Chirality, genera and simplicity of orientably-regular maps

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Partly joint work with Jozef Siráň & Tom Tucker

Regular maps

A map M is a 2-cell embedding of a connected graph or multigraph on a surface.

The faces of the map are the simply-connected components of the complement of the graph or multigraph in the surface.

Every automorphism of a map M is uniquely determined by its effect on a given flag (incident vertex-edge-face triple), and it follows that $|\text{Aut}\, M| \leq 4|E|$ where E is the edge set.

A map M is regular if $\text{Aut}\, M$ is transitive on flags, and is orientably-regular (or rotary) if the group of all its orientation-preserving automorphisms is transitive on the ordered edges (arcs) of M.

Some history:

- Platonic solids — the tetrahedron, cube, octahedron, dodecahedron & icosahedron are regular maps on the sphere

The Platonic solids. Scotland, ca. 2000 B.C. (Photo, courtesy Graham Challifour, from "Time Stands Still", K. Critchlow.)

- Theory developed by Brahana (1920s), Coxeter, Wilson, Jones & Singerman, and many others

- Connections with algebraic geometry & Galois theory — Belyi (1980), Grothendieck (1997), Jones et al (2006)

Classification of regular maps:

Regular and orientably-regular maps have been the subject of attention from three main perspectives:

- Classification by *underlying graph*

— e.g. embeddings of K_n or $K_{n,n}$ as a regular map

- Classification by *surface*

— orientable genus 2 to 100 inclusive (chiral and reflexible)
— non-orientable genus 2 to 200 inclusive

- Classification by *group*

— rotation group or full automorphism group.

Regular maps (cont.)

If M is (orientably) regular then every face has the same number of edges (say m) and every vertex has the same valency (say k), and M has type $\{k, m\}$.

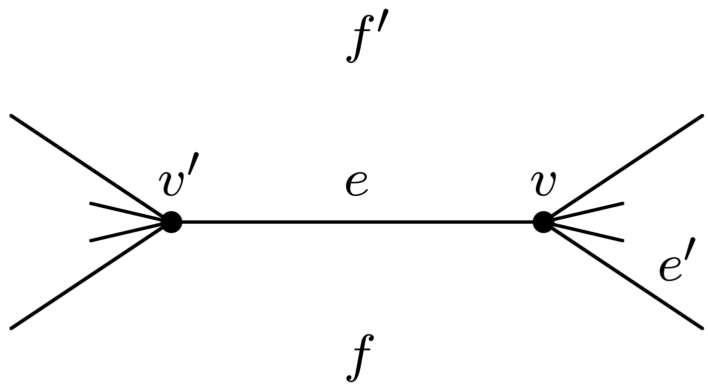
Examples: Graphs of Platonic solids embedded on the sphere, of types $\{3, 3\}$, $\{3, 4\}$, $\{4, 3\}$, $\{3, 5\}$, $\{5, 3\}$, and embeddings of honeycomb graphs on the torus, of types $\{3, 6\}$ and $\{6, 3\}$.

Reflexible vs chiral

If the orientable map M admits automorphisms that *reverse* orientation, then M is called reflexible, and otherwise chiral.

If M is a regular map of type $\{k, m\}$, then for any given flag (v, e, f) there exist automorphisms a, b and c such that

- a fixes (e, f) but takes v to the other incident vertex v'
- b fixes (v, f) but takes e to the other incident edge e'
- c fixes (v, e) but takes f to the other incident face f'



These generate the automorphism group Aut M, and satisfy the $(2, k, m)$ triangle group relations

$$a^2 = b^2 = c^2 = (ab)^m = (bc)^k = (ac)^2 = 1.$$

Conversely ...

Given any group epimorphism $\theta\colon (2,k,m) \to G$ with torsion-free kernel K, a regular map M may be constructed with automorphism group G:

Take as vertices of M the right cosets of $V = K\langle b,c\rangle$, edges the right cosets of $E = K\langle a,c\rangle$, and faces the right cosets of $F = K\langle a,b\rangle$, and let incidence be non-empty intersection. Then $(2,k,m)$ acts on M by right multiplication, inducing the automorphism group $(2,k,m)/K \cong G$.

