Chaos and stability in learning random two-person games

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Introduction

- game theory often assumes perfect rationality
- all agents know all payoff structures
- they assume their opponents play fully rationally
- outcomes: Nash equilibria
- no player has an incentive to deviate unilaterally

Learning dynamics

- agents with bounded rationality
- need to learn which strategies to use
- dynamical behaviour
- might fail to converge to NE

Questions to be addressed here

- circumstances under which learning process converges
- in case of no convergence: chaotic behaviour ?
- influence of finite-memory of agents
- statistics of strategy use: all or only a few ?

➡ modified replicator equations

➡ analysis with tools from statistical mechanics

The Model

- 2 players, X and Y
- each have N strategies at their disposal
- payoff matrices A and B
- say X plays strategy i and Y plays strategy j
- payoff for X will be a_{ij}
- payoff for Y will be b_{ji}

Example

- rock-papers-scissors game
- N=3 strategies $i, j \in \{R, P, S\}$
- payoff matrices are 3x3

$$A = -B^T = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Learning dynamics:
Sato-Crutchfield replicator equations

- player keeps a 'score' for each of his strategies

$$u_i(t+1) = u_i(t) + a_{i,j(t)} - \alpha u_i(t)$$

memory loss rate

- plays strategy i with probability

$$p_i(t) = \frac{e^{\beta u_i(t)}}{Z}$$

Learning dynamics:

Leads to modified replicator equations

$$\frac{d}{dt}p_i^X = p_i^X \left[\sum_j a_{ij} p_j^Y - \alpha \ln p_i^X - f^X + \alpha S^X \right]$$

$$\frac{d}{dt}p_j^Y = p_j^Y \left[\sum_i b_{ji} p_i^X - \alpha \ln p_j^Y - f^Y + \alpha S^Y \right]$$

[Sato+Crutchfield, PRE 2003]

