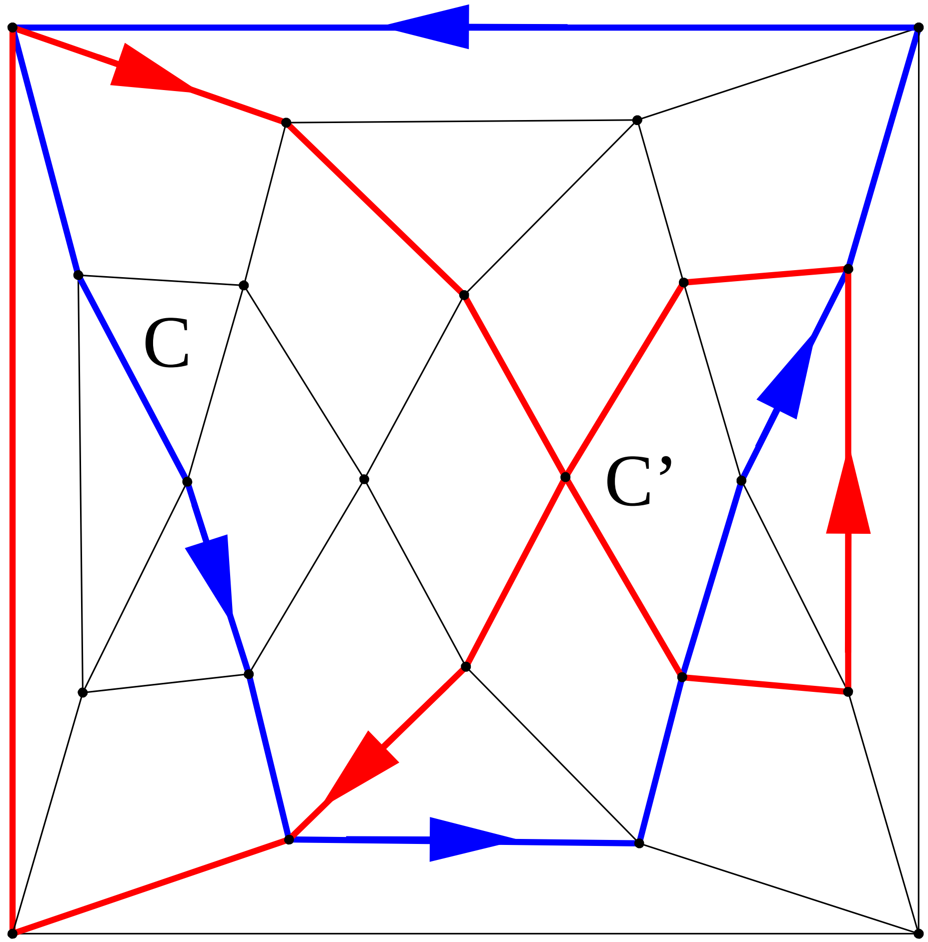
Intersection types for central circuits

Take $\mathcal{C}_1$( ), $\mathcal{C}_2$( ) a bipartition of the face-set of G



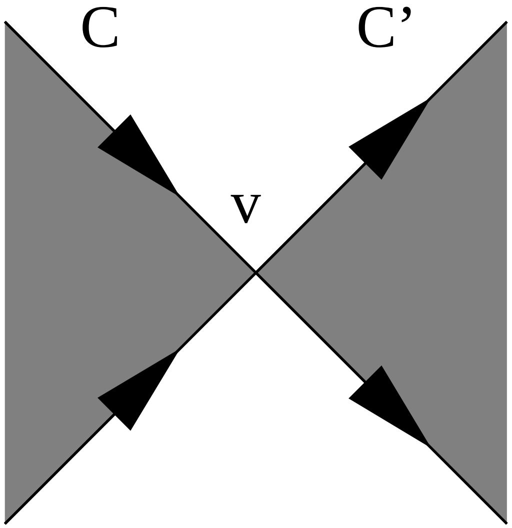
Intersection types for central circuits

Let C and C' be two central circuits of G and let an orientation be selected on them.

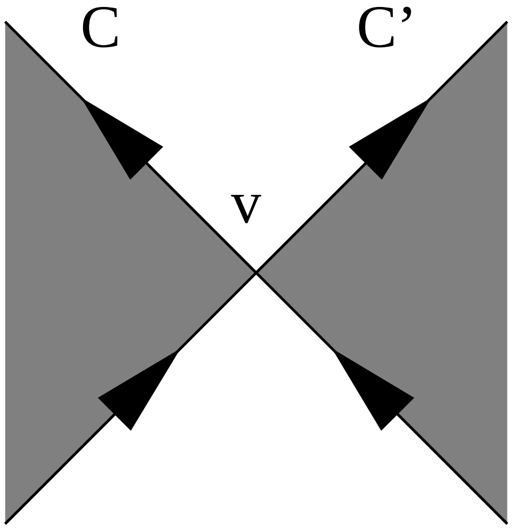


Intersection types for central circuits

Local View on a vertex v and type



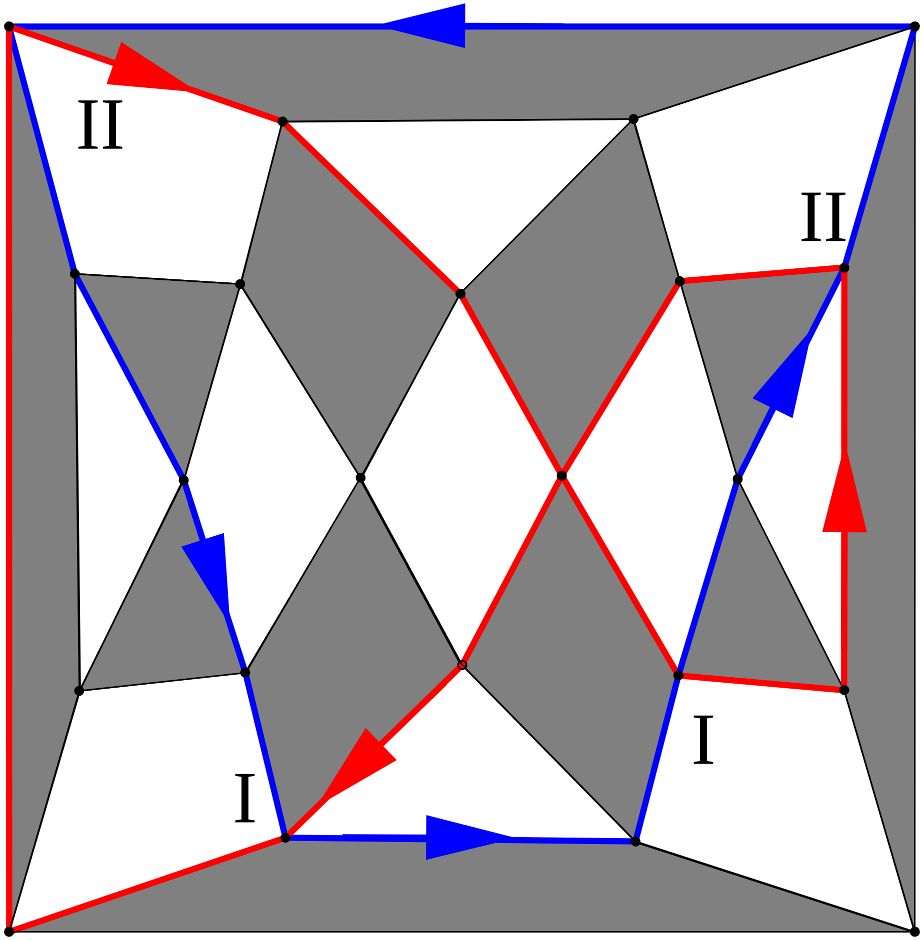
Type I



Type II

Intersection types for central circuits

C and C' have 2 intersections I and 2 intersection II



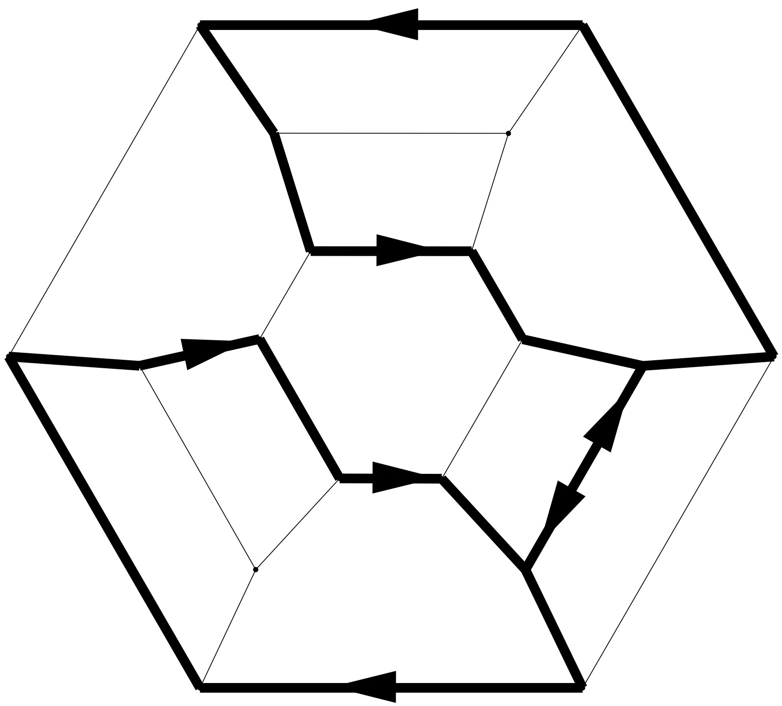
Duality and types

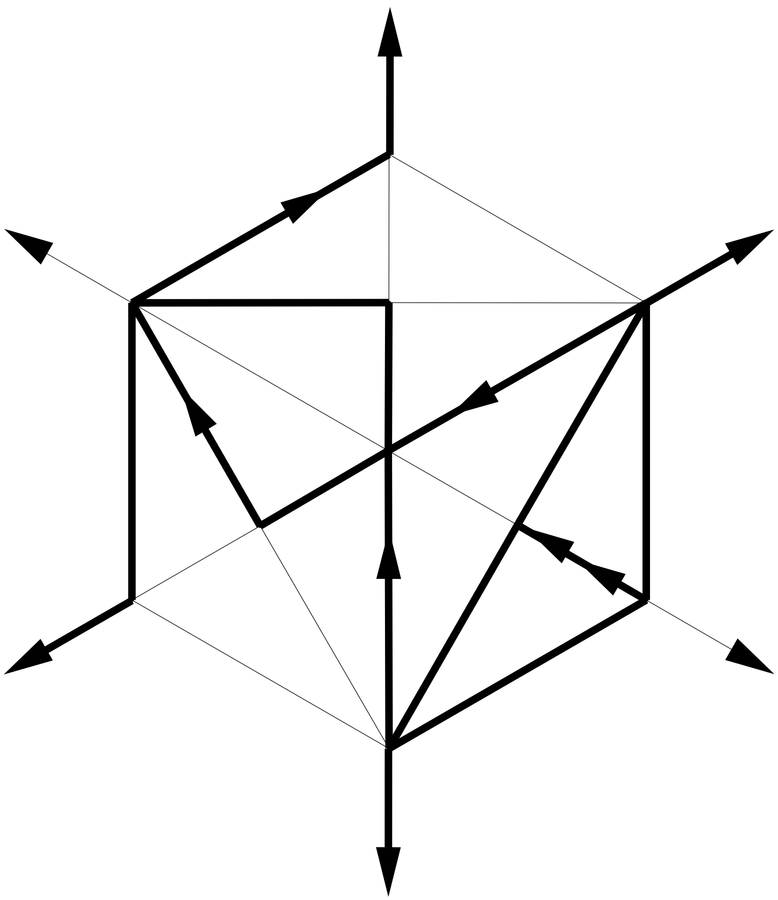
Theorem
The zigzags of a plane graph G *are in one-to-one correspondence with zigzags of* G^**. The length is preserved, but intersection of type I and II are interchanged.*

Theorem
Let G *be a plane graph; for any orientation of all zigzags of* G*, we have:*
(i) The number of edges of type II, which are incident to any fixed vertex, is even.
(ii) The number of edges of type I, which are incident to any fixed face, is even.

Medial, zigzags and central circuits

Zigzags of a plane graph G are in one-to-one correspondence with zigzags of G^*.

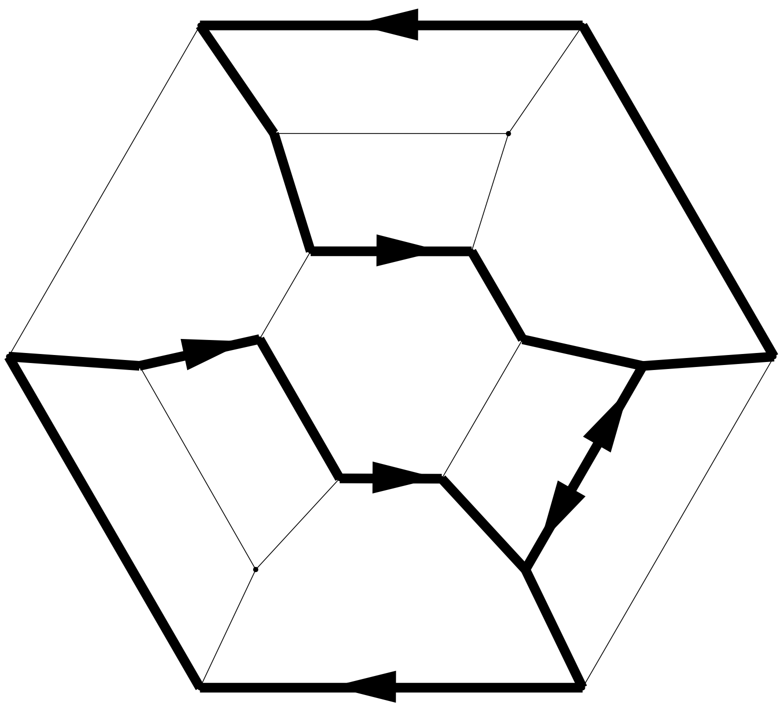


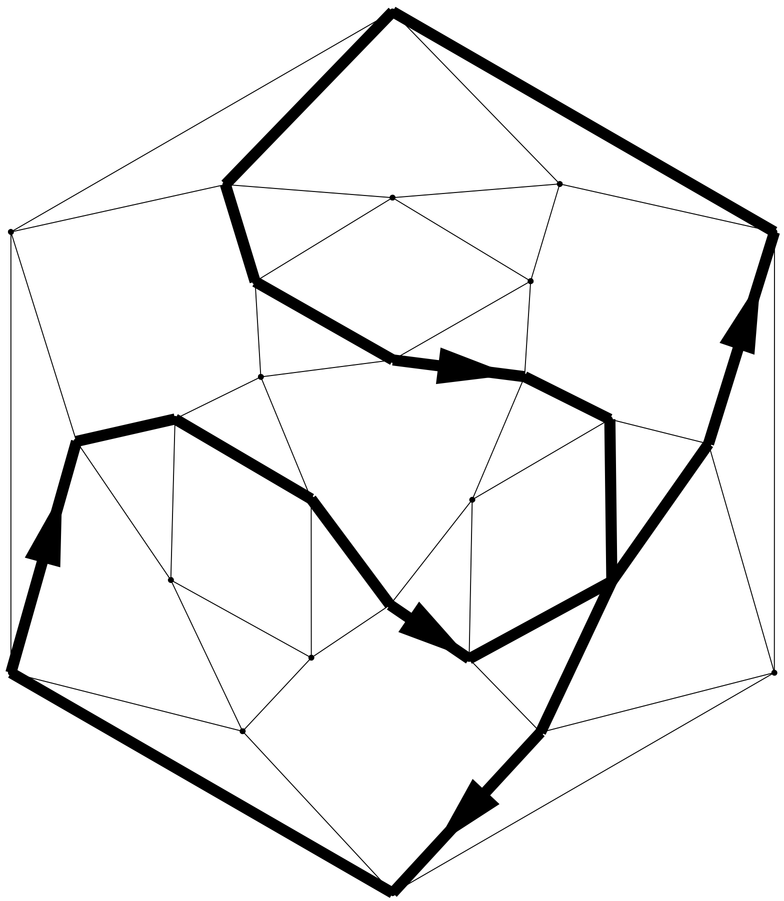


Types are interchanged.

Medial, zigzags and central circuits

Zigzags of a plane graph G are in one to one correspondence with central circuits of $Med(G)$.





Notation

- ZC-circuit stands for “zigzag or central circuit” in 3- or 4-valent plane graphs.
- The length of a ZC-circuit is the number of its edges.
- The ZC-vector of a 3- or 4-valent plane graph G_0 is the vector $\dots, c_k^{m_k}, \dots$ where m_k is the number of ZC-circuits of length c_k.
- A graph is ZC-transitive if its group of automorphism is transitive on the set of ZC-circuits.

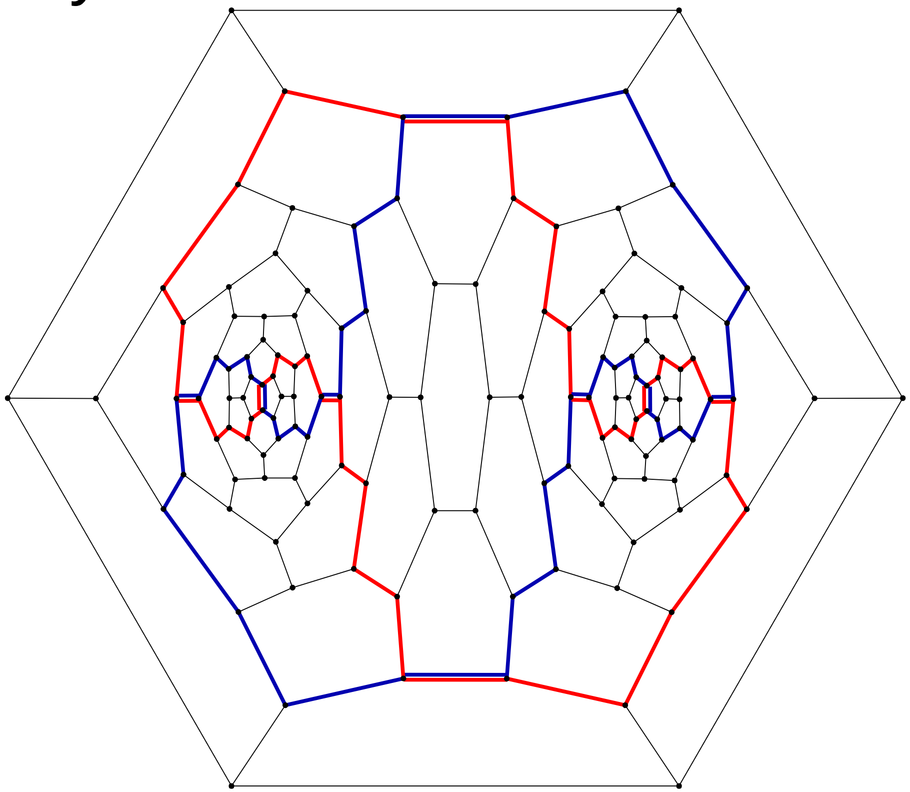
- Zigzags are also called left-right paths (Shank) or Petri paths, from Petri polygons of polytopes (Coxeter).

Zigzags versus spanning trees

- Given a plane graph G with vertices $v_1, \ldots v_n$ and the agjacency matrix $A(G)$, the Laplacian of G is the matrix $L(G) = D(G) - A(G)$, where $D(G)$ is a diagonal matrix with d_{ii} being the degree of vertex v_i.
- Kirchhoff: the the number of spanning trees of G is equal to the determinant of any minor of $L(G)$.
- Shank: to every zigzag z of G corresponds an element $x(z) \in \{0, 1\}^n$ which is a basic element of the kernel of $L(G)$ (equation $Lx = 0$ over $\{0, 1\}$).
- So, the number of zigzags $Z(G)$ is equal to corank of $L(G)$ over $\mathbb{Z}^2$; co-rank of any minor of $L(G)$ is $Z(G)$-1.
- Godsil-Royle: the number of spanning trees of G is odd if and only if G is z-knotted.

Intersection of two simple ZC-circuits

- For the class of graph 4_n the size of the intersection of two simple zigzags belongs to $\{0, 2, 4, 6\}$.
- For classes of octahedrites, graph 3_n or graph 5_n the size of the intersection of two simple ZC-circuits can be any even number.



Two simple zigzags of a graph 5_n with $|Z \cap Z'| = 8$.