Thus regular maps of type $\{k,m\}$ correspond to non-degenerate homomorphic images of the $(2,k,m)$ triangle group.

Genus calculations

- If M is an orientably-regular map M of type $\{k, m\}$ that has $|V|$ vertices, $|E|$ edges and $|F|$ faces, then

$$k|V| = 2|E| = m|F| = |\mathrm{Aut}^o M|$$

so its genus g and Euler characteristic χ are given by

$$2 - 2g = \chi = |V| - |E| + |F| = |\mathrm{Aut}^o M|\,(1/k - 1/2 + 1/m).$$

- Similarly, if the map M of type $\{k, m\}$ is regular then

$$2k|V| = 4|E| = 2m|F| = |\mathrm{Aut}\, M|$$

so its Euler characteristic χ is given by

$$\chi = |V| - |E| + |F| = |\mathrm{Aut}\, M|\,(1/2k - 1/4 + 1/2m).$$

Questions about the genus spectrum

• Is there an orientably-regular map of every genus?
Answer: Yes, for every $g > 1$ there exists a regular map of type $\{4g, 4g\}$ with dihedral automorphism group of order $8g$ — but with only one vertex and one face, and multiple edges

• What are the genera of orientably-regular maps that have simple underlying graphs (with no multiple edges)?

• Are there non-orientable regular maps of all but finitely many genera?
Answer: No, since Breda, Nedela and Siráň (2005) proved that there's only one such map of genus $p+2$ where p is a prime congruent to 1 mod 12 (viz. one map of genus 15)

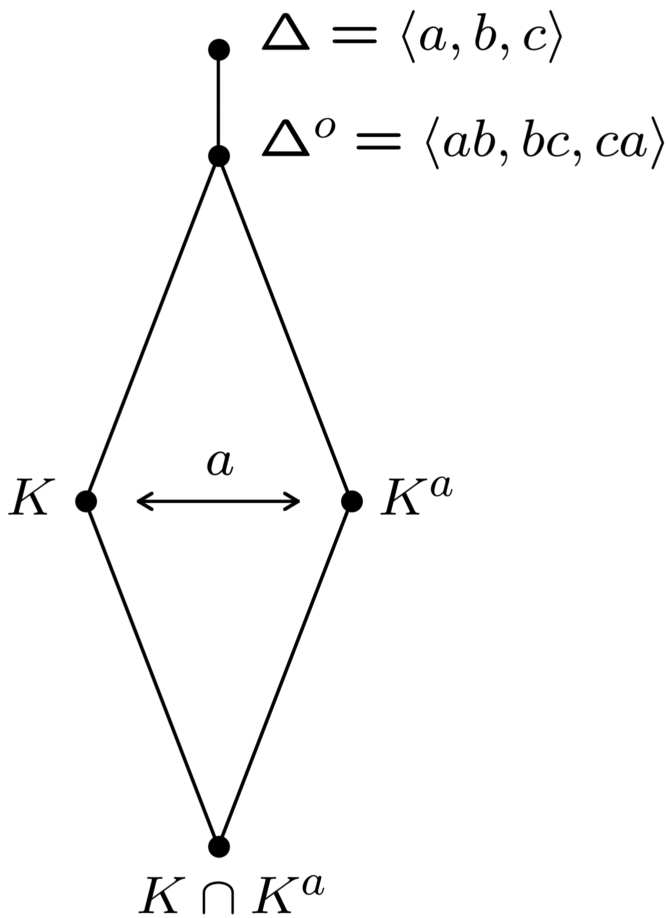
• What are the genera of orientably-regular but <u>chiral</u> maps?

Chirality

Recall that if an orientably-regular map M has no orientation-reversing automorphisms, then M is chiral (or irreflexible).