Rock-paper-scissors game

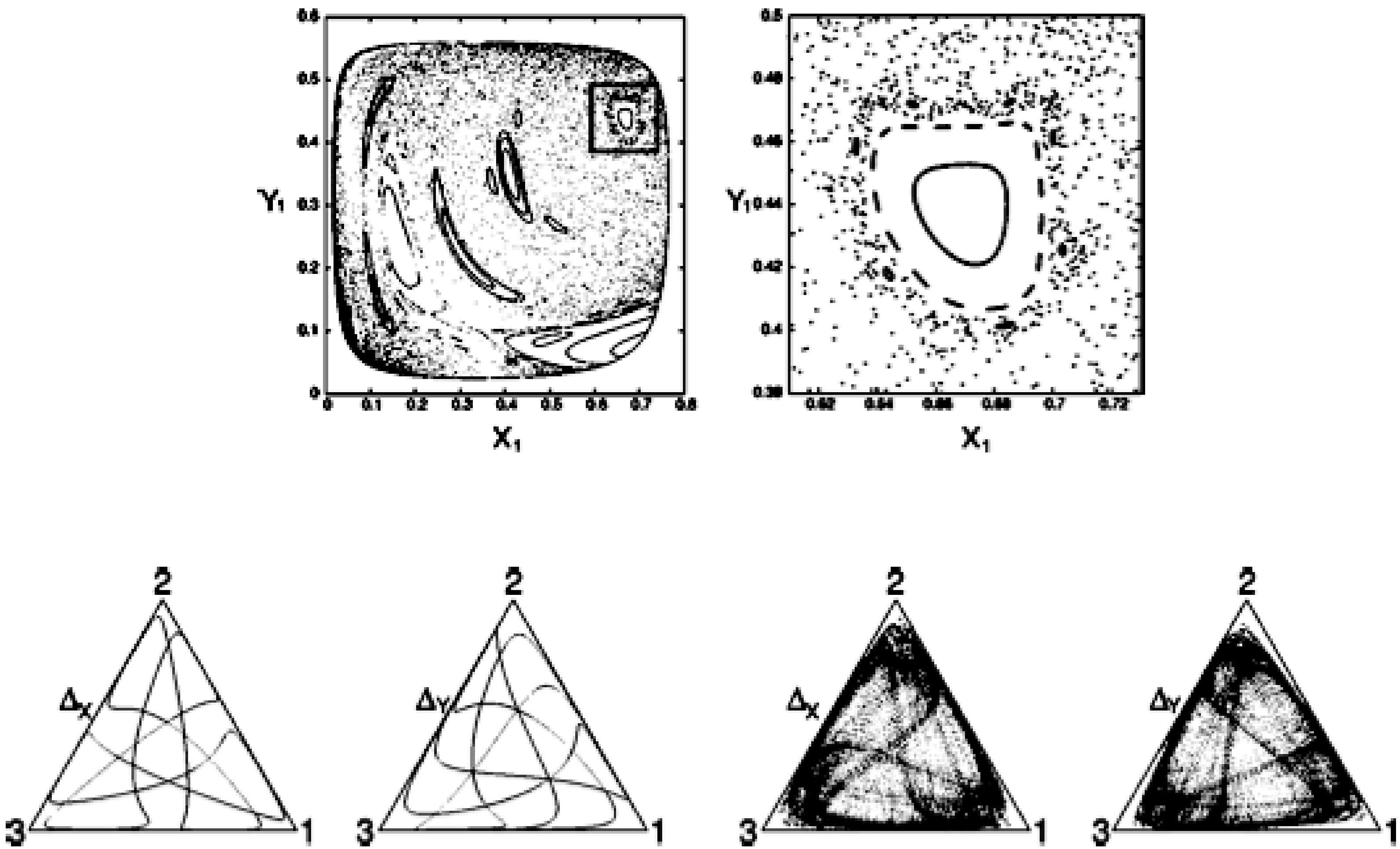


FIG. 2. Limit cycle (left, $\epsilon_Y = 0.025$) and chaotic attractor (right, $\epsilon_Y = -0.365$), with $\epsilon_X = 0.5$, $\alpha_X = \alpha_y = 0.01$, and $\beta_X = \beta_Y$.

Sato+Akiyama+Farmer ‘02, Crutchfield +Sato ‘03, Sato+Akiyama+Crutchfield ‘05

Question: what about 'generic games' ?

Study S-C equations with random payoff matrices

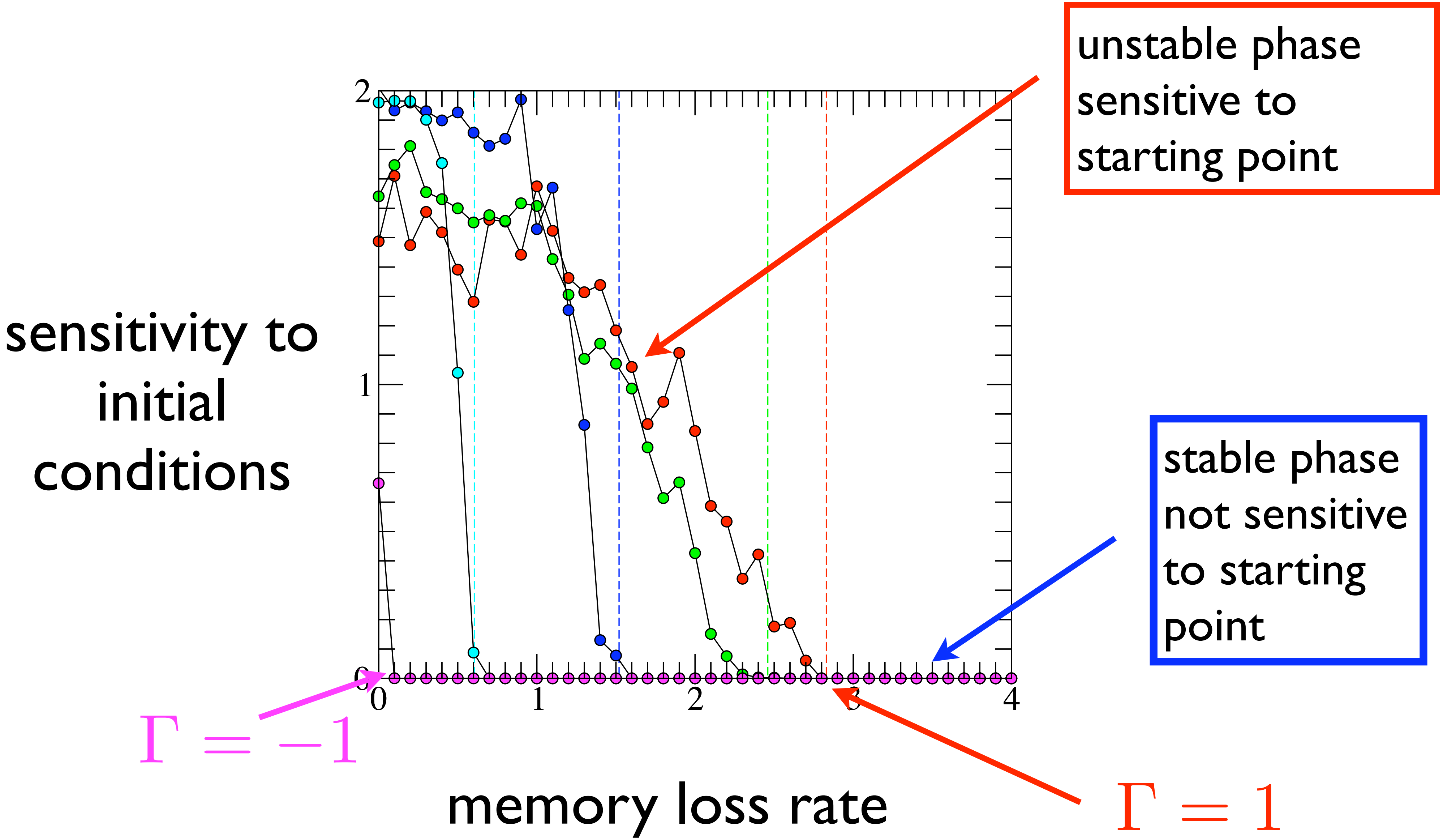
$$\frac{d}{dt}p_i^X = p_i^X \left[\sum_j a_{ij} p_j^Y - \alpha \ln p_i^X - f^X + \alpha S^X \right]$$