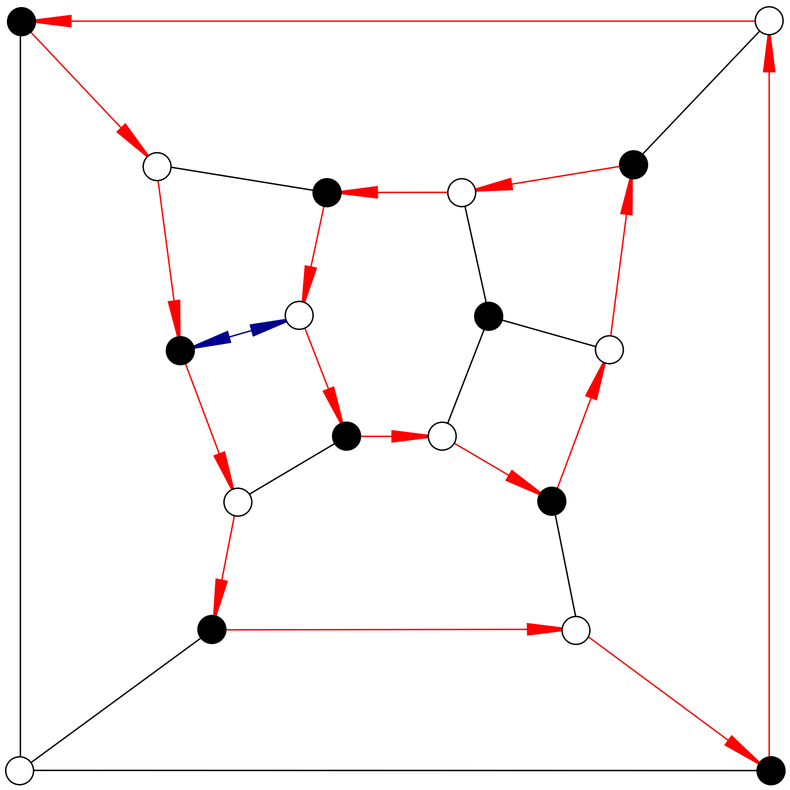
On surfaces of genus $g \geq 1$, the intersection can be odd.

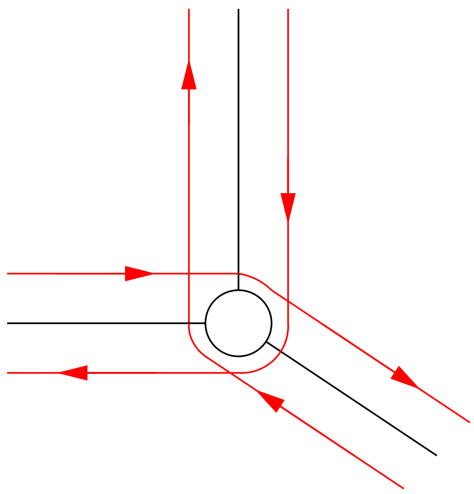
Bipartite graphs

Remark *A plane graph is* *bipartite* *if and only if its faces have even gonality.*

Theorem (*Shank-Shtogrin*)

For any planar bipartite graph G there exist an orientation of zigzags, with respect to which each edge has type I.

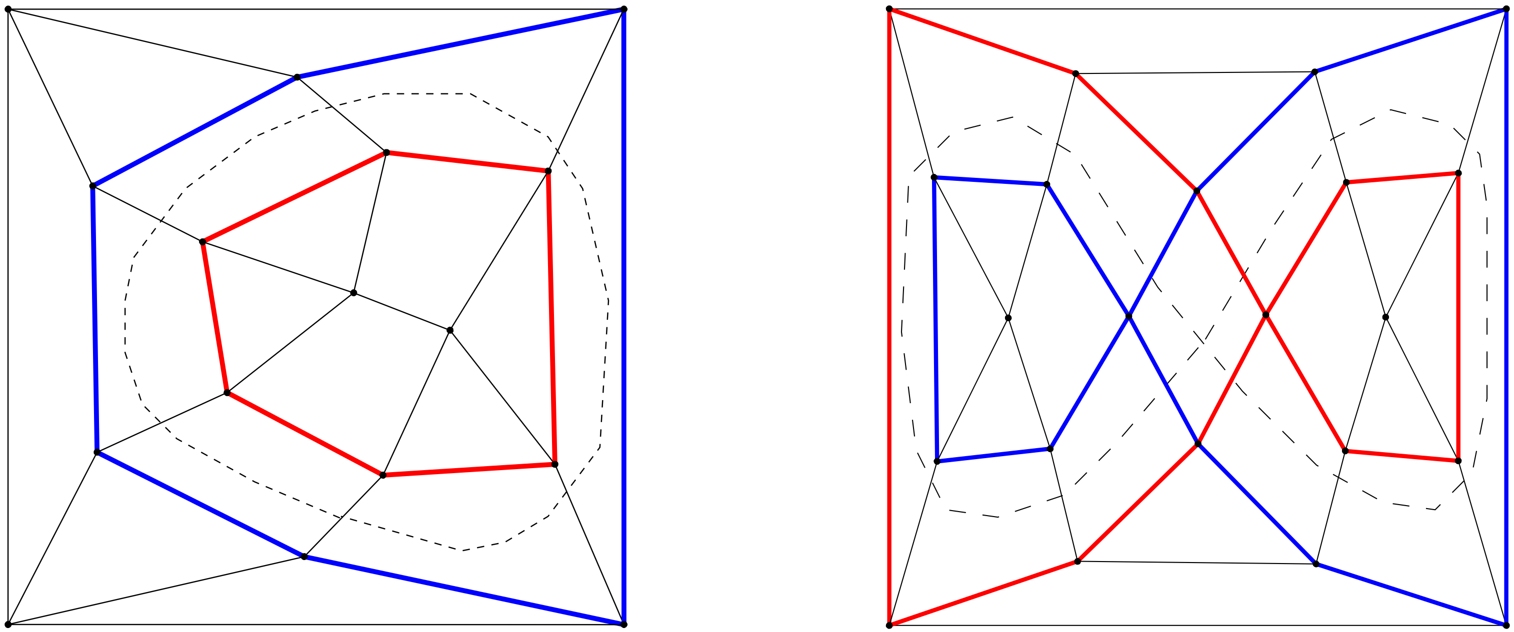




III. Railroad structure and tightness

Railroads, 4-valent case

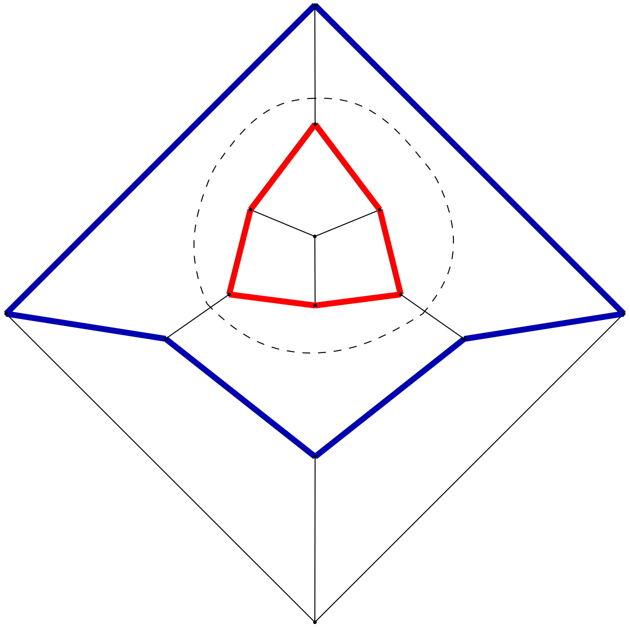
A railroad in an octahedrite is a circuit of square faces, such that any of them is adjacent to its neighbors on opposite faces. Any railroad is bordered by two central circuits.



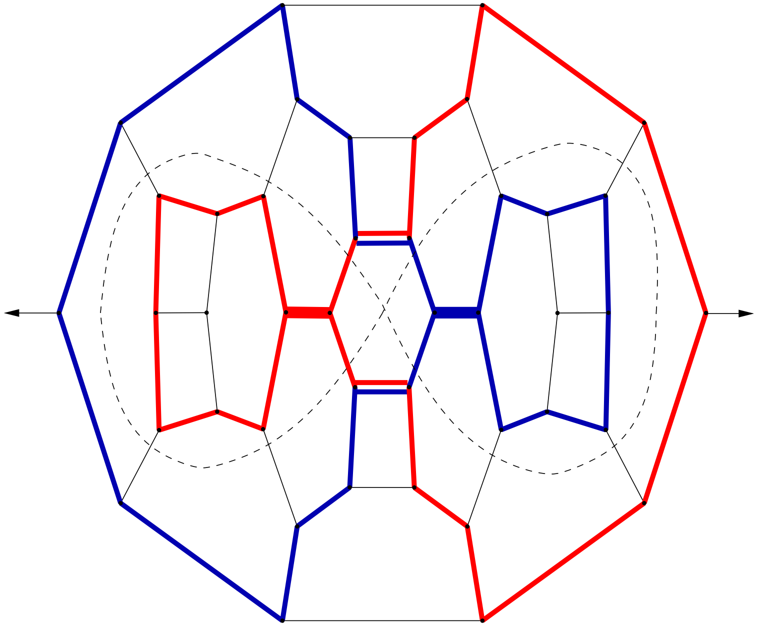
Railroads, as well as central circuits, can self-intersect.
An octahedrite is called tight if it has no railroad.

Railroads, 3-valent case

A railroad in graph q_n, $q = 3, 4, 5$ is a circuit of hexagonal faces, such that any of them is adjacent to its neighbors on opposite faces. Any railroad is bordered by two zigzags.



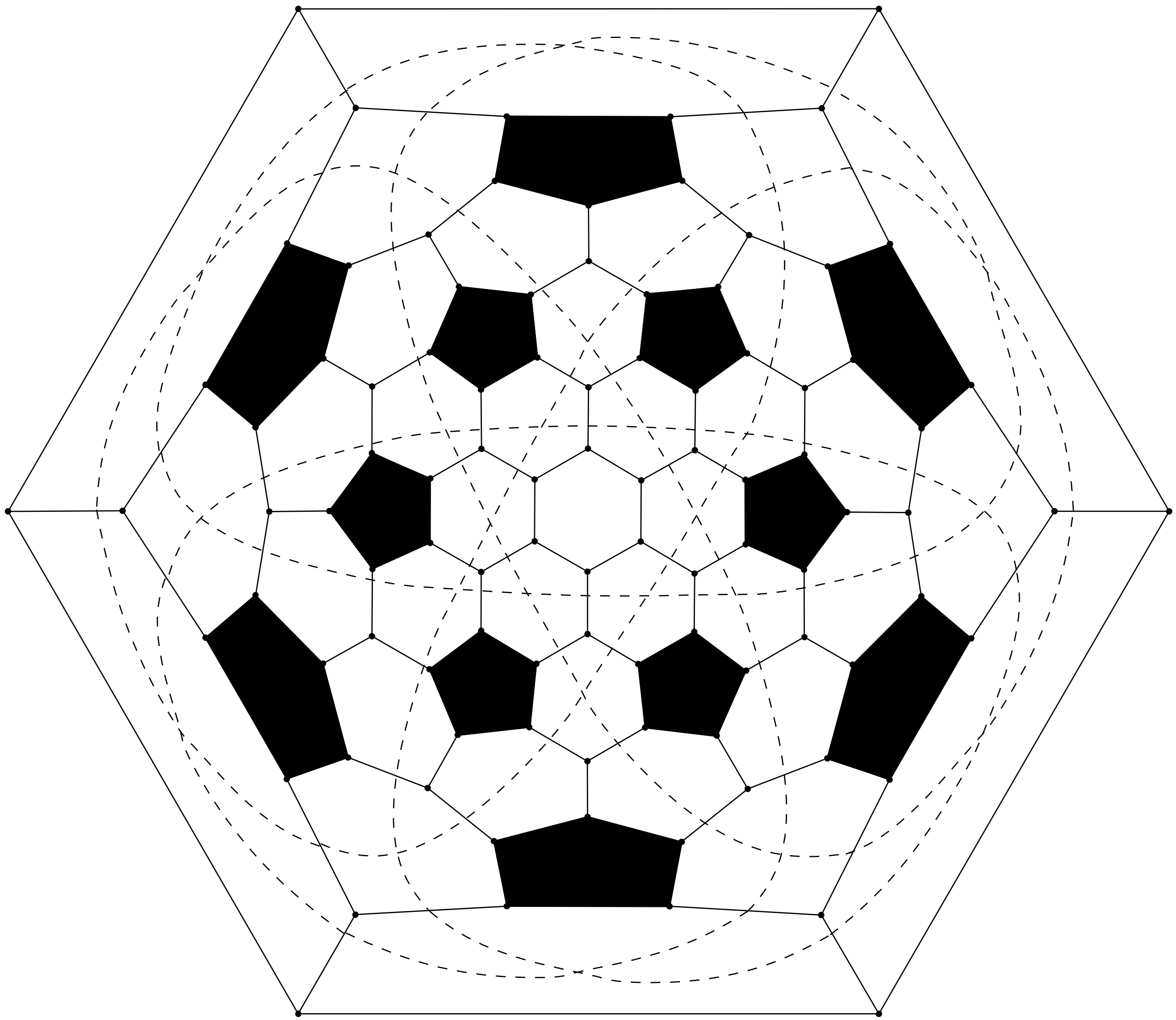
$4_{14}(D_{3h})$



$4_{42}(C_{2v})$

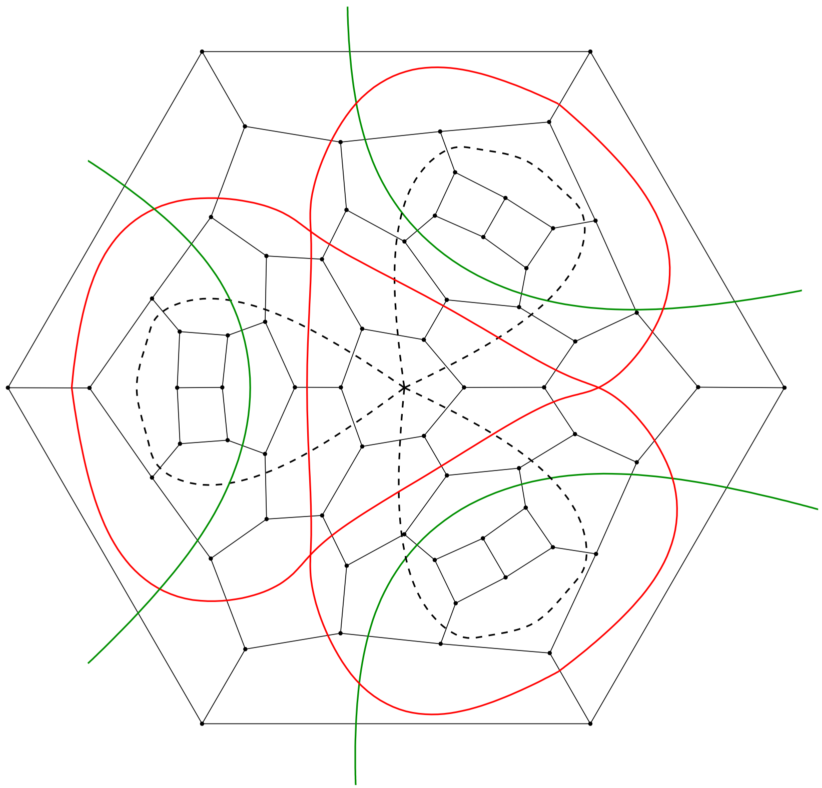
Railroads, as well as zigzags, can self-intersect (doubly or triply). A graph q_n is called tight if it has no railroad.

First IPR fullerene with self-int. railroad

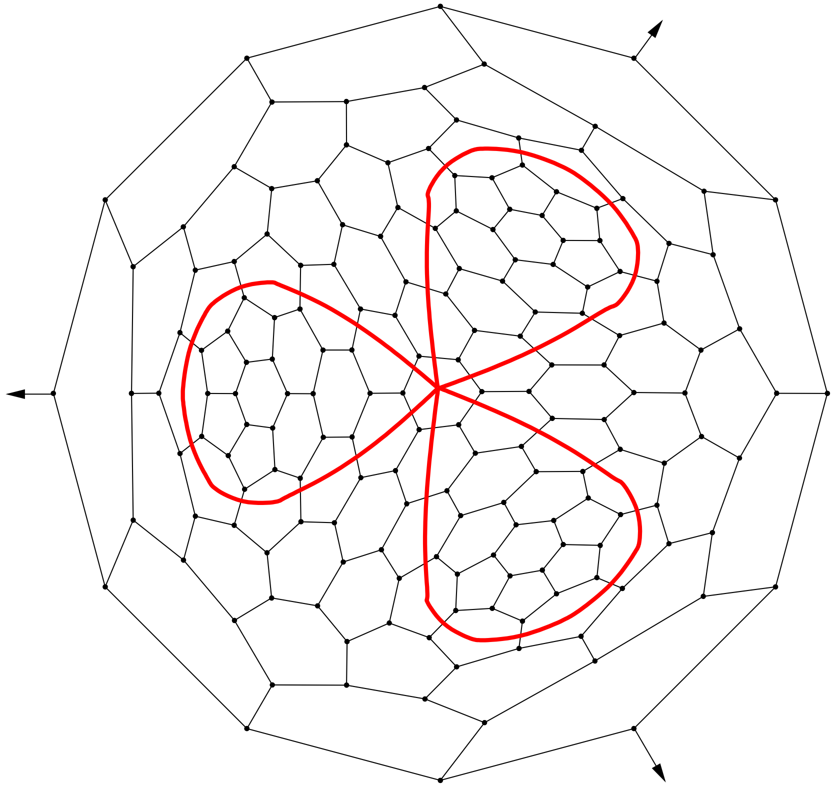


$F_{96}(D_{6d})$; realizes projection of Conway knot $(4 \times 6)^*$

Triple self-intersection

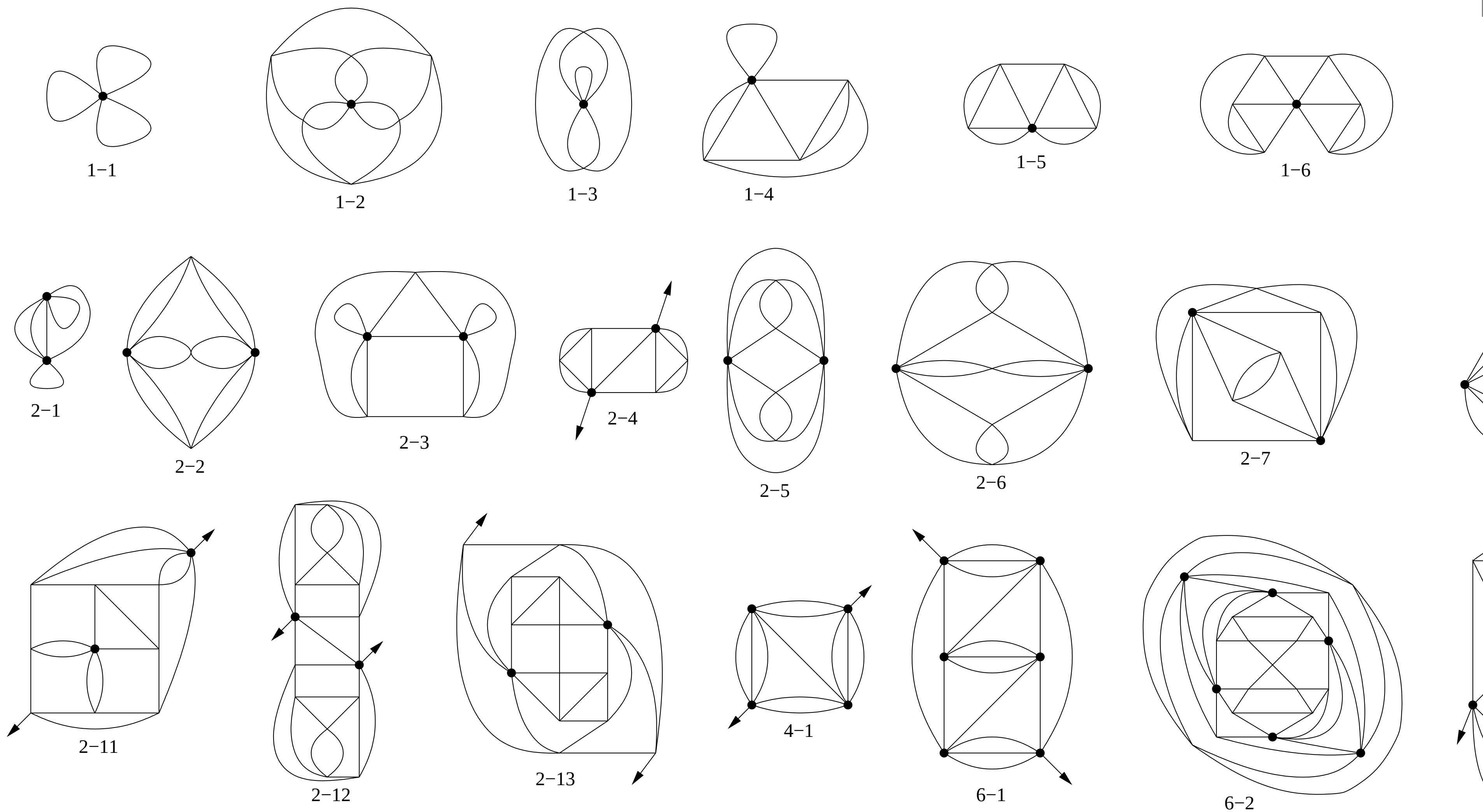


$4_{66}(D_{3h})$
It is smallest such 4_n graph.



