EMERGENCE

SOME FEATURES:

Convergence to a consensus.

Local Averaging

(Decentralized) Mechanism

Learning (inductive)

Agents

Quantitative

THREE EXAMPLES:

Primitive Language

Flocking

Price Systems

Example: Origin of a primitive language. This was a joint work with F. Cucker and D-X. Zhou and can be found on my TTI-C website.

A language is a function $f : M \longrightarrow S$ where M is a finite set interpreted as a space of meanings and S is a convex set in Euclidean space interpreted as a space of sounds (and especially vowel sounds).

It is supposed that there are k linguistic agents, which we label as i going from 1 to k.

The states of this system are denoted by

$(f_1, f_2, \ldots, f_k)$ in $L \times L \times L \ldots \times L = L^k$

Where f_i is the language of agent i and L is the space of languages.

To describe a dynamics $T : L^k \circlearrowleft$ we use a k by k matrix of non-negative entries interpreted as the measure of linguistic encounters between agents and not necessarily symmetric. The matrix should have an irreducibility property. One could equivalently use a graph.

T is defined by local averaging relative to the linguistic encounter matrix. Moreover the dynamics has a learning mechanism. Learning theory uses a set of examples which in this case amounts to learning a language. The examples are meaning-sound pairs (m,s) in MxS emitted by each agent and transmitted by the encounter matrix to the listening agents.

Theorem. The dynamics starting with an arbitrary initial set of k languages converges to a common language.

Remark. This result confirms numerical simulations of several linguists as Bill Wang.

MECHANISMS

Learning theory uses examples or data:

For $i = 1, \quad \ldots, \quad m, \quad (x_i, y_i)$ in $X x Y$, $\quad Y = R$ (the reals).

It is inductive learning.

Find an appropriate function $\quad f : X \to Y \quad$ which mimics $f(x_i) = y_i$.

Here is a version of curve fitting or interpolation.

Learning theory combines approximation theory and probability.

Goal: combine learning theory with local averaging to obtain a quantitative model of emergence, with universality.

Eventually one may obtain some universal laws of emergence.

Learning theory can supply a mechanism, as it relies on examples.

Moreover there exists a rather developed subject of learning theory in the last decades.

With Felipe Cucker,

Flocking. Birds $i = 1, \ldots, k$

Adjacency matrix $A = (a_{ij})$ where

$$a_{ij} = \frac{1}{(1 + \|x_i - x_j\|^2)^\beta}, \quad x_i \in E^3, \quad \beta \geq 0$$

"Laplacian" $L = D - A$, $D_{ii} = \sum_{j=1}^{k} a_{ij}$

~~Main~~ Equations of flocking on $(E^3)^k \times (E^3)^k$

$$x' = v$$
$$v' = -Lv$$

Derived from

$$v_i(t+h) - v_i(t) = h \sum_{j=1}^{k} a_{ij}(v_j - v_i)$$

Question, Is there flocking?
Do solutions $v_i(t)$ converge to a common $v^* \in E^3$?

__Theorem.__ For the equations of flocking, one has existence and uniqueness for all time.

If $\beta < \frac{1}{2}$, $v_i(t) \to v^* \in E^3$ as $t \to \infty$, v^* independent of i, and $x_i - x_j \longrightarrow \hat{x}_{ij}$ as $t \to \infty$, so that the relative positions stay bounded.

If $\beta \geq \frac{1}{2}$ dispersal is possible. But flocking as above will occur provided certain explicit initial conditions are satisfied.

Suppose that G is a graph, defined by vertices and edges.

Let A be its adjacency matrix so that the entries satisfy

a(i, j) is 1 if (i, j) is an edge and zero otherwise.

Let D be the diagonal matrix defined by $d(i, i) = \sum_j a(i, j)$.

Then the Laplacian of G is

$L = L(G) = D - A$

The eigenvalues of L may be expressed by

$$0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \quad \ldots$$

The Fiedler number $F = F(G) = \lambda_2$

Fact: $F \neq 0$ if and only if G is connected.