In that case $\text{Aut}\,M = \text{Aut}^o M$ is not a quotient of the full $(2, k, m)$ triangle group $\Delta = \langle\, a, b, c \mid a^2 = b^2 = c^2 = (ab)^m = (bc)^k = (ca)^2 = 1\,\rangle$, but is a quotient Δ^o/K of its index 2 subgroup $\Delta^o = \langle\, x, y, z \mid x^2 = y^k = z^m = xyz = 1\,\rangle$ where $x = ac$, $y = cb$ and $z = ba$. This group Δ^o is the ordinary $(2, k, m)$ triangle group.

Chiral maps occur in pairs, with each map being a 'mirror image' of the other (and with the corresponding normal subgroups of $\Delta^o(2, k, m)$ interchanged by the generator a).



$K = K^a$ iff map M is reflexible

(when $\text{Aut}^o M \cong \Delta^o / K$)

Simplicity

Let M be an orientably-regular map of type $\{k, m\}$, so that $\text{Aut}^o M$ is a quotient of the ordinary $(2, k, m)$ triangle group $\Delta^o(2, k, m) = \langle\, x, y, z \mid x^2 = y^k = z^m = xyz = 1 \,\rangle$.

Then the underlying graph of M or its dual is simple if and only if the subgroup generated by the image of y or z in $\text{Aut}^o M$ is core-free — that is, contains no non-trivial normal subgroup of $\text{Aut}^o M$.

Low index normal subgroups

Small homomorphic images of a finitely-presented group G can be found as the groups of permutations induced by G on cosets of subgroups of small index. This gives G/K where K is the core of H, but produces only images that have small degree faithful permutation representations.

Alternatively, the low index subgroups method can be adapted to produce only normal subgroups (of small index in G).

A new method has been developed recently by Derek Holt and his student David Firth, which systematically enumerates the possibilities for the composition series of the factor group G/K, for any normal subgroup K of small index in G.

Determination of regular maps of small genus

Genus 0: regular polyhedra (incl. “dihedra” and their duals)

Genus 1 and 2: Brahana [1927] and Coxeter [1957]

Genus 3: Sherk [1959]

Genus 4, 5 and 6: Garbe [1969]

MC & Peter Dobcsányi [2001]:
- All orientably-regular maps of genus 2 to 15
- All non-orientable regular maps of genus 2 to 30.

MC [2006]:
- All orientably-regular maps of genus 2 to 101
- All non-orientable regular maps of genus 2 to 202.

Digression: Symmetric cubic graphs

The same computational techniques applied to different families of finitely-presented groups (such as the modular group $\langle\, x, y \,|\, x^2 = y^3 = 1 \,\rangle$) can be used to find all arc-transitive 3-valent graphs of small order, extending the Foster census:

MC & Peter Dobcsányi [2001]: up to 768 vertices

MC [2006]: up to 2048 vertices

Bonus find: largest known 3-valent graph of diameter 10
[This has 1250 vertices, and is a cover of the Petersen graph with covering group $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$]

Summary of maps data for small genus

Orientably-regular maps (up to isomorphism & duality)
Genus 2: 6 reflexible, 0 chiral
Genus 3: 12 reflexible, 0 chiral
Genus 4: 12 reflexible, 0 chiral
Genus 5: 16 reflexible, 0 chiral
Genus 6: 13 reflexible, 0 chiral
Genus 7: 12 reflexible, 2 chiral pairs
Genus 2 to 101: 3378 reflexible, 594 chiral pairs

Non-orientable regular maps (up to isomorphism & duality)
Genus 2 or 3: 0
Genus 4: 2
Genus 14: 3
Genus 2 to 202: 862

Observations

- There is no orientably-regular but chiral map of genus 2, 3, 4, 5, 6, 9, 13, 23, 24, 30, 36, 47, 48, 54, 60, 66, 84, 95, 108, 116, 120, 139, 150, 167, 168, 174, 180, 186 or 198

- There is no regular orientable map of genus 20, 32, 38, 44, 62, 68, 74, 80 or 98 with simple underlying graph

- A lot of these exceptional genera are of the form $p + 1$ where p is prime.