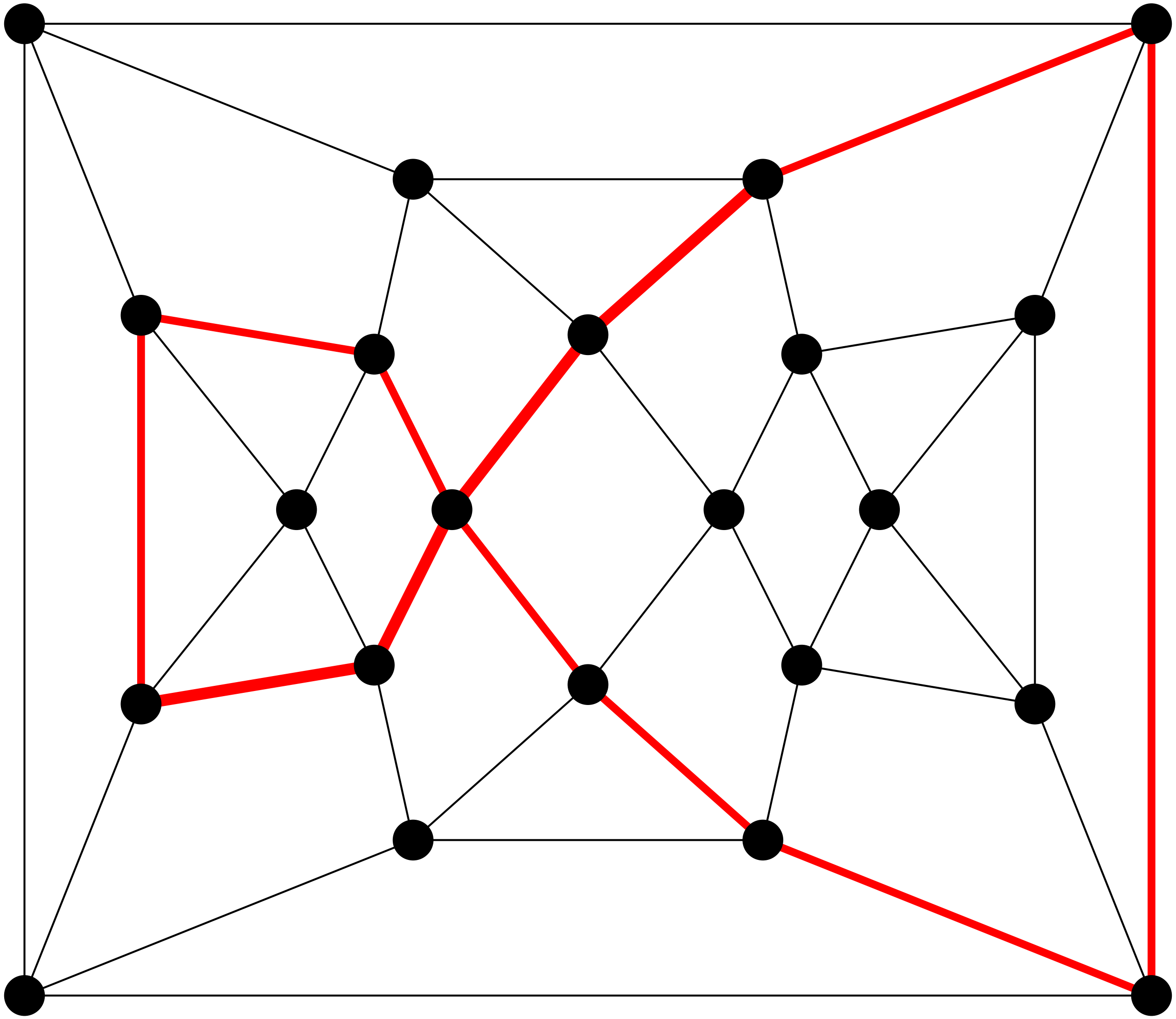
$5_{176}(C_{3v})$
Conjecture: It is smallest such 5_n graph.

Railroads with triple points in small 4_n



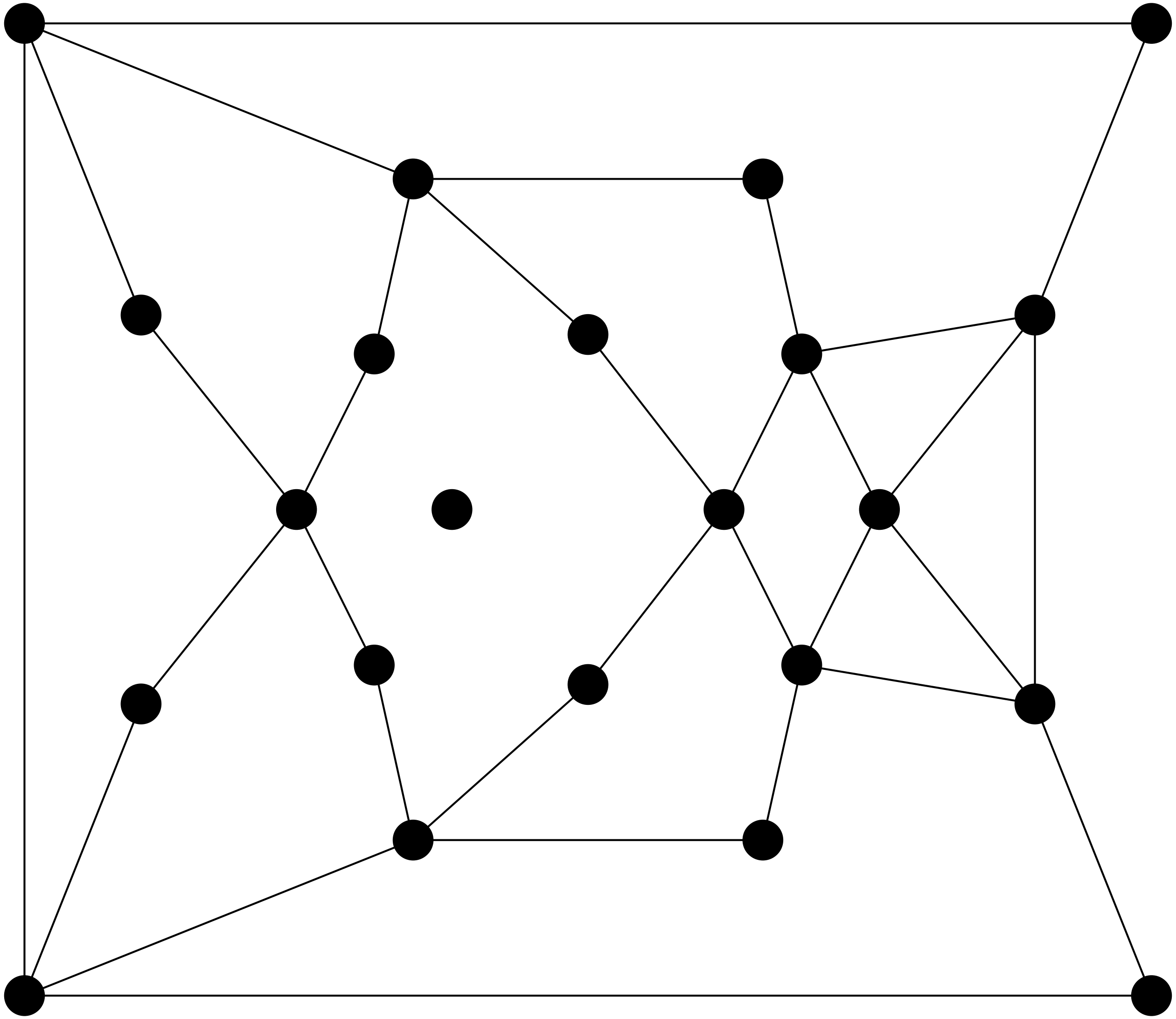
Removing central circuits

Take a 4-valent plane graph G and a central circuit.



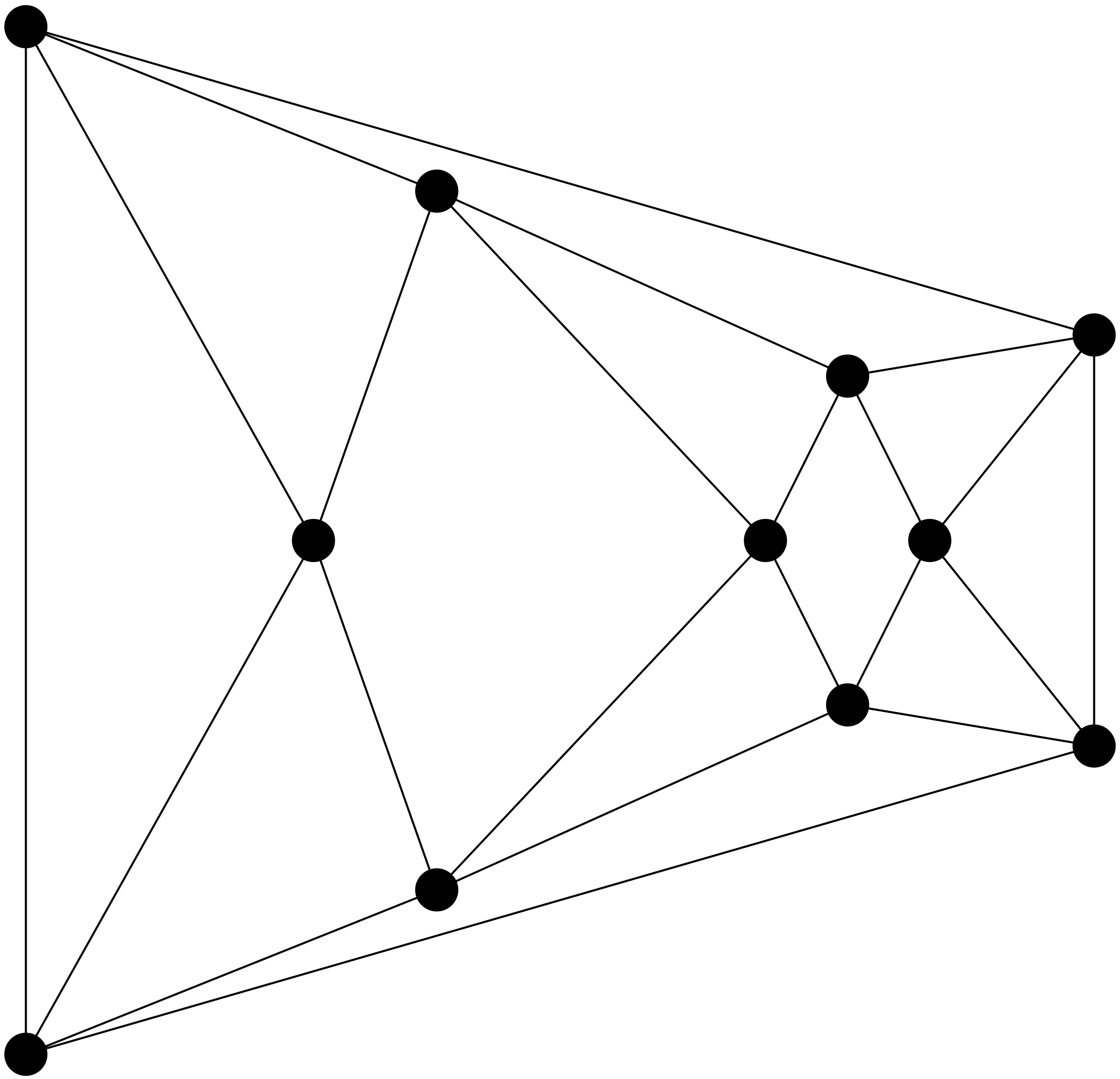
Removing central circuits

Remove the edges of the central circuit.



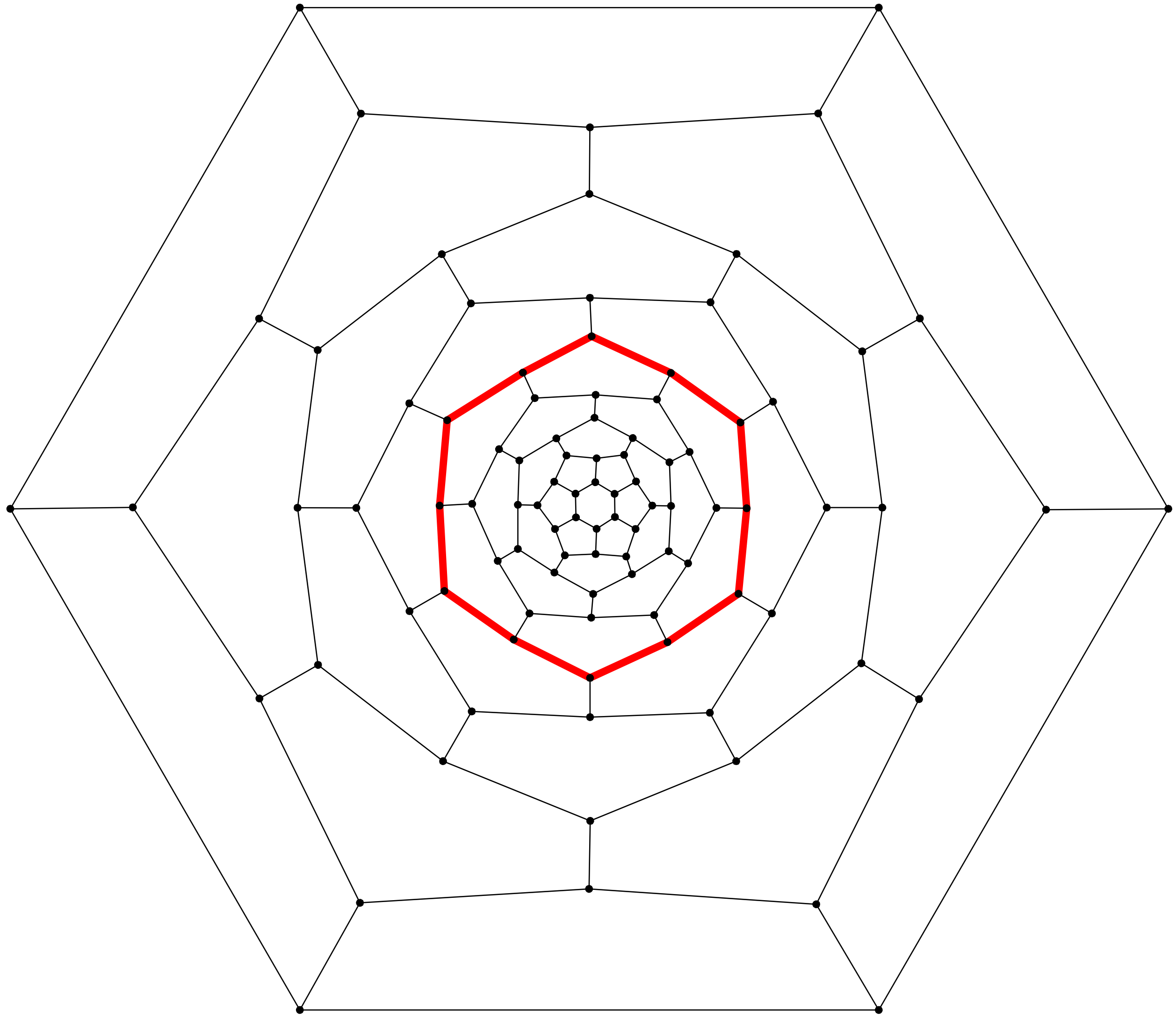
Removing central circuits

Remove the vertices of degree 0 or 2.



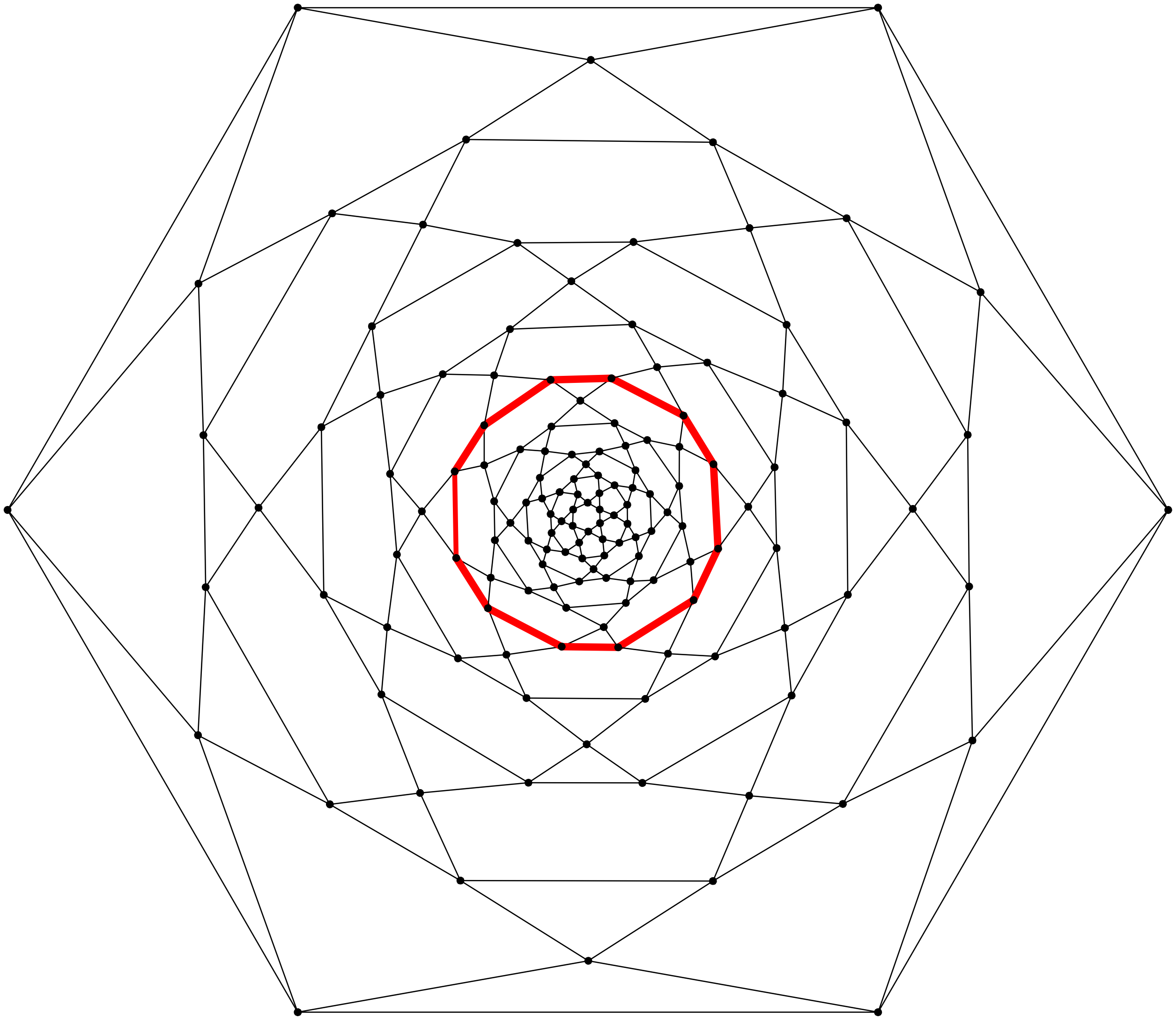
Removing zigzags

Take a plane graph G and a zigzag.