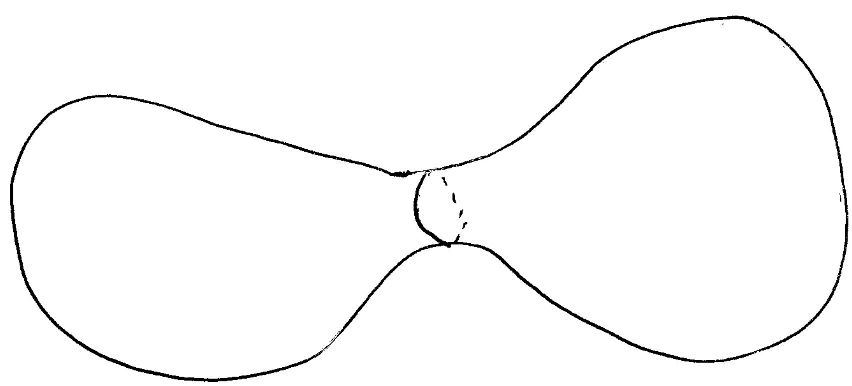
One extends these definitions to a general weighted graph

a(i, j) (or such a matrix A).

For the flocking case F depends on the positions x and thus also on time. In fact if $F(x(t)) \geqslant$ constant > 0 one obtains flocking; otherwise the birds disperse. This way the Fiedler number F is a crucial invariant for the study of emergence.

In the above we may see an extension of the flocking model to cases where some $a(i, j)$ could be zero and one relaxes the condition of a complete graph.

The Fiedler number is inspired by the Cheeger Theorem in differential geometry, and the second eigenvalue of the Laplace-Beltrami operator.



For the proof of the flocking theorem it is useful to consider x in X where X is the orthogonal space to the diagonal in the product space $(\mathbb{E}^3)^k$, and similarly v in V for the velocities.

Then the ordinary differential equations of flocking make sense in X x V and in this space flocking means $v(t) \to 0$ in V and $x(t) \to x^*$ in X as time goes to infinity.

Our conclusion in the flocking theorem above can be stated in these terms.

The first proposition in the proof is this.

Let the norm squared of $v(t)$ be denoted by N. Then

$$((d/dt)N)(t) \leq ((d/dt)N)(0) \exp -t\, F^*(t)$$

where F^* is the minimum of the Fiedler numbers $F(s)$ over

$0 \leq s \leq t$.

The proof goes by $d/dt\, N = -\langle Lv, v \rangle \leq -\ F N$.

Divide by N and integrate etc.

FROM FLOCKING TO THE GENERAL EMERGENCE PROBLEM

Let X be some general Euclidean space.

Let V be the product space modulo the diagonal, just as before.

$L : X \longrightarrow M(k \times k)$ (the k by k matrices) satisfy:

$$F(x) = \min_{v \in V} \frac{\langle L(x)\, v, v \rangle}{\langle v, v \rangle} \geq \frac{1}{(1 + \|x\|^2)^{\beta}}$$

The differential equations are: $dx/dt = f(x, v)$

$$dv/dt = -L(x)\, v$$

Here we suppose $\|f(x, v)\| \leq K \|v\|^d$

THEOREM. If $\beta < \frac{1}{2}$ there is emergence.

ie. $v(t) \longrightarrow 0$, etc, etc.

THE PRICE ADJUSTMENT PROBLEM

Playing the role of the previous velocity we use a corresponding "Beliefs of a price system", or function p from commodity space to a space of prices. Every economic agent will have at a given time such a belief.

A learning mechanism is based on signals of the form (c_i, p_i) which could be interpreted as offers, bids, and even exchanges.

This is very preliminary and not in the framework of general equilibrium theory. A space of economic characteristics as wealth, utility and economic relations will be our new X.