Theorems

- If M is an irreflexible (chiral) orientably-regular map of genus $p+1$ where p is prime, then

  either $p \equiv 1 \bmod 3$ and M has type $\{6,6\}$,
  or $p \equiv 1 \bmod 5$ and M has type $\{5,10\}$,
  or $p \equiv 1 \bmod 8$ and M has type $\{8,8\}$.

  In particular, there are no such maps of genus $p+1$ whenever p is a prime such that $p-1$ is not divisible by 3, 5 or 8.

  [MC & Jozef Siráň (October 2006)]

- There is no regular map M with simple underlying graph on an orientable surface of genus $p+1$ where p is a prime congruent to 1 mod 6, for $p > 13$.

  [MC & Tom Tucker (December 2006)]

In fact, even more ...

• A complete classification of all regular and orientably-regular maps M for which $|\text{Aut}\,M|$ is coprime to the Euler characteristic χ (if χ is odd) or to $\chi/2$ (if χ is even)

[MC, Jozef Siráň & Tom Tucker (January 2007)]

This has all three main results to date as corollaries:

• No chiral orientably-regular maps of genus $p+1$ for primes p not congruent to 1 mod 3, 5 or 8

• No regular orientable maps with simple underlying graph and genus $p+1$ for primes $p > 13$ congruent to 1 mod 6,

• No non-orientable regular maps of genus $p+2$ for primes $p > 13$ congruent to 1 mod 12.

Coprime classification: $|G|$ coprime to χ or $\chi/2$

• If M has type $\{k,m\}$ and G is the subgroup of Aut M generated by vertex- and face-stabilizers, then

$$-\chi = |G|(1/2 - 1/k - 1/m) = |G|(km - 2k - 2m)/2km$$

where $1/84 \leq (km - 2k - 2m)/2km < 1/2$ for $-\chi > 0$

• The coprime assumption gives $km - 2k - 2m = (-\chi)td$ or $(-\chi/2)td$ for some t, where $d = \gcd(k,m)$, and hence

$$|G| = 2\,\mathrm{lcm}(k,m)/t \quad \text{or} \quad 4\,\mathrm{lcm}(k,m)/t$$

where $t = 1, 2$ or 4

• Every cyclic subgroup of G odd order is conjugate to a subgroup of the vertex-stabilizer (of order k) or the face-stabilizer (of order m), and hence G is 'almost Sylow-cyclic'

Coprime classification (cont.)

• 'Almost Sylow-cyclic' groups have been classified: by Zassenhaus (1936) for solvable groups, and by Suzuki (1955) and Wong (1966) for non-solvable groups

• Let X and Y be generators of the stabilizers of a face and an incident vertex of M, so that X and Y generate G and satisfy $X^m = Y^k = (XY)^2 = 1$

• The map M (or its topological dual) has simple underlying graph if and only if $\langle Y \rangle$ (resp. $\langle X \rangle$) is 'core-free' in G

• When $\langle X \rangle \cap \langle Y \rangle$ is trivial, we have $|G| \geq |\langle X \rangle \langle Y \rangle| = km$, and since also $|G| = 2\,\mathrm{lcm}(k,m)/t$ or $4\,\mathrm{lcm}(k,m)/t$ where $t \in \{1,2,4\}$, this gives us only a small number of cases to consider, according to the values of $d = \mathrm{gcd}(k,m)$ and t

Cases where $\langle X \rangle \cap \langle Y \rangle$ is trivial

We use fairly standard combinatorial group theory to deduce the following possibilities in these cases:

1) $k = 2$, and G is dihedral of order $2m$ where m is odd,

2) $|G| = km$, with $\gcd(k, m) = 2$, and $\langle X^2, Y^2 \rangle$ is a cyclic normal subgroup, with $(X^2)^Y = X^{-2}$ and $(Y^2)^X = Y^{-2}$, and with quotient $C_2 \times C_2$

3) $k = 4$ or 8, m is divisible by 3, and $\gcd(k, m) = 1$, and G is an extension of $C_{m/3}$ by $\mathrm{PGL}(2, 3)$ or $\mathrm{GL}(2, 3)$

4) $k = m = 3$, and $G \cong A_4$

5) $k, m = \{3, 5\}$, and $G \cong A_5$.