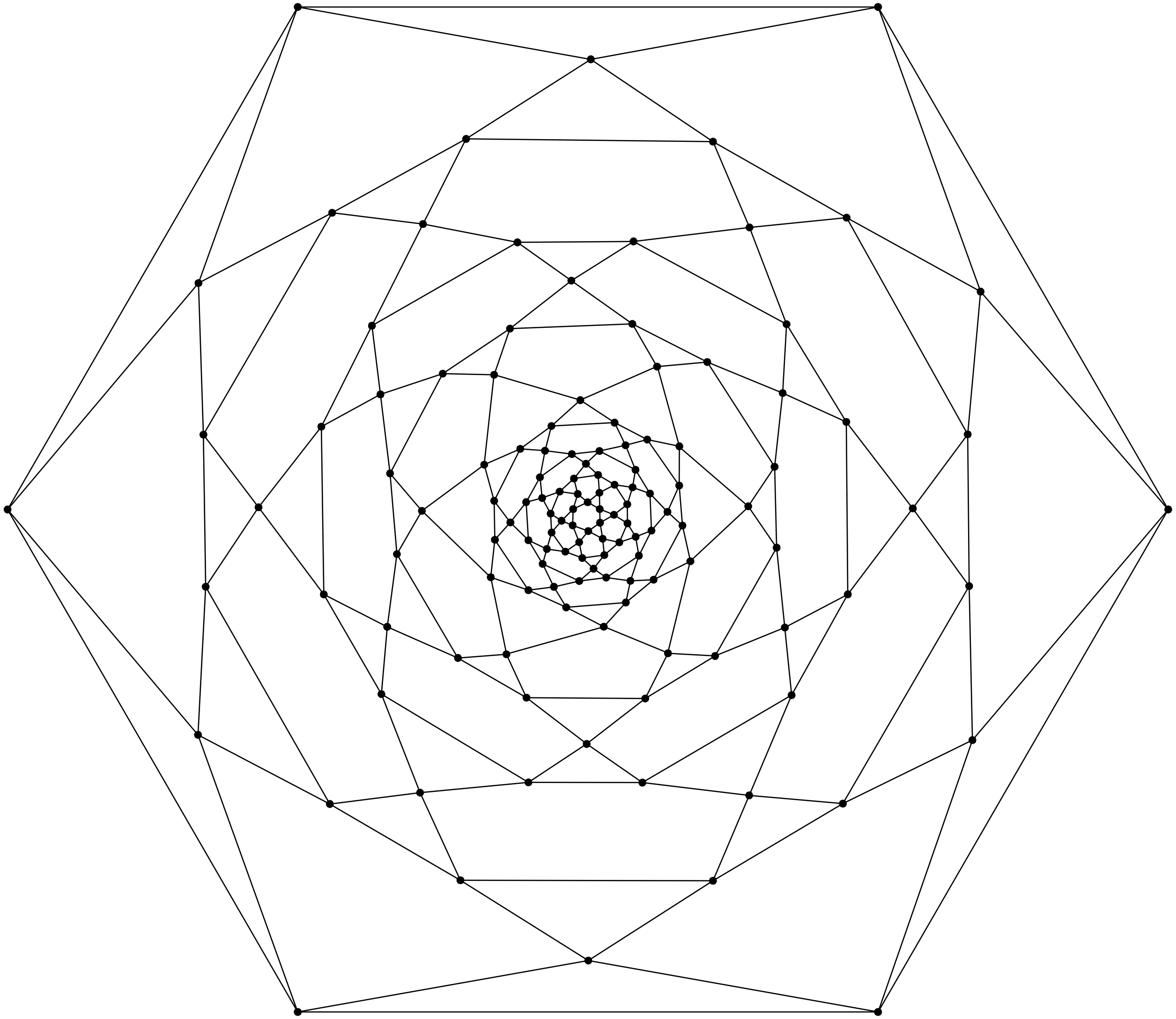
Removing zigzags

Go to the medial.



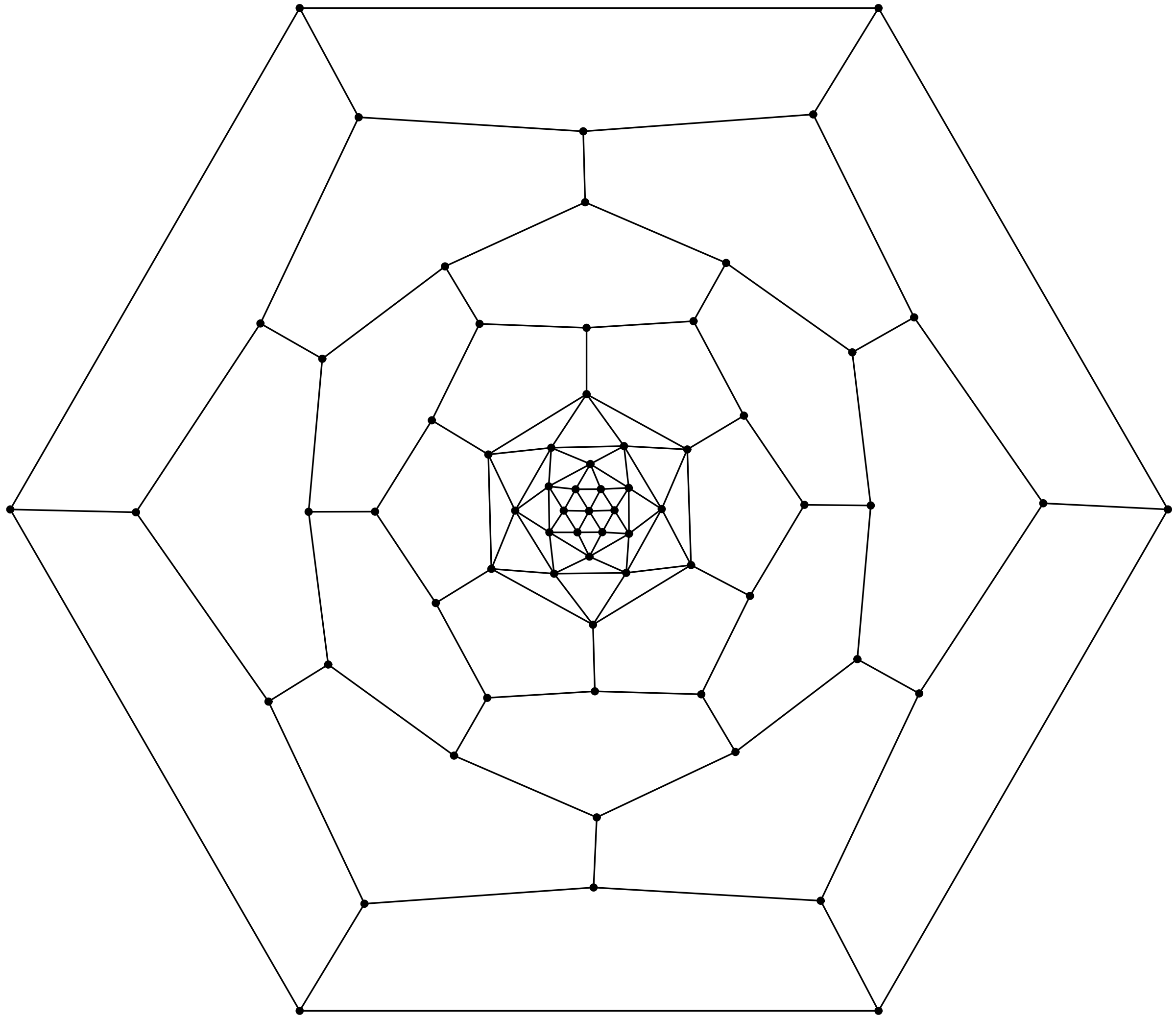
Removing zigzags

Remove the central circuit.



Removing zigzags

Take one (out of two) inverse medial graph.



Extremal problem

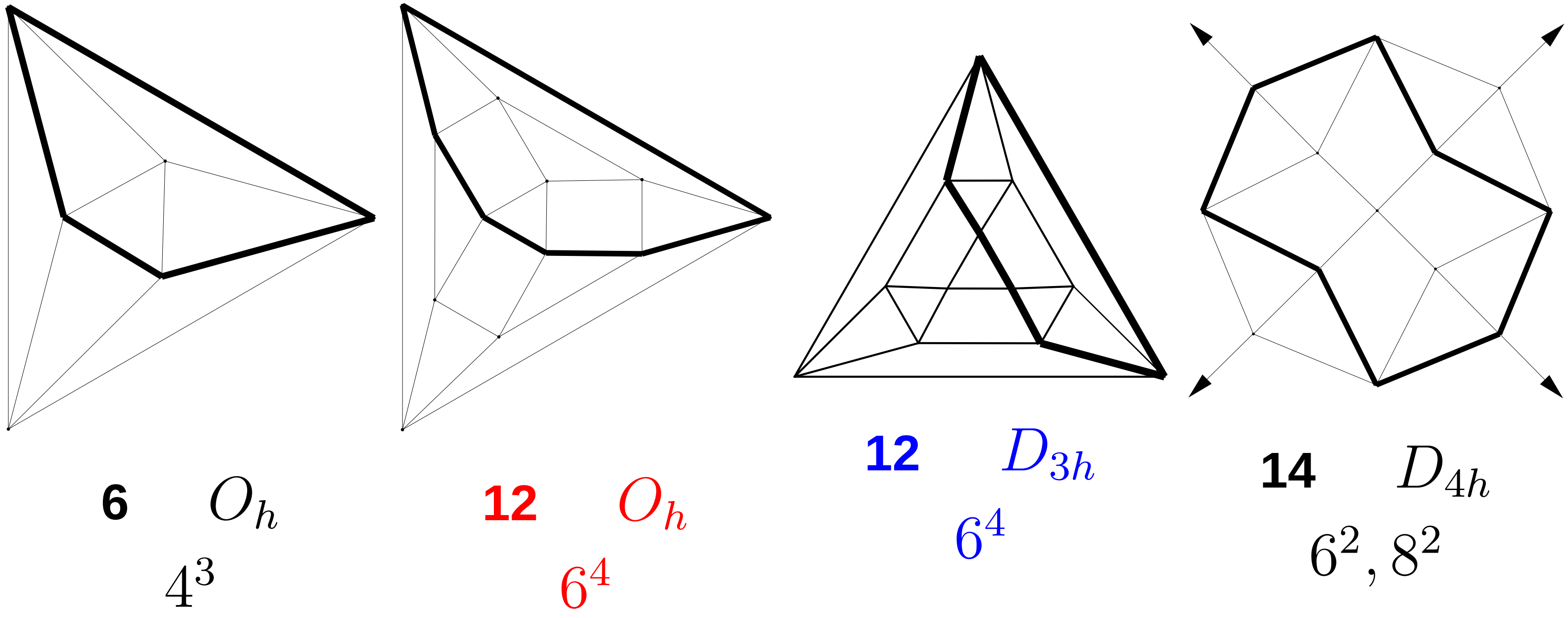
Given a class of tight graphs (octahedrites, graphs q_n), there exist a constant C such that any element of the class has at most C ZC-circuits.

- Every tight octahedrite has at most 6 central circuits.
  Proof method: Local analysis + case by case analysis.
- Every tight 3_n has exactly 3 zigzags.
  Proof method: Uses an algebraic formalism on the graphs 3_n.
- Every tight 4_n has at most 9 zigzags.
  Conjecture: The correct upper bound is 8. Checked for $n \leq 400$.
- Every tight 5_n has at most 15 zigzags.
  Attempted proof: Uses a local analysis on zigzags.

Tight with simple central circuits

Theorem 1 *There is exactly* 8 *tight octahedrites with simple central circuits.*

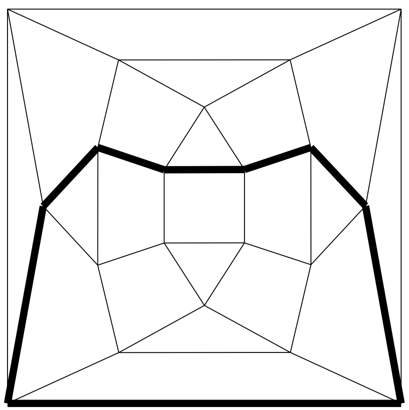
Proof method: After removing a central circuit, the obtained graph has faces of gonality at most 4.



Tight with simple central circuits

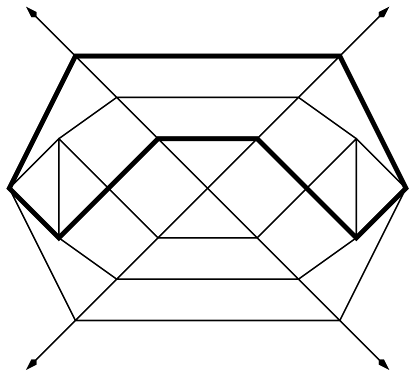
Theorem 2 *There is exactly* 8 *tight octahedrites with simple central circuits.*

Proof method: After removing a central circuit, the obtained graph has faces of gonality at most 4.



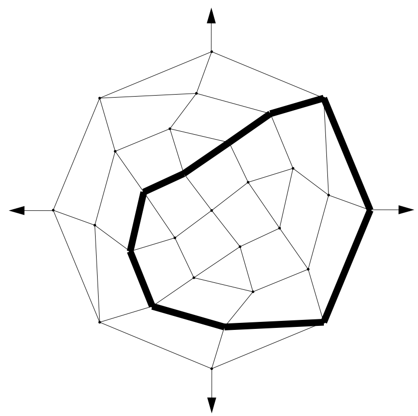
20 D_{2d}

8^5



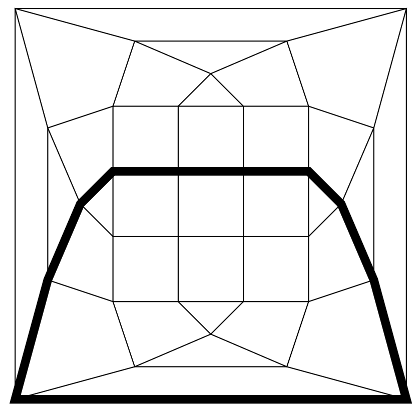
22 D_{2h}

$8^3, 10^2$



30 O

10^6

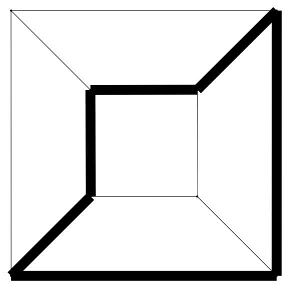


32 D_{4h}

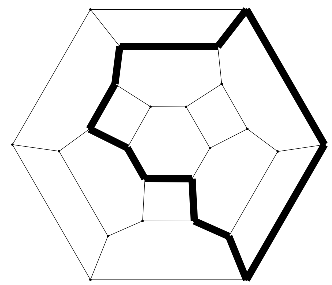
$10^4, 12^2$

Tight with simple zigzags

- All tight 3_n have simple zigzags
  ➠Infinity of such graphs
- There are exactly 2 tight graph 4_n with simple zigzags: Cube and Truncated Octahedron=$GC_{1,1}(Cube)$.
  Proof method: The size of intersection of two simple zigzags is at most 6. There is at most 9 zigzags.
  ➠Upper bound on n.



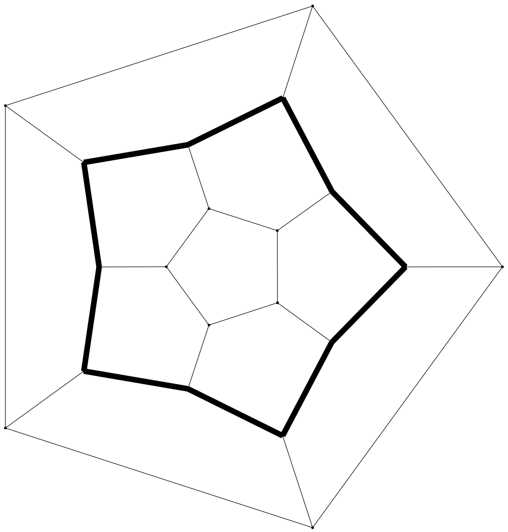
6 O_h, 6^4



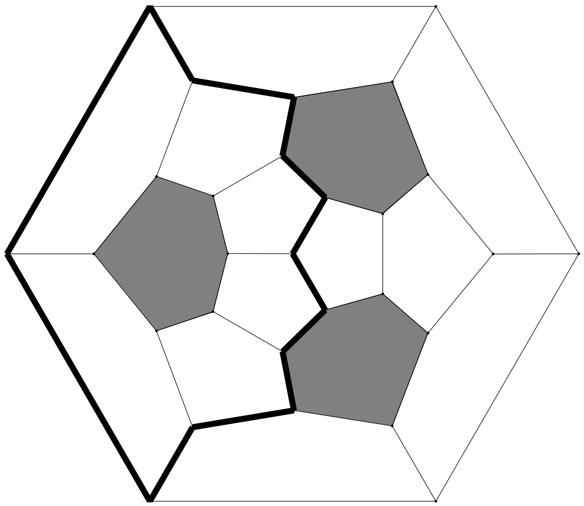
24 O_h, 10^6

- There is at least 9 tight graphs 5_n with simple zigzags. G. Brinkmann and T. Harmuth computation of fullerenes with simple zigzags up to 200 vertices.