$$\frac{d}{dt}p_j^Y = p_j^Y \left[\sum_i b_{ji} p_i^X - \alpha \ln p_j^Y - f^Y + \alpha S^Y \right]$$

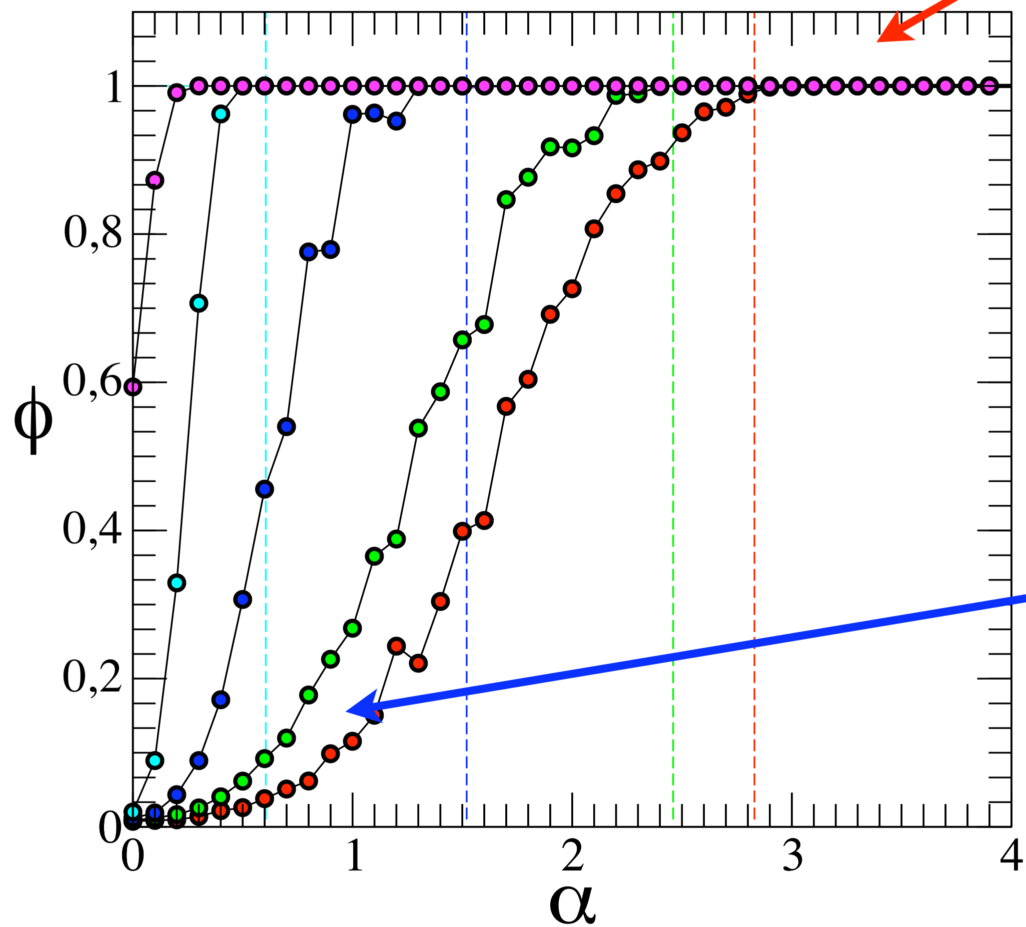
$a_{ij}\,,\, b_{ij}$ Gaussian with correlations:

$$\overline{a_{ij}^2} = \overline{b_{ij}^2} = \frac{1}{N} \qquad \overline{a_{ij} b_{ij}} = \frac{\Gamma}{N}$$

→ path integral analysis