Cases where $\langle X \rangle \cap \langle Y \rangle$ is non-trivial

- Here the subgroup $N = \langle X \rangle \cap \langle Y \rangle$ is centralized by X and Y and hence by all of $\langle X, Y \rangle = G$

- We use the transfer homomorphism $h \mapsto h^{|G:N|}$ from G to N (and Schur's theorem, which says that the order of every element of the derived group G' divides the index $|G : Z(G)|$) to determine all possibilities in each of the five cases for G/N

- This completes the classification.

Note: We require $\langle X \rangle \cap \langle Y \rangle$ to be trivial if the map or its dual has simple underlying graph, and those cases were dealt with previously.

Also: In all of these cases, the map M is reflexible!

Type	Genus	$\lvert G \rvert$	Comments
$\{8n, 8n\}$	$2n$	$8n$	G cyclic
$\{4n+1, 8n+2\}$	$2n$	$8n+2$	G cyclic, $n \not\equiv 2 \bmod 3$
$\{2n, vn\}$	$v(n-1)/2$	$2vn$	$G \cong C_n \times D_v$, $n \equiv 1 \bmod 4$
$\{2rn, 2sn\}$	$rsn-r-s+1$	$4rsn$	G has quotient $C_2 \times C_2$
$\{4n, 3vn\}$	$6vn-3v-3$	$24vn$	G has quotient S_4
$\{8n, 3vn\}$	$12vn-3v-7$	$48vn$	G has genus 2 quotient
$\{3n, 3n\}$	$3n-3$	$12n$	$G \cong C_n \rtimes A_4$, n odd
$\{3n, 5n\}$	$15n-15$	$60n$	$G \cong C_n \times A_5$, $\gcd(n, 60) = 1$

Approach when $-\chi = p$ or $2p$ for p prime

For such a map M, let G be the subgroup of $\mathrm{Aut}\, M$ generated by vertex- and face-stabilizers. Then:

- For small p, we know all examples

- For large p when p divides $|G|$, we can use Sylow theory to reduce the case of a quotient G/P acting on a map of small genus

- For large p when p does not divides $|G|$, we can use the 'coprime classification'.

Summary: new theorems

- If M is an irreflexible (chiral) orientably-regular map of genus $p+1$ where p is prime, then
  either $p \equiv 1$ mod 3 and M has type $\{6, 6\}$,
  or $p \equiv 1$ mod 5 and M has type $\{5, 10\}$,
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In particular, there are no such maps of genus $p+1$ whenever p is a prime such that $p-1$ is not divisible by 3, 5 or 8.

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How prevalent is chirality?

Orientably-regular maps of small genus:

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What about for larger genera?

Another viewpoint ....

Theorem [Liebeck and Shalev (2004)]
Let Δ be a Fuchsian group, let H be a randomly chosen subgroup of index n in Δ, and let K be the core of H in Δ — that is, the kernel of the natural permutation representation of Δ on (right) cosets of H. Then the probability that Δ/K is the alternating group A_n or the symmetric group S_n tends to 1 as $n \to \infty$.

Consequence: Almost all orientably-regular maps of a given hyperbolic type $\{k, m\}$ have an alternating or symmetric group as their orientation-preserving automorphism group.

Further consequence for chirality?

Also for n large with respect to k and m, one cannot expect a given epimorphism from the ordinary $(2, k, m)$ triangle group $\Delta^o(2, k, m)$ to A_n or S_n to be 'reflexible' — that is, one cannot expect an extension to to a homomorphism from the full triangle group $\Delta(2, k, m)$ — so the corresponding map will be chiral in almost all cases. Thus:

Hence in some respects, for large maps of a given type, chirality occurs quite frequently!

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Abstract