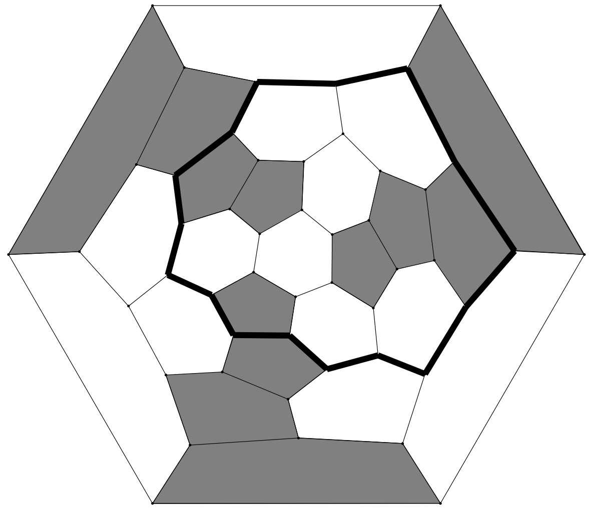
Tight 5_n with simple zigzags



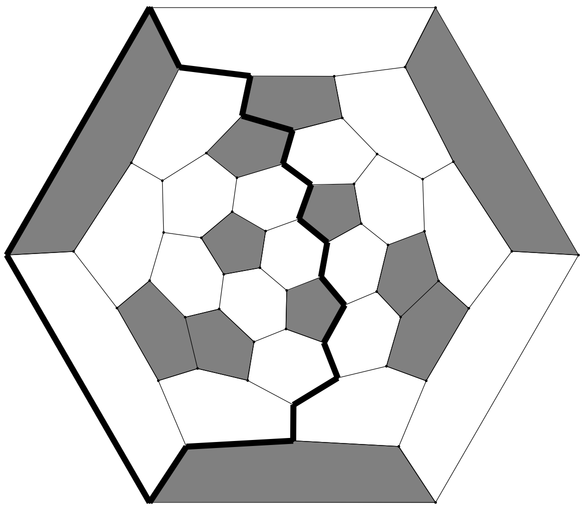
$20 \quad I_h, 20^6$



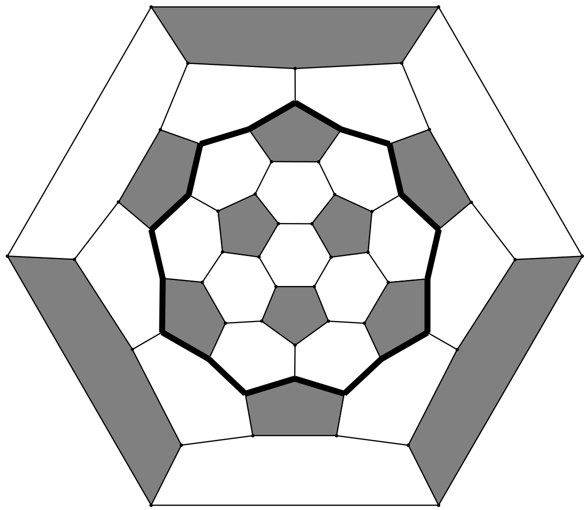
$28 \quad T_d, 12^7$



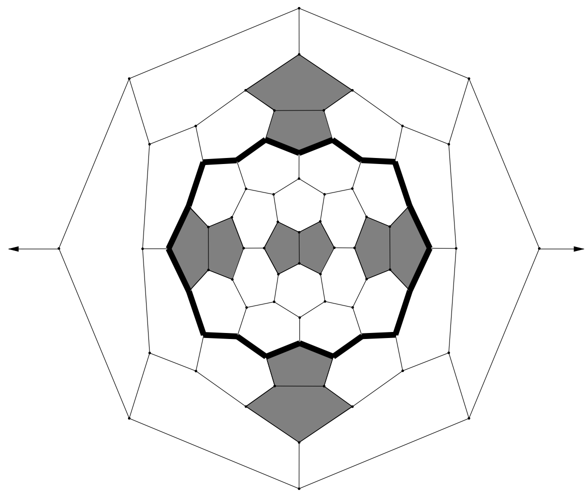
$48 \quad D_3, 16^9$



$60 \quad D_3, 18^{10}$

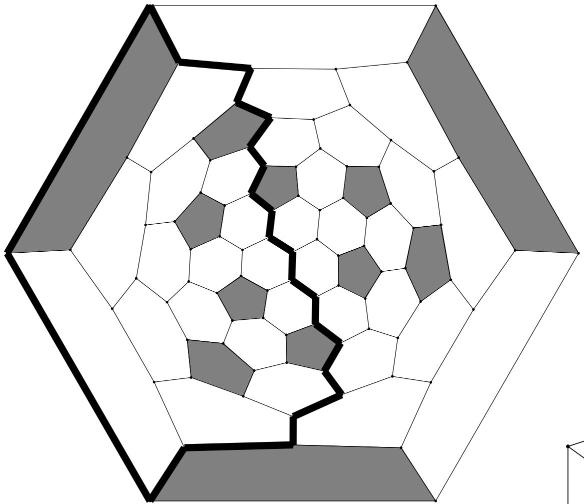


$60 \quad I_h, 18^{10}$

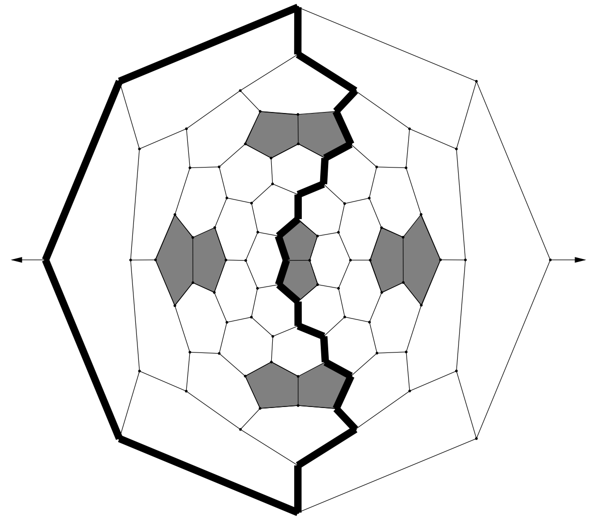


$76 \quad D_{2d}, 22^4, 20^7$

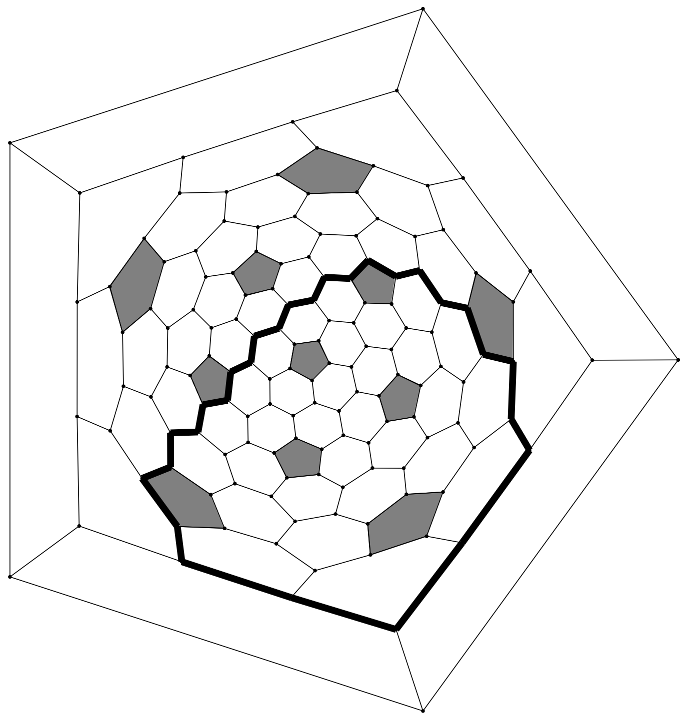
Tight 5_n with simple zigzags



$88 \quad T, 22^{12}$



$92 \quad T_h, 24^6, 22^6$



$140 \quad I, 28^{15}$

Tight F_n with only simple zigzags

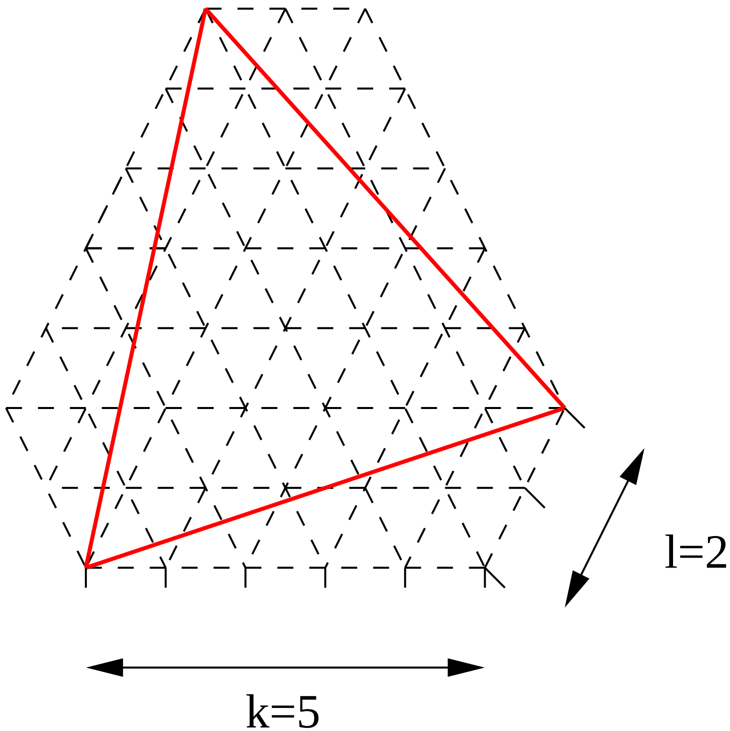
n	group	z-vector	orbit lengths	int. vector
20	I_h	10^6	6	2^5
28	T_d	12^7	3,4	2^6
48	D_3	16^9	3,3,3	2^8
60, IPR	I_h	18^{10}	10	2^9
60	D_3	18^{10}	1,3,6	2^9
76	D_{2d}	$22^4, 20^7$	1,2,4,4	$4, 2^9$ and 2^{10}
88, IPR	T	22^{12}	12	2^{11}
92	T_h	$22^6, 24^6$	6,6	2^{11} and $2^{10}, 4$
140, IPR	I	28^{15}	15	2^{14}

Conjecture: this list is complete (checked for $n \leq 200$).
It gives 7 Grünbaum arrangements of plane curves.

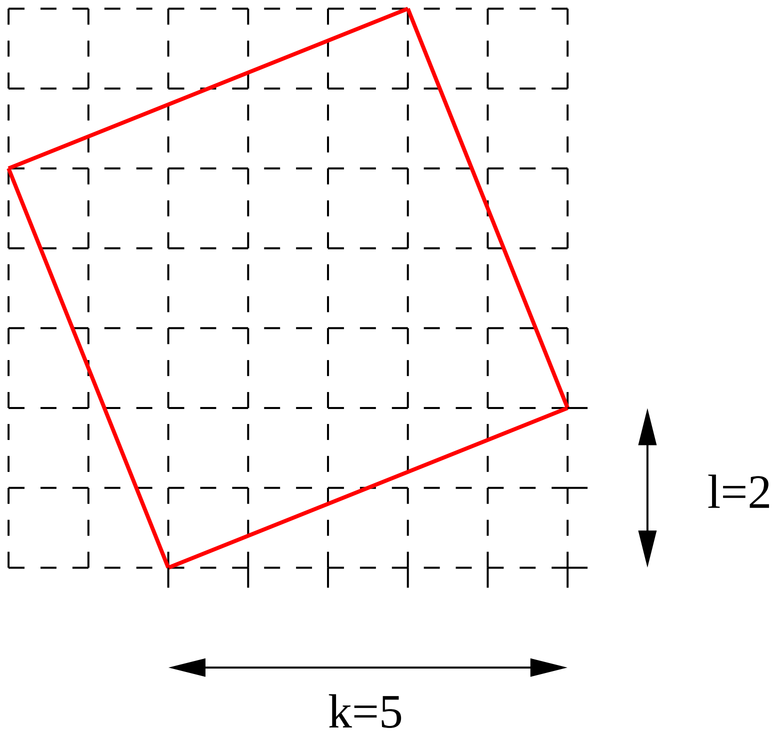
IV. Goldberg-Coxeter construction

The construction

- Take a 3- or 4-valent plane graph G_0. The graph G_0^* is formed of triangles or squares.
- Break the triangles or squares into pieces according to parameter (k, l).



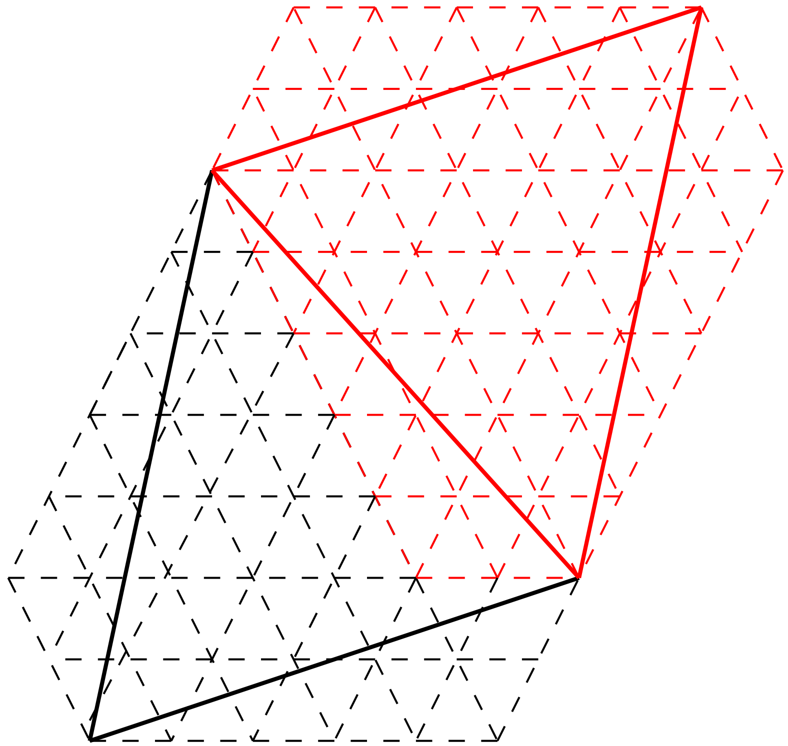
3−valent case



4−valent case

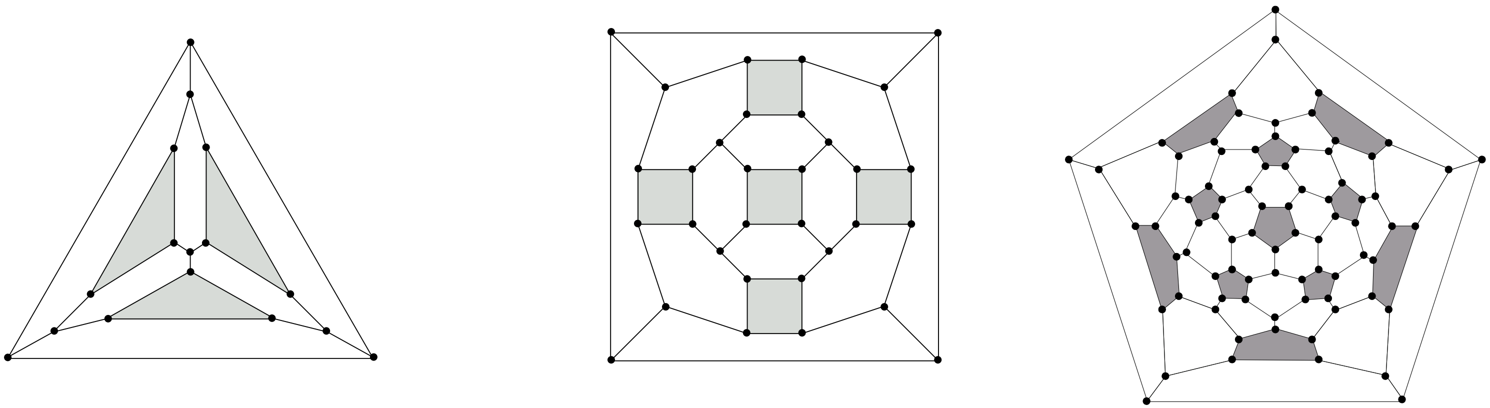
Gluing the pieces

- Glue the pieces together in a coherent way.
- We obtain another triangulation or quadrangulation of the plane.



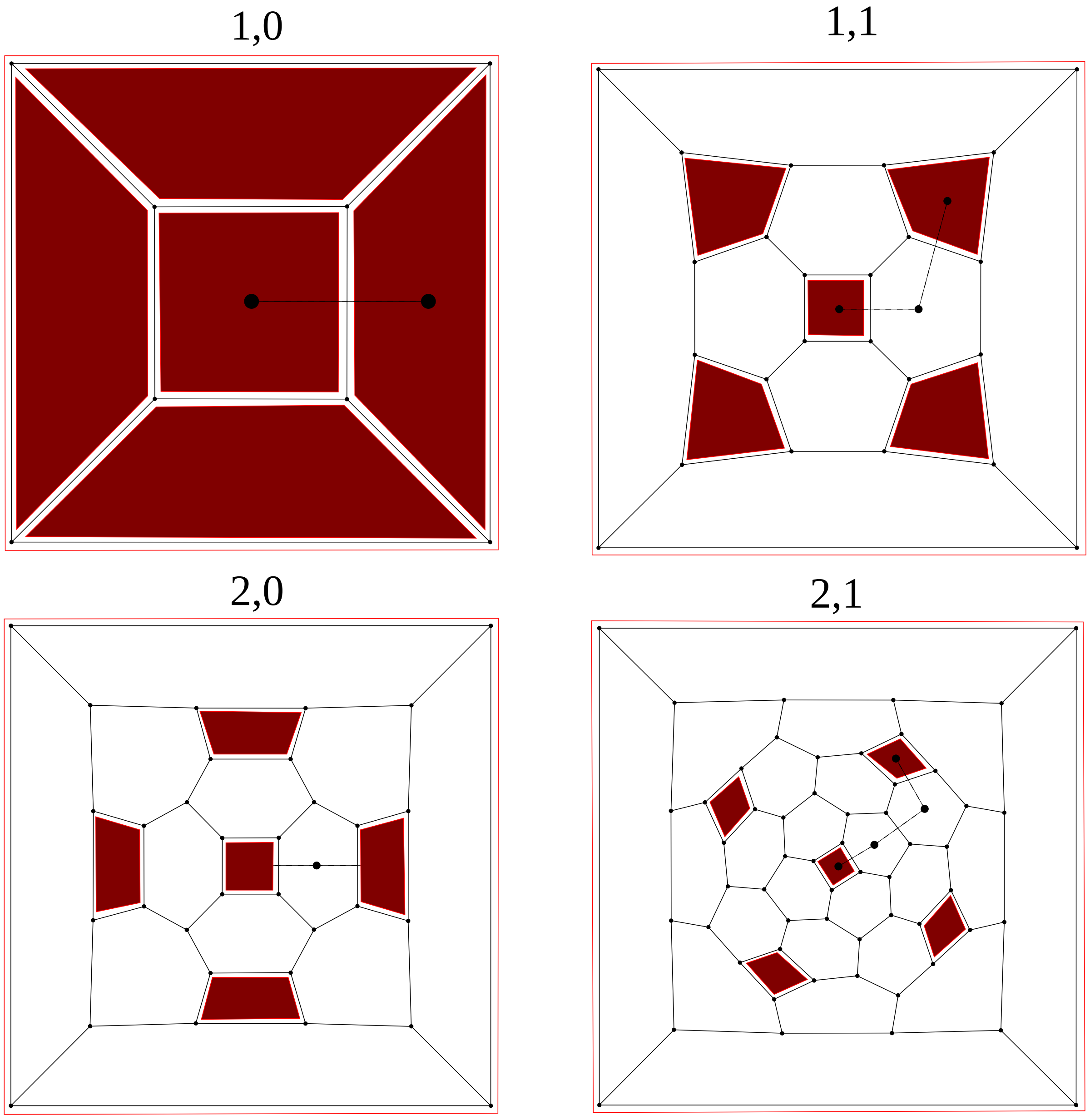
Final steps

- Go to the dual and obtain a 3- or 4-valent plane graph, which is denoted $GC_{k,l}(G_0)$ and called "Goldberg-Coxeter construction".
- The construction works for any 3- or 4-valent map on oriented surface.



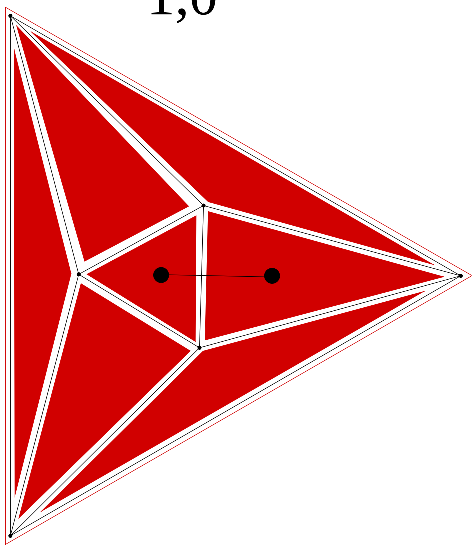
Operation $GC_{2,0}$ on Tetrahedron, Cube and Dodecahedron

Goldberg-Coxeter for Cube

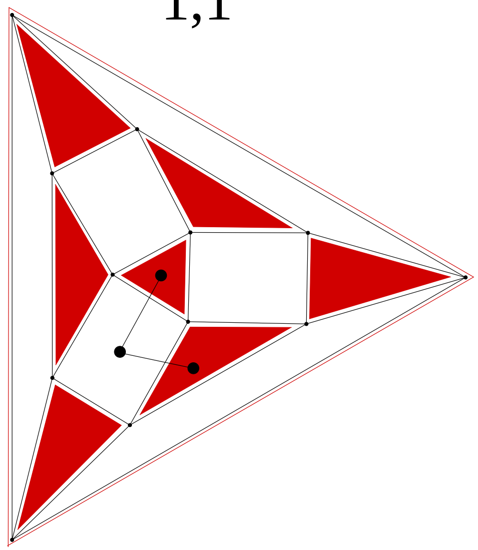


Goldberg-Coxeter for Octahedron

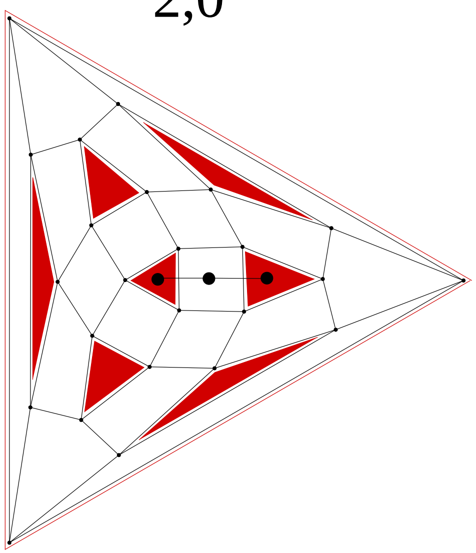
1,0



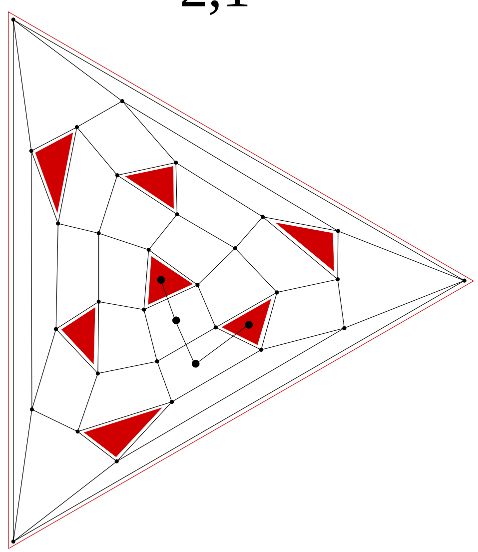
1,1



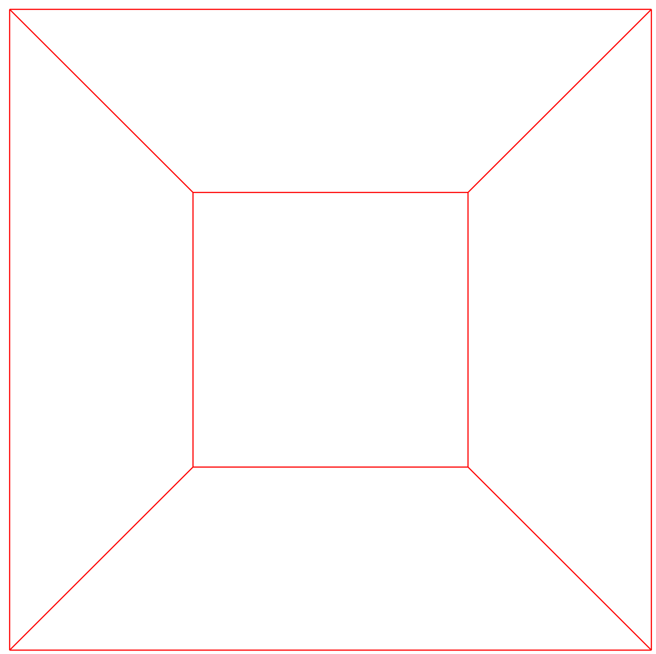
2,0



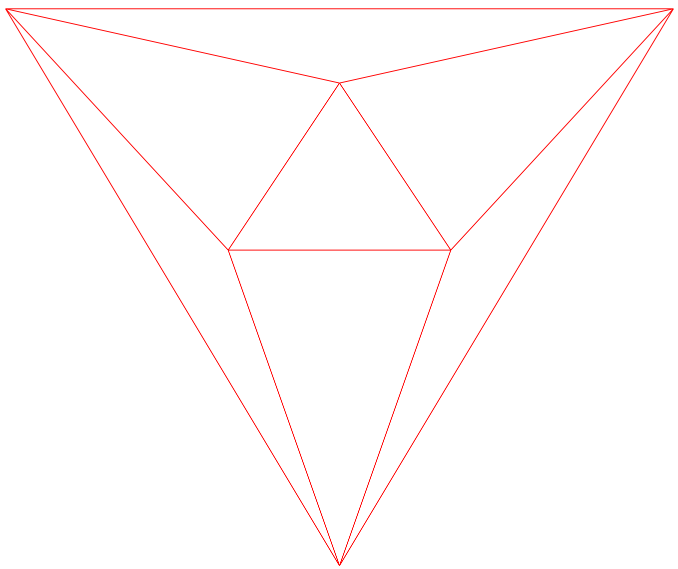
2,1



The case $(k, l) = (1, 1)$

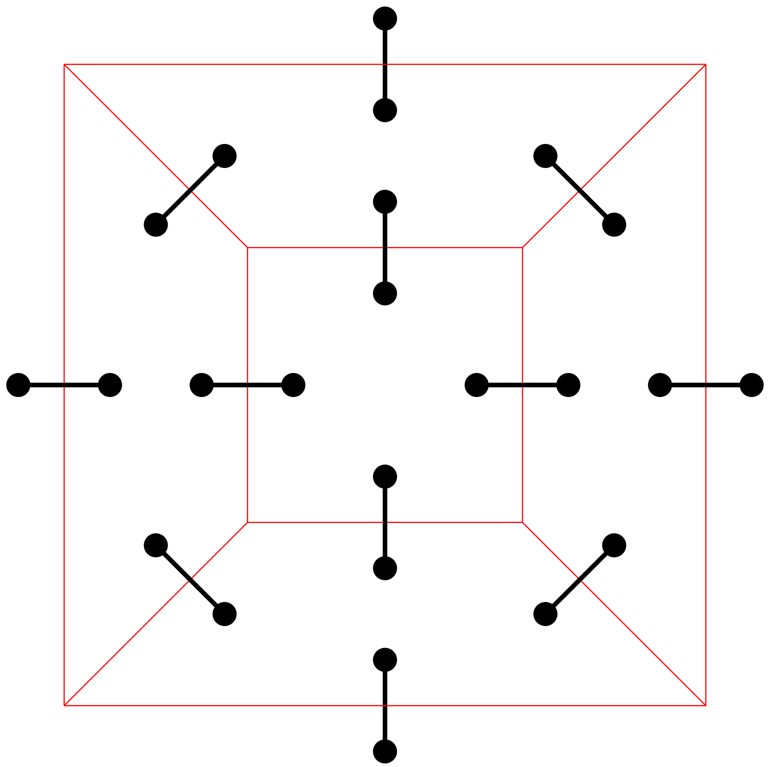


Case 3-valent

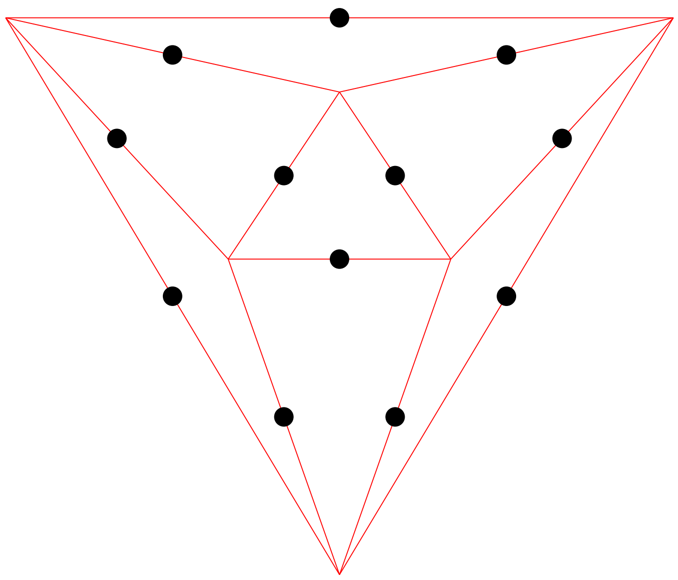


Case 4-valent

The case $(k, l) = (1, 1)$

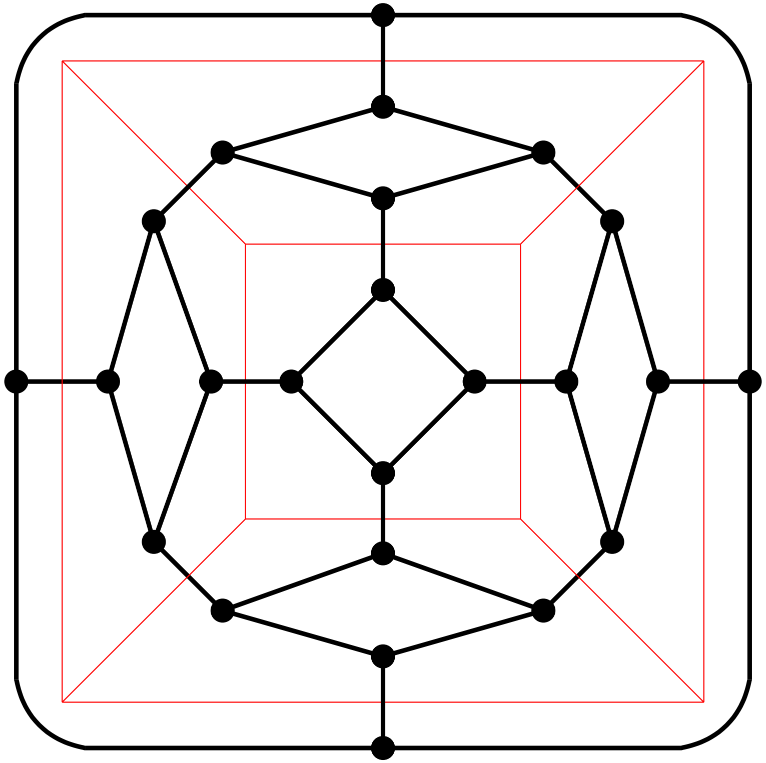


Case 3-valent

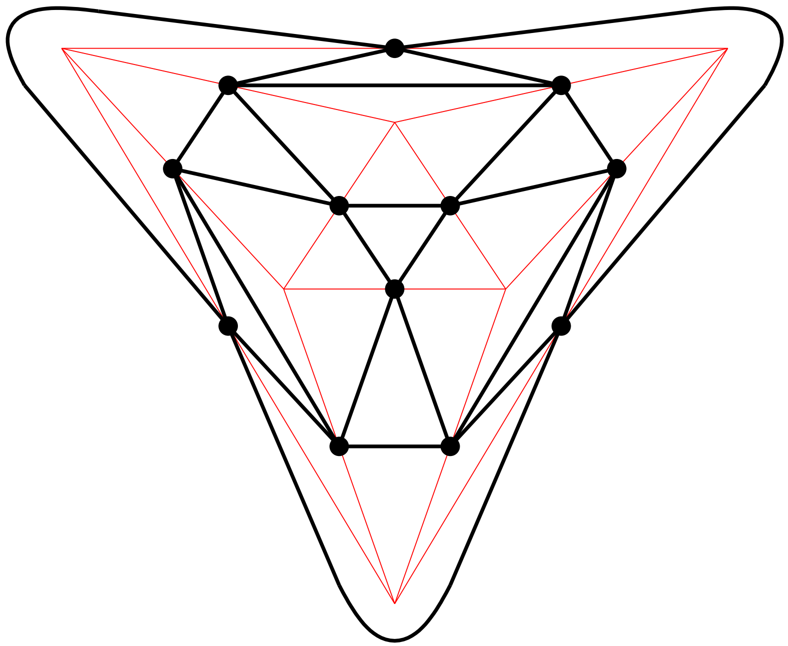


Case 4-valent

The case $(k, l) = (1, 1)$



Case 3-valent
$GC_{1,1}$ is called leapfrog
(=Truncation of the dual)



Case 4-valent
$GC_{1,1}$ is called medial

Properties

- One associates $z = k + le^{i\frac{\pi}{3}}$(Eisenstein integer) or $z = k + li$(Gaussian integer) to the pair (k, l) in 3- or 4-valent case.
- If one writes $GC_z(G_0)$ instead of $GC_{k,l}(G_0)$, then one has:

$$GC_z(GC_{z'}(G_0)) \quad = \quad GC_{zz'}(G_0)$$

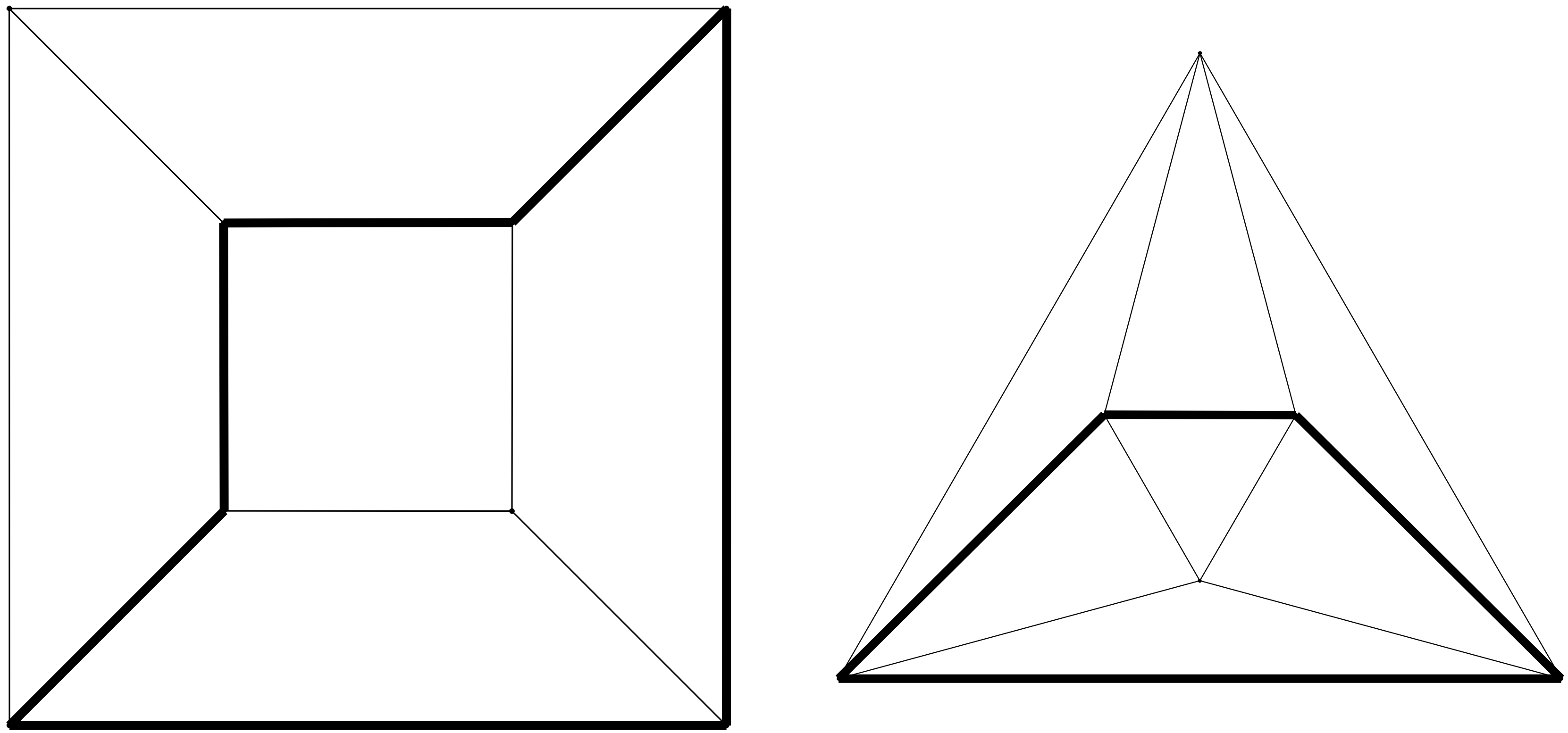
- If G_0 has n vertices, then $GC_{k,l}(G_0)$ has

$$n(k^2 + kl + l^2) = n|z|^2 \text{ vertices} \quad \text{if} \quad G_0 \text{ is } 3\text{-valent},$$
$$n(k^2 + l^2) = n|z|^2 \text{ vertices} \quad \text{if} \quad G_0 \text{ is } 4\text{-valent}.$$

- If G_0 has a symmetry plane, then $GC_z(G_0) = GC_{\overline{z}}(G_0)$.
- $GC_z(G_0)$ has all rotational symmetries of G_0 and all symmetries if $l = 0$ or $l = k$.

The special case $GC_{k,0}$

- Any ZC-circuit of G_0 corresponds to k ZC-circuits of $GC_{k,0}(G_0)$ with length multiplied by k.
- If the ZC-vector of G_0 is $\dots, c_l^{m_l}, \dots$, then the ZC-vector of $GC_{k,0}(G_0)$ is $\dots, (kc_l)^{km_l}, \dots$.



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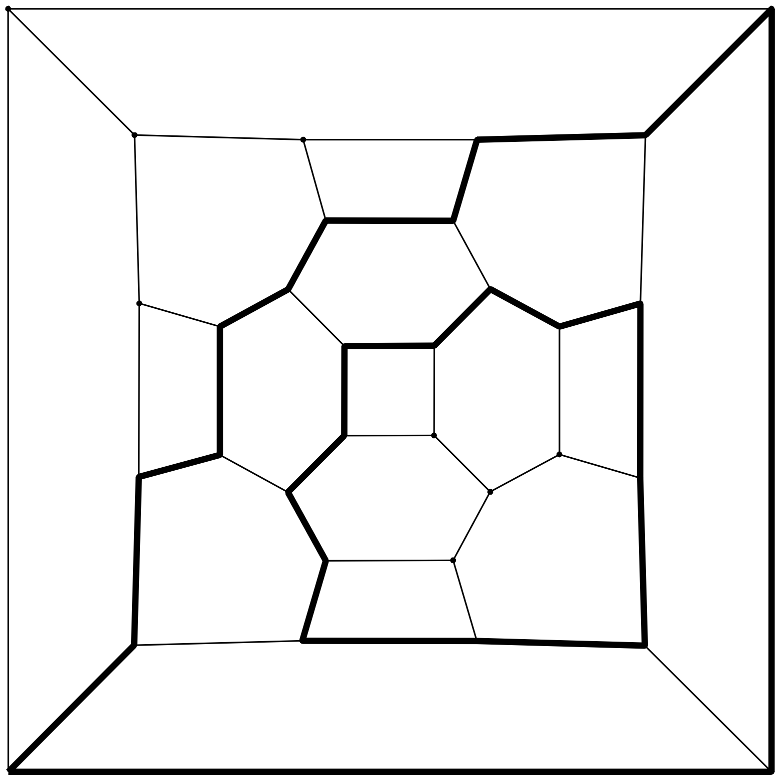
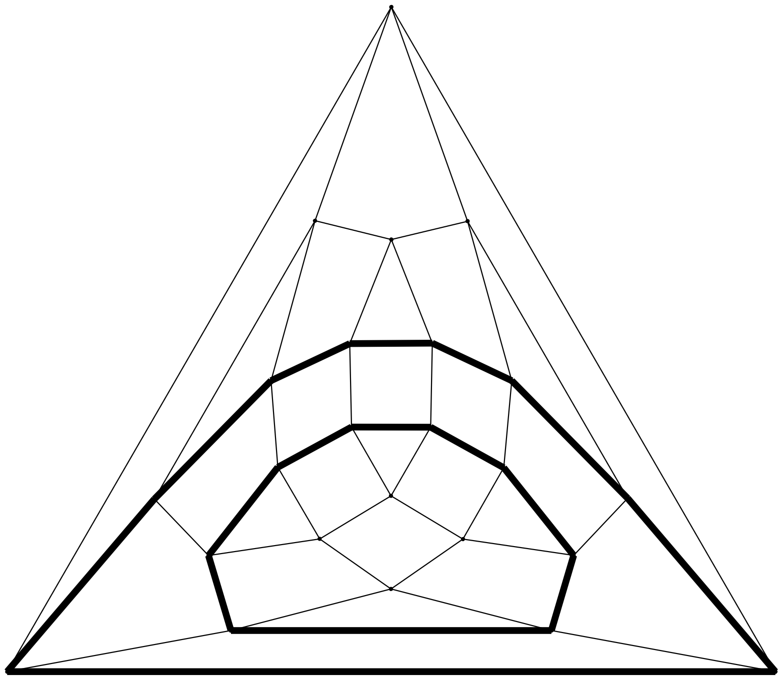
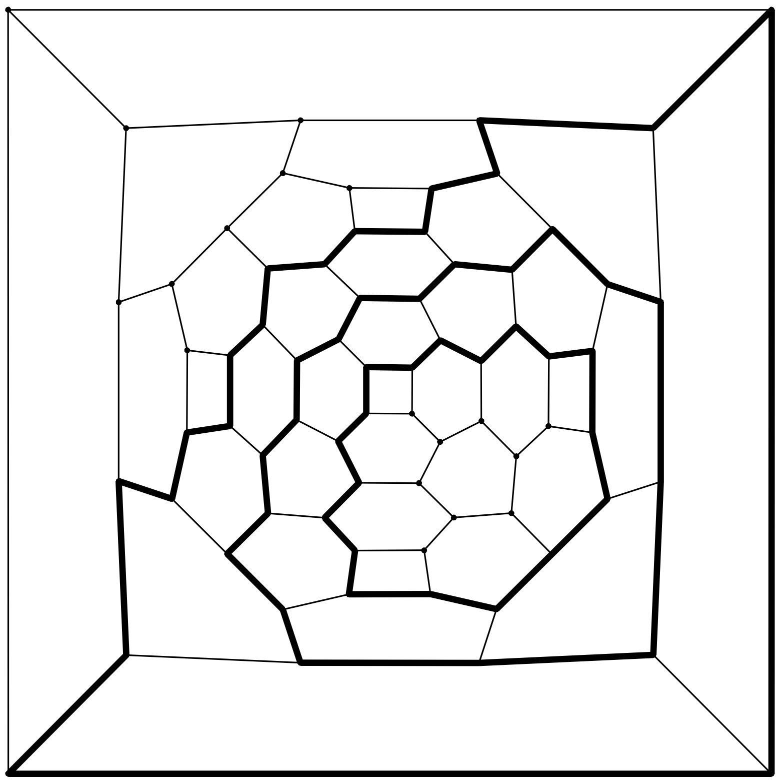
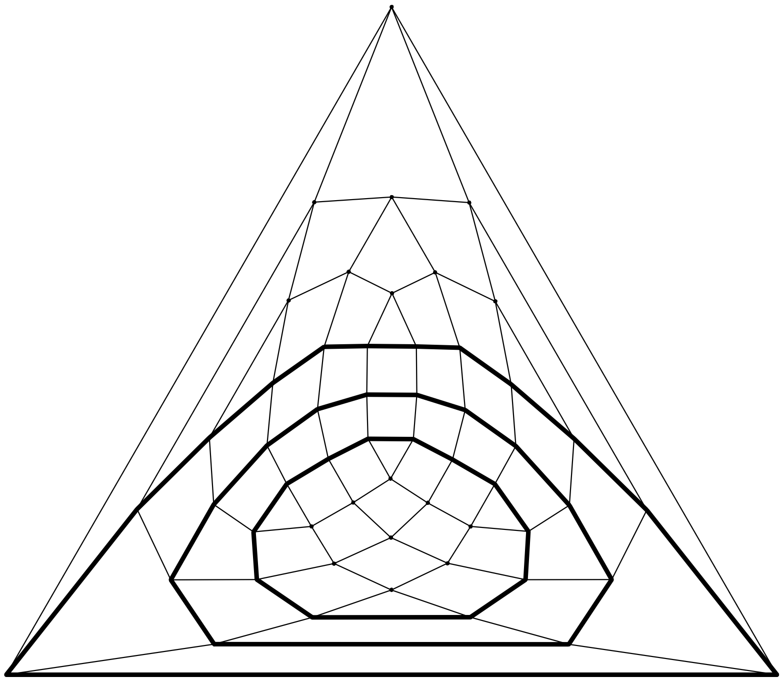




The special case $GC_{k,0}$

- Any ZC-circuit of G_0 corresponds to k ZC-circuits of $GC_{k,0}(G_0)$ with length multiplied by k.
- If the ZC-vector of G_0 is $\dots, c_l^{m_l}, \dots$, then the ZC-vector of $GC_{k,0}(G_0)$ is $\dots, (kc_l)^{km_l}, \dots$.





(k,l)-product formalism

Given a 3-valent plane graph G, the zigzags of the Goldberg-Coxeter construction of $GC_{k,l}(G)$ are obtained by:

- associating to G two elements L and R of a group called moving group,
- computing the value of the (k,l)-product $L \odot_{k,l} R$,
- the lengths of zigzags are obtained by computing the cycle structure of $L \odot_{k,l} R$.

For $GC_{k,l}(Dodecahedron)$ with $gcd(k,l) = 1$, this gives 6, 10 or 15 zigzags.

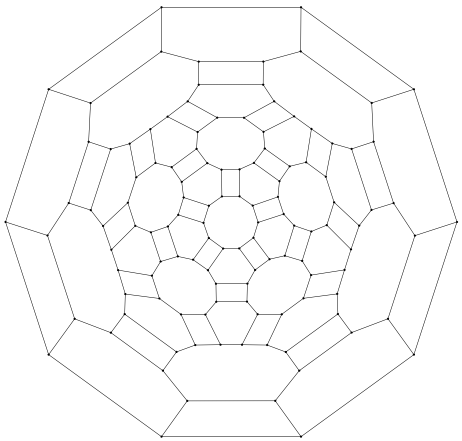
Illustration

- For any ZC-circuit of $GC_{k,l}(G_0)$, there exist $\alpha \geq 1$

  $\text{length}(ZC)$=$2(k^2+kl+l^2)\alpha$ 3-valent case

  $\text{length}(ZC)$=$(k^2+l^2)\alpha$ 4-valent case

  The [ZC]-vector of $GC_{k,l}(G_0)$ is the vector $\dots, \alpha_k^{m_k}, \dots$ where m_k is the number of ZC-circuits with order α_k.
- If $gcd(k,l)=1$, then $GC_{k,l}(Cube)$ has 6 zigzags if $k \equiv l \pmod 3$ and 4 otherwise.
- For Truncated Icosidodecahedron, possible [ZC]:



$2^{30}, 3^{40}$	$2^{30}, 5^{24}$	$3^{20}, 5^{24}$
$2^{60}, 3^{20}$	$2^{60}, 5^{12}$	$3^{40}, 5^{12}$
2^{90}	3^{60}	5^{36}
9^{20}	6^{30}	15^{12}

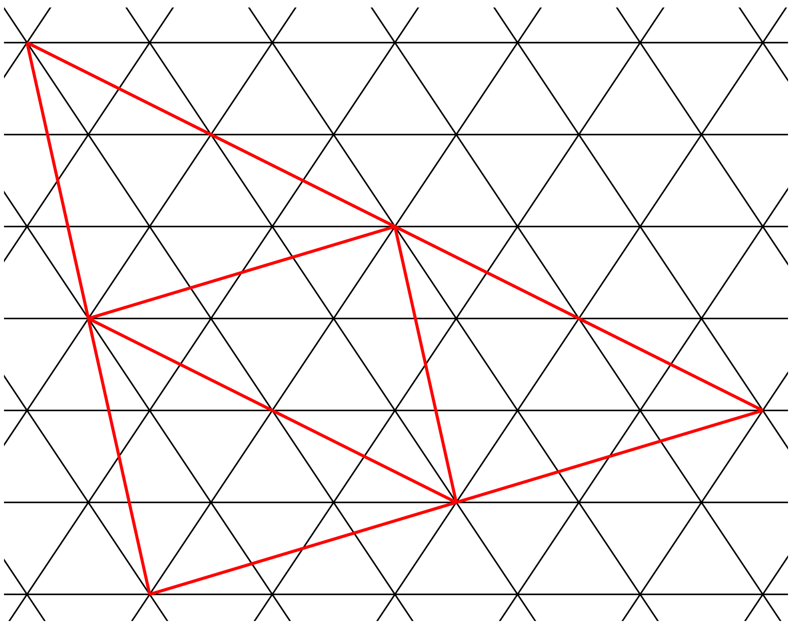
V. Parametrizing graphs

Parametrizing graphs q_n

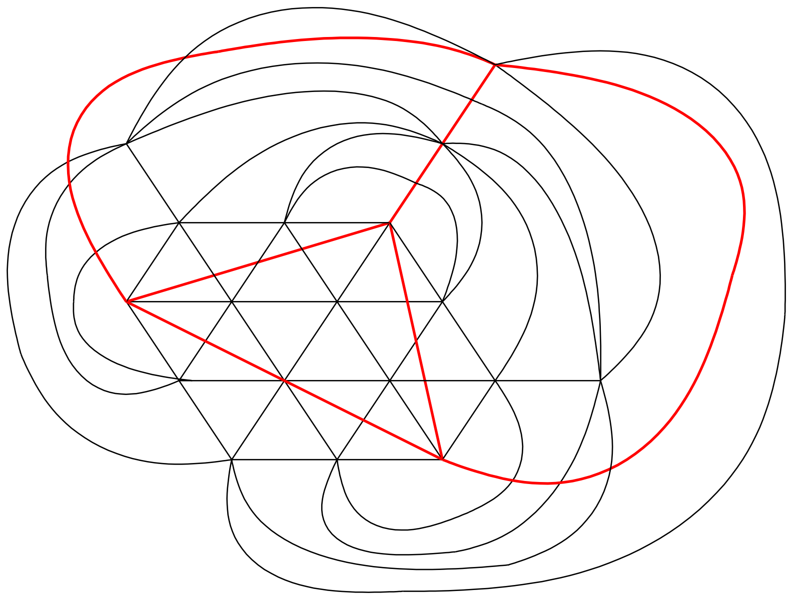
Idea: the hexagons are of zero curvature, it suffices to give relative positions of faces of non-zero curvature.

- Goldberg (1937) All 3_n, 4_n or 5_n of symmetry (T, T_d), (O, O_h) or (I, I_h) are given by Goldberg-Coxeter construction $GC_{k,l}$.
- Fowler and al. (1988) All 5_n of symmetry D_5, D_6 or T are described in terms of 4 parameters.
- Graver (1999) All 5_n can be encoded by 20 integer parameters.
- Thurston (1998) The 5_n are parametrized by 10 complex parameters.
- Sah (1994) Thurston's result implies that the Nrs of 3_n, 4_n, $5_n \sim n$, n^3, n^9.

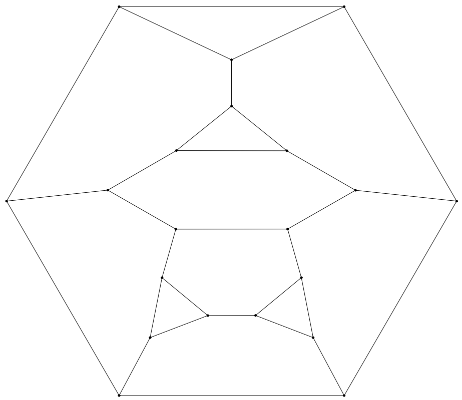
The structure of graphs 3_n



4 triangles in $Z[\omega]$



The corresponding triangulation



The graph $3_{20}(D_{2d})$

z- and railroad-structure of graphs 3_n

All zigzags and railroads are simple.

- The z-vector is of the form

$$(4s_1)^{m_1}, (4s_2)^{m_2}, (4s_3)^{m_3} \quad \text{with} \quad s_i m_i = \frac{n}{4};$$

  the number of railroads is $m_1 + m_2 + m_3 - 3$.
- G has ≥ 3 zigzags with equality if and only if it is tight.
- If G is tight, then $z(G) = n^3$ (so, each zigzag is a Hamiltonian circuit).
- All 3_n are tight if and only if $\frac{n}{4}$ is prime.
- There exists a tight 3_n if and only if $\frac{n}{4}$ is odd.

General theory

Extensions:

- 3-valent or 4-valent graphs.
- Classes of graphs with fixed p_i, $i \neq 6$.
- Classes with a fixed symmetry.
- Maps on surfaces.

Dictionnary

	3-valent graph G_0	4-valent graph G_0
ring	Eisenstein integers $\mathbb{Z}[\omega]$	Gaussian integers $\mathbb{Z}[i]$
Euler formula	$\sum_i (6-i)p_i = 12$	$\sum_i (4-i)p_i = 8$
zero-curvature	hexagons	squares
ZC-circuits	zigzags	central circuits
Operation	leapfrog graph	medial graph

Number of parameters

Octahedrites:

Group	$\#param.$
C_1	6
C_2	4
D_2	3
D_3	2
D_4	2
O	1

Graphs 3_n:

Groups	$\#param.$
D_2	2
T	1

Graphs 4_n:

Group	$\#param.$
C_1	4
C_2	3
D_2	2
D_3	2
O	1

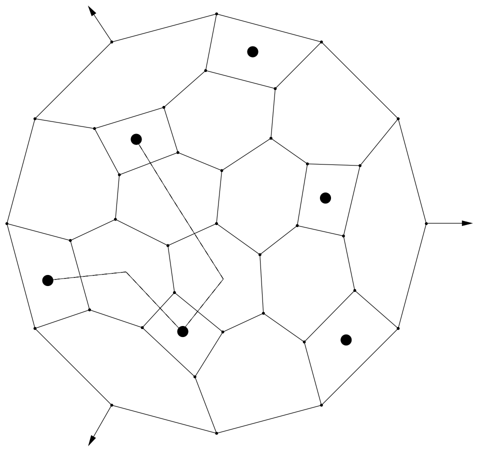
Graph 5_n:

Group	$\#param.$
C_1	10
C_2	6
C_3	4
D_2	4
D_3	3
D_5	2
D_6	2
T	2
I	1

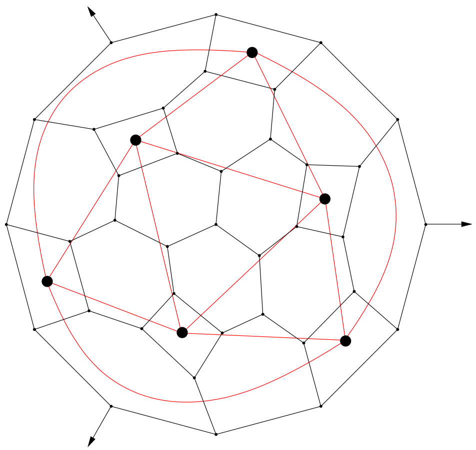
If there is just one parameter, then this is Goldberg-Coxeter construction (of Octahedron, Tetrahedron, Cube, Dodecahedron for octahedrite, 3_n, 4_n, 5_n, respectively).

Conjecture on $4_n(D_{3h}, D_{3d}\ or\ D_3)$

- $4_n(D_3 \subset D_{3h}, D_{3d}, D_6, D_{6h}, O, O_h)$ are described by two complex parameters. They exists if and only if $n \equiv 0, 2 \pmod 6$ and $n \geq 8$.



$4_n(D_3)$ with one zigzag



The defining triangles

- $4_n(D_{3d} \subset O_h, D_{6h})$ exists if and only if $n \equiv 0, 8 \pmod{12}$, $n \geq 8$.
- If n increases, then part of $4_n(D_3)$ amongst $4_n(D_{3h}, D_{3d}, D_3)$ goes to 100%.

More conjectures

- All 4_n with only simple zigzags are:
  - $GC_{k,0}(Cube)$, $GC_{k,k}(Cube)$ and
  - the family of $4_n(D_3 \subset \dots)$ with parameters $(m,0)$ and $(i, m-2i)$ with $n = 4m(2m-3i)$ and $z = (6m-6i)^{3m-3i}, (6m)^{m-2i}, (12m-18i)^i$.

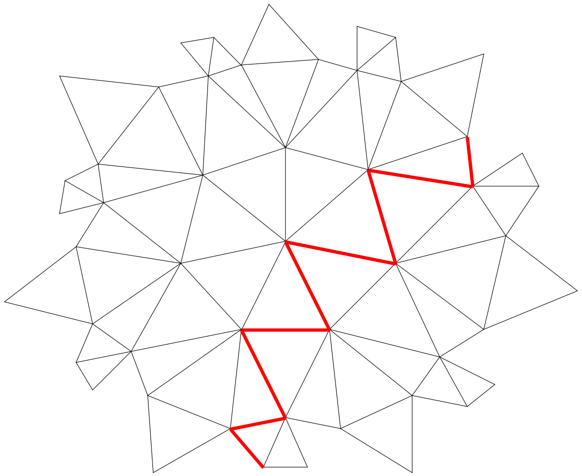
  They have symmetry D_{3d} or O_h or D_{6h}
- Any $4_n(D_3 \subset \dots)$ with one zigzag is a $4_n(D_3)$.
- For tight graphs $4_n(D_3 \subset \dots)$ the z-vector is of the form a^k with $k \in \{1,2,3,6\}$ or a^k, b^l with $k,l \in \{1,3\}$.
- Tight $4_n(D_{3d})$ exist if and only if $n \equiv 0 \pmod{12}$, they are z-transitive with:
  - $z = (n/2)^6_{n/36,0}$ iff $n \equiv 24 \pmod{36}$ and, otherwise,
  - $z = (3n/2)^2_{n/4,0}$ iff $n \equiv 0, 12 \pmod{36}$

VI. Zigzags

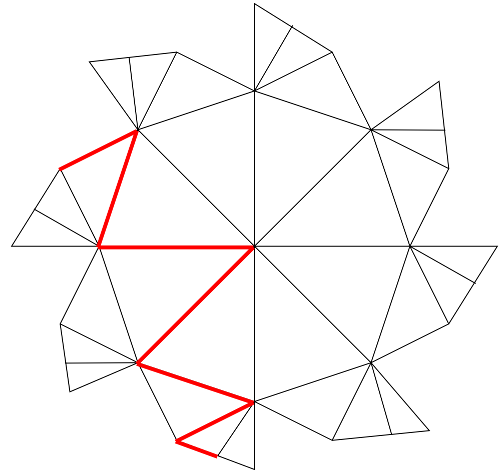
on

surfaces

Zigzags of 2-complexes (surface maps)



Klein map: $z = 8^{21}$



Dyck map: $z = 6^{16}$

- Zigzags (and central circuits), being local notions, are defined on any surface, even on a non-orientable one.
- Zigzags are also called left-right paths (Shank) or Petri paths, from Petri polygons of polytopes (Coxeter).
- A map and its dual have the same zigzag vector z.

Zigzags of regular maps

A flag-transitive map is called regular.
Zigzags of regular maps are simple
(i.e., not self-intersecting).

map	n	rot. group	z	$z(GC_{k,l})/k^2+kl+l^2$
Dod. $\{5^3\}$	20	A_5	10^6	10^6 or 6^{10} or 4^{15}
Klein* $\{7^3\}$	56	$PSL(2,7)$	8^{21}	8^{21} or 6^{28}
Dyck* $\{8^3\}$	32	$(*)$	6^{16}	6^{16} or 8^{12}
$\{11^3\}$	220	$PSL(2,11)$	10^{66}	10^{66} or 6^{110} or 12^{55}

$(*)$ is a solvable group of order 96 generated by two elements R, S subject to the relations $R^3=S^8=(RS)^2=(S^2R^{-1})^3=1$.

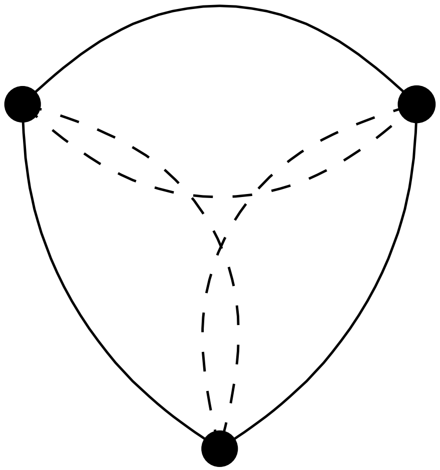
Lins trialities

$(v, f, z) \to$	notation in [2]	notation in [1]	
(v, f, z)	$\mathcal{M}$	**G**raph-**E**coded **M**ap	$\mathcal{M}$
(f, v, z)	$\mathcal{M}^*$	dual gem	$\mathcal{M}^*$
(z, f, v)	$phial(\mathcal{M})$	phial gem	$s((s(\mathcal{M}))^*)$
(f, z, v)	$(phial(\mathcal{M}))^*$	skew-dual gem	$s(\mathcal{M}^*)$
(v, z, f)	$skew(\mathcal{M})$	skew gem	$s(M)$
(z, v, f)	$(skew(\mathcal{M}))^*$	skew-phial gem	$(s(\mathcal{M}))^*$

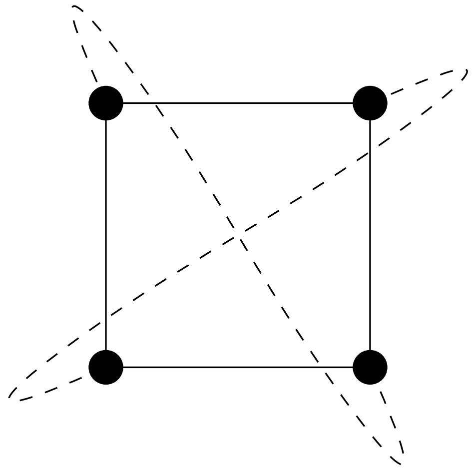
Jones, Thornton, 1987: those are only "good" dualities.
The group S_3 of trialities is isomorphic to $Sym(3)$.

1. S. Lins, *Graph-Encoded Maps*, J. Comb. Theory **B-32** (1982) 171–181
2. D. and M.Dutour, *Zigzag Structure of Complexes*, SEAMS Math. Bull. **29-2** (2005), 301–320; papers/math.CO/0405279 of LANL archive.

Example: Tetrahedron



$phial(Tetrahedron)$

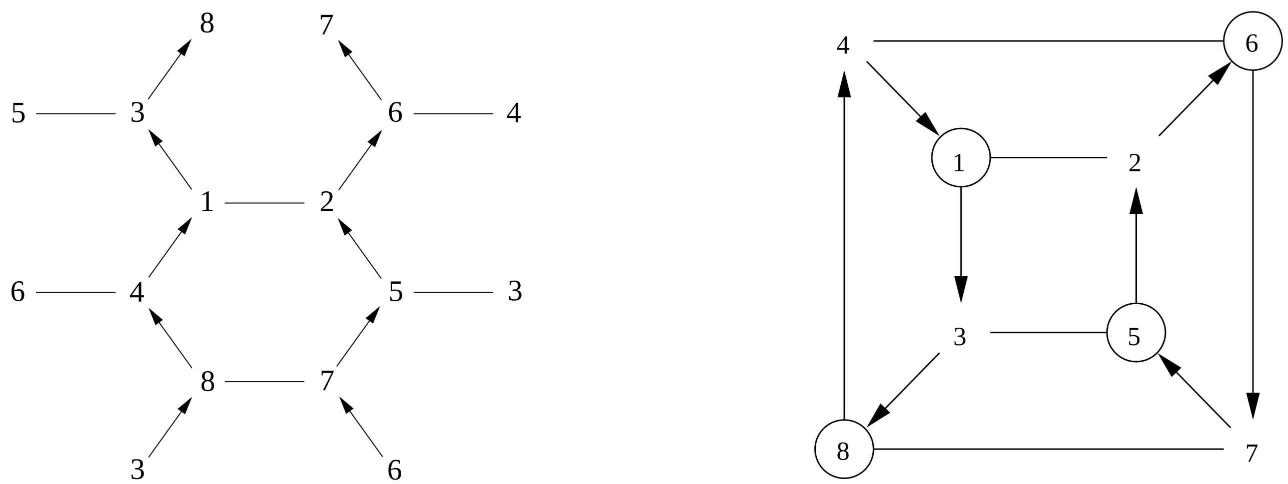


$skew(Tetrahedron)$

Two Lins maps on projective plane.

- The $phial(Tetrahedron)$ is the complex obtained by taking the octahedron and identifying opposite points.
- $skew(Cube)$ and $phial(Octahedron)$ are toric maps. $phial(Cube)$ and $skew(Octahedron)$ are maps on a non-oriented surface of genus 4, i.e., with $\chi = 2$.

Bipartite skeleton case



Two representation of $\color{red}{skew}(Cube)$: on Torus and as a Cube with cyclic orientation of vertices (marked by ◯) reversed.

Theorem

For bipartite graph embedded in oriented surface, the skew operation is, in fact, reversing orientation of one of the part of the bipartition.

Trialities of prisms and antiprisms

Let χ denotes the Euler characteristic.
Conjecture (checked up to $n = 100$):

- $skew(Prism_m)$ has $\chi = gcd(m, 4) - m$ and is oriented iff m is even;
- $phial(Prism_m)$ has $\chi = 2 + gcd(m, 4) - 2m$ and is non-oriented.

- $skew(APrism_m)$ has $\chi = 1 + gcd(m, 3) - 2m$ and is non-oriented;
- $phial(APrism_m)$ has $\chi = 3 + gcd(m, 3) - 2m$ and is oriented.

VII. Zigzags on n-dimensional complexes

Zigzags on n-dimensional polytopes

A (maximal) flag $u = (f_0, \dots, f_{n-1})$ is a sequence of i-faces f_i (of polytope P) with $f_i \subset f_{i+1}$.
Given a flag u, there exist an unique flag $\sigma_i(u)$, which differs from u only in position i, i.e., in $f'_i \neq f_i$, $f_{i-1} \in f_i, f'_i \in f_{i+1}$.

A zigzag z is a circuit of flags $(u_j)_{0 \leq j \leq l}$, such that $u_0 = u$, $u_j = \sigma_n \dots \sigma_1(u_{j-1})$; so, $u_1 = (f'_0, \dots, f'_{n-1})$.
The number of flags is called its length.

The zigzags partition the flag-set of P.
z-vector of P is a vector, listing zigzags with their lengths.

Proposition: *if the dimension of polytope is odd, then the length of any zigzag is even.*

Problem: generalize Lins triality of maps on d-complexes.

Zigzags of regular/semiregular polytopes

d	d-polytope	z-vector
3	Dodecahedron	10^6
4	24-cell	12^{48}
4	600-cell	30^{240}
d	d-simplex=α_d	$(n+1)^{n!/2}$
d	d-cross-polytope=β_d	$(2n)^{2^{n-2}(n-1)!}$
4	octicosahedric 4-polytope	45^{480}
4	snub 24-cell	20^{144}
4	0_{21}=Med(α_4)	15^{12}
5	1_{21}=Half-5-Cube	12^{240}
6	2_{21}=Schläfli polytope (in E_6)	18^{4320}
7	3_{21}=Gosset polytope (in E_7)	90^{48384}
8	4_{21} (240 roots of E_8)	$36^{29030400}$

Zigzags of reg. and semireg. polyhedra

# edges	polyhedron	z-vector	int. vector
6	Tetrahedron	4^3	$(1,1)^2$
12	Cube, Octahedron	6^4	$(0,2)^3$
30	Dodecahedron, Icosahedron	10^6	$(0,2)^5$
24	Cuboctahedron	8^6	$(0,2)^4, (0,0)$
60	Icosidodecahedron	10^{12}	$(0,2)^5, (0,0)^6$
48	Rhombicuboctahedron	12^8	$(0,2)^6, (0,0)$
120	Rhombicosidodecahedron	20^{12}	$(0,2)^{10}, (0,0)$
72	Truncated Cuboctahedron	18^8	$(0,6), (0,2)^6$
180	Truncated Icosidodecahedron	30^{12}	$(0,10), (0,2)^{10}$
18	Truncated Tetrahedron	12^3	$(3,3)^2$
36	Truncated Octahedron	12^6	$(0,4), (0,2)^4$

36	Truncated Cube	18^4	$(2,4)^3$
90	Truncated Icosahedron	18^{10}	$(0,2)^9$
90	Truncated Dodecahedron	30^6	$(2,4)^5$
60	Snub Cube	$30^4_{3,0}$	$(4,4)^3$
150	Snub Dodecahedron	$50^6_{5,0}$	$(4,4)^5$
3m	$Prism_m, m \equiv 0 \pmod 4$	$(\frac{3m}{2})^4$	$(0, \frac{m}{2})^3$
3m	$Prism_m, m \equiv 2 \pmod 4$	$(3m_{\frac{m}{2},0})^2$	$(0, 2m)$
3m	$Prism_m, m \equiv 1,3 \pmod 4$	$6m_{m,2m}$	
4m	$APrism_m, m \equiv 0 \pmod 3$	$(2m)^4$	$(0, \frac{2m}{3})^3$
4m	$APrism_m, m \equiv 1,2 \pmod 3$	$2m; 6m_{0,2m}$	
84	Klein map(oriented, genus 3 surface)	8^{21}	$(0,1)^8, 0^{12}$
48	Dyck map(oriented, genus 3 surface)	6^{16}	$(0,1)^6, 0^9$

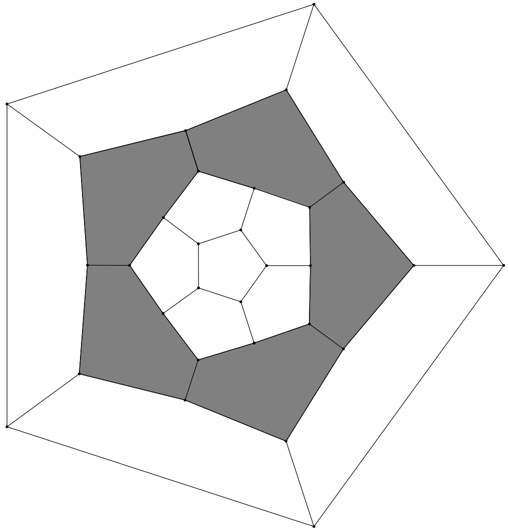
Regular-faced and Conway's polytopes

d	d-polytope	z-vector
4	$Pyr(Icosahedron)$	25^{12}
4	$BPyr(Icosahedron)$	40^{12}
4	$0_{21} + Pyr(\beta_3)$	42^6
d	$Pyr(\beta_{d-1})$, $d \geq 4$	$(\frac{2(d^2-1)}{gcd(d,2)})^x$
d	$BPyr(\alpha_{d-1})$, $d \geq 5$	$(\frac{2d^2}{gcd(d,2)})^y$
4	Grand Antiprism	$30^{20}, 50^{40}, 90^{20}$
4	$C_p \times C_q$	$(\frac{2pq}{t})^{2t}, (\frac{4pq}{t})^{2t}$
	(put $t = gcd(p, q)$)	if both, p and q, are odd
		$(\frac{2pq}{t})^{6t}$, otherwise

VIII. Special fullerenes 5_n

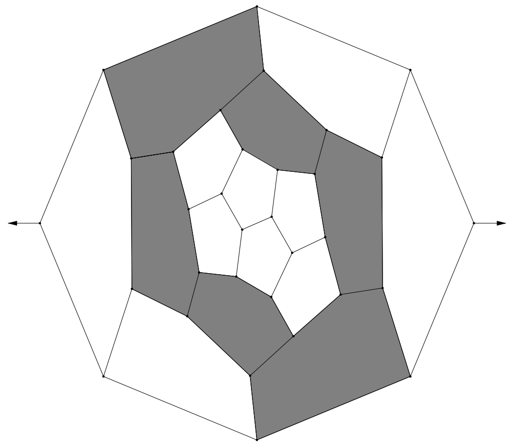
1. All (8) with every hexagon being in a ring
2. z-uniform and z-transitive fullerenes
3. z-knotted fullerenes

All 5_n with hexagons in 1 ring

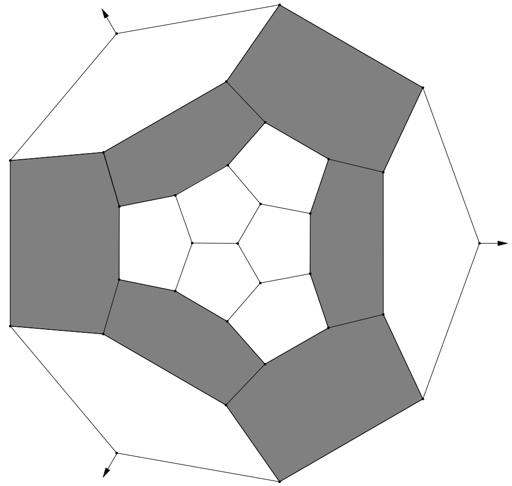


30 D_{5h}

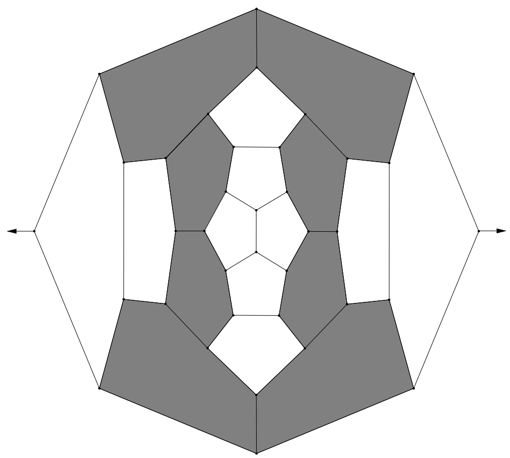
railroad



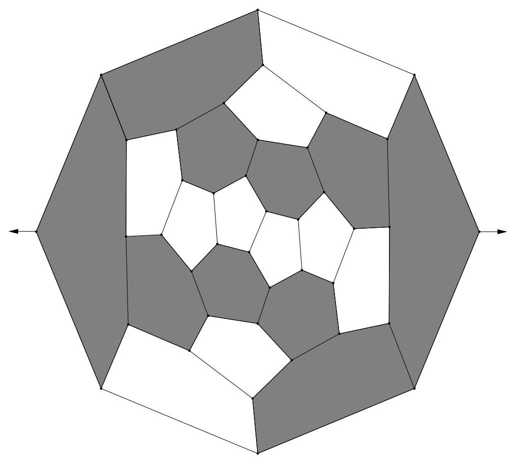
32 D_2



32 D_{3d}

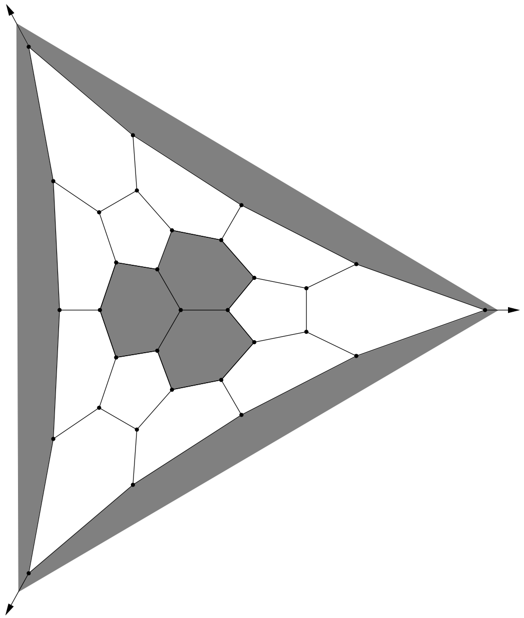


36 D_{2d}

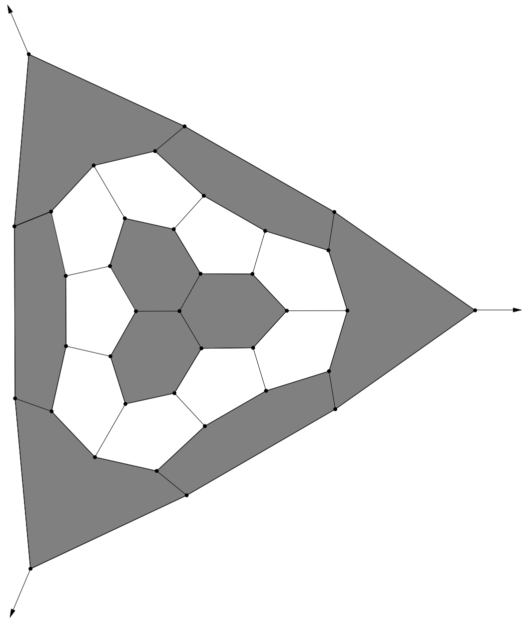


40 D_2

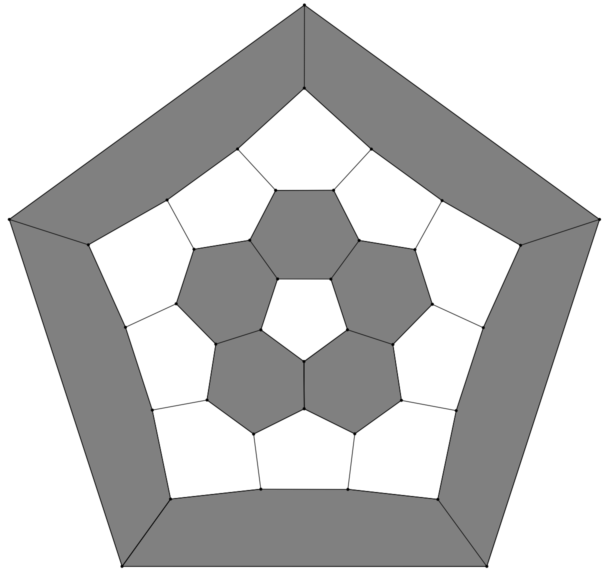
All 5_n with hexagons in (>1) rings



32 D_{3h};
6-gons in two 3-rings



38 C_{3v};
6-gons in 3- and 6-ring
5-gons in 3- and 9-ring

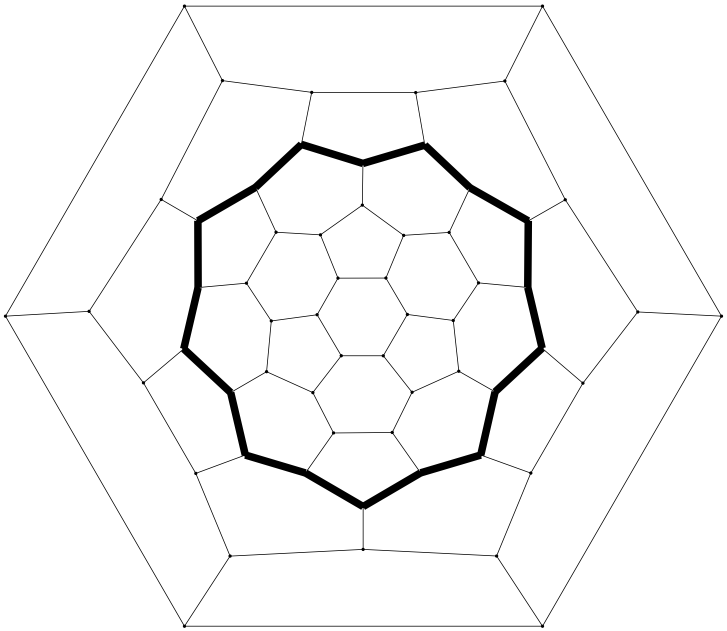


40 D_{5h};
6-gons in two 5-rings

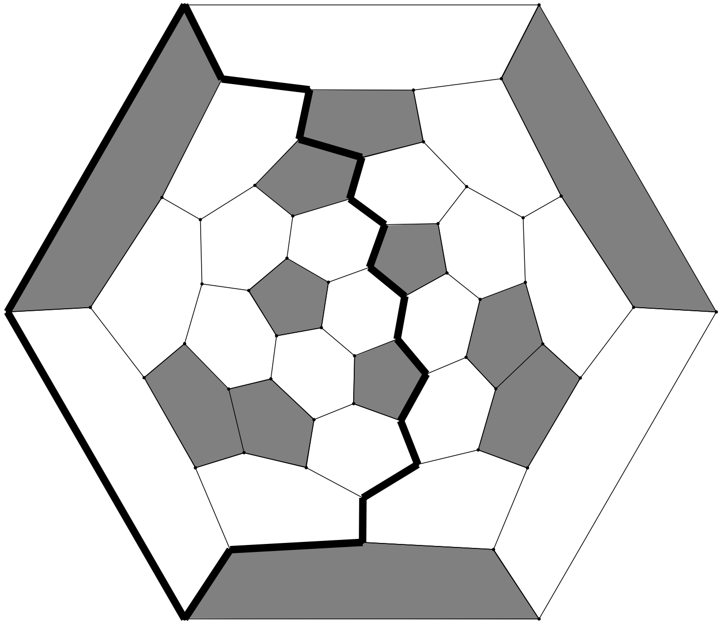
z-uniform 5_n with $n \leq 60$

n	isomer	orbit lengths	z-vector	int. vector
20	I_h:1	6	$10^6_{0,0}$	2^5
28	T_d:2	4,3	$12^7_{0,0}$	2^6
40	T_d:40	4	$30^4_{0,3}$	8^3
44	T:73	3	$44^3_{0,4}$	18^2
44	D_2:83	2	$66^2_{5,10}$	36
48	C_2:84	2	$72^2_{7,9}$	40
48	D_3:188	3,3,3	$16^9_{0,0}$	2^8
52	C_3:237	3	$52^3_{2,4}$	20^2
52	T:437	3	$52^3_{0,8}$	18^2
56	C_2:293	2	$84^2_{7,13}$	44
56	C_2:349	2	$84^2_{5,13}$	48
56	C_3:393	3	$56^3_{3,5}$	20^2
60	C_2:1193	2	$90^2_{7,13}$	50
60	D_2:1197	2	$90^2_{13,8}$	48
60	D_3:1803	6,3,1	$18^{10}_{0,0}$	2^9
60	I_h:1812	10	$18^{10}_{0,0}$	2^9

Two 5_{60} with z-vector $18^{10}_{0,0}$



$C_{60}(I_h)$



$F_{60}(D_3)$

This pair was first answer on a question in B.Grunbaum "Convex Polytopes" (Wiley, New York, 1967) about non-existance of simple polyhedra with the same p-vector but different zigzags.

z-uniform IPR 5_n with $n \leq 100$

n	isomer	orbit lengths	z-vector	int. vector
80	I_h:7	12	$20^{12}_{0,0}$	$0, 2^{10}$
84	T_d:20	6	$42^{6}_{0,1}$	8^5
84	D_{2d}:23	4,2	$42^{6}_{0,1}$	8^5
86	D_3:19	3	$86^{3}_{1,10}$	32^2
88	T:34	12	$22^{12}_{0,0}$	2^{11}
92	T:86	6	$46^{6}_{0,3}$	8^5
94	C_3:110	3	$94^{3}_{2,13}$	32^2
100	C_2:387	2	$150^{2}_{13,22}$	80
100	D_2:438	2	$150^{2}_{15,20}$	80
100	D_2:432	2	$150^{2}_{17,16}$	84
100	D_2:445	2	$150^{2}_{17,16}$	84

IPR means the absence of adjacent pentagonal faces;
IPR enhanced stability of putative fullerene molecule.

IPR z-knotted 5_n with $n \leq 100$

n	signature	isomers
86	$43, 86$*	C_2:2
90	$47, 88$	C_1:7
	$53, 82$	C_2:19
	$71, 64$	C_2:6
94	$47, 94$*	C_1:60; C_2:26, 126
	$65, 76$	C_2:121
	$69, 72$	C_2:7
96	$49, 95$	C_1:65
	$53, 91$	C_1:7, 37, 63

98	$49, 98$*	C_2:191, 194, 196
	$63, 84$	C_1:49
	$75, 72$	C_1:29
	$77, 70$	C_1:5; C_2:221
100	$51, 99$	C_1:371, 377; C_3:221
	$53, 97$	C_1:29, 113, 236
	$55, 95$	C_1:165
	$57, 93$	C_1:21
	$61, 89$	C_1:225
	$65, 85$	C_1:31, 234

The symbol * above means that fullerene forms a Kékule structure, i.e., edges of self-intersection of type I cover exactly once the vertex-set of the fullerene graph (in other words, they form a perfect matching of the graph).

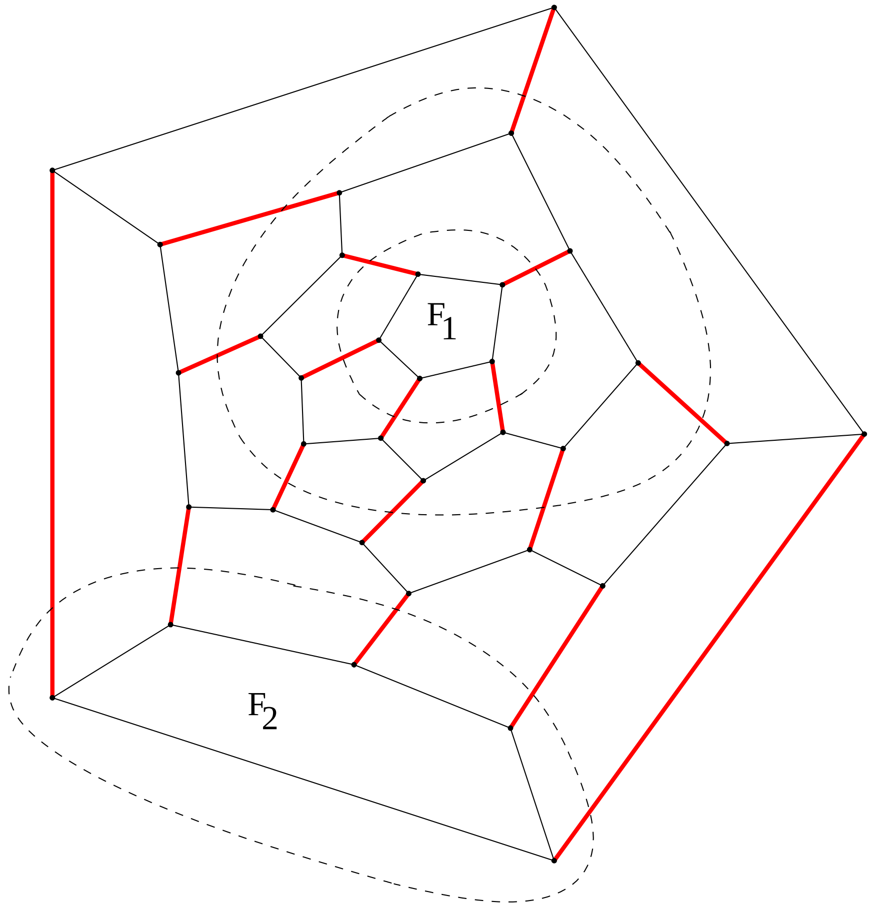
Perfect matching on 5_n graphs

Let G be a graph 5_n with one zigzag with self-intersection numbers (α_1, α_2).

(i) $\alpha_1 \geq \frac{n}{2}$. If $\alpha_1 = \frac{n}{2}$ then the edges of self-intersection of type I form a perfect matching PM

(ii) every face incident to 0 or 2 edges of PM

(iii) two faces, F_1 and F_2 are free of PM, PM is organized around them in concentric circles.



Statistics of z-knotted 5_n with $n \leq 74$

n	# of 5_n	# of z-knotted
34	6	1
36	15	0
38	17	4
40	40	1
42	45	6
44	89	9
46	116	15
48	199	23
50	271	30
52	437	42

54	580	93
56	924	87
58	1205	186
60	1812	206
62	2385	341
64	3465	437
66	4478	567
68	6332	894
70	8149	1048
72	11190	1613
74	14246	1970

It will be interesting to estimate the relative order of magnitude of z-knotted fullerenes among all 5_n.
From Schaeffer and Zinn-Justin, 2004: the proportion, among all 3-valent n-vertex plane graphs, of those having $\leq m$ zigzags (for any fixed m) goes to 0 with $n \to \infty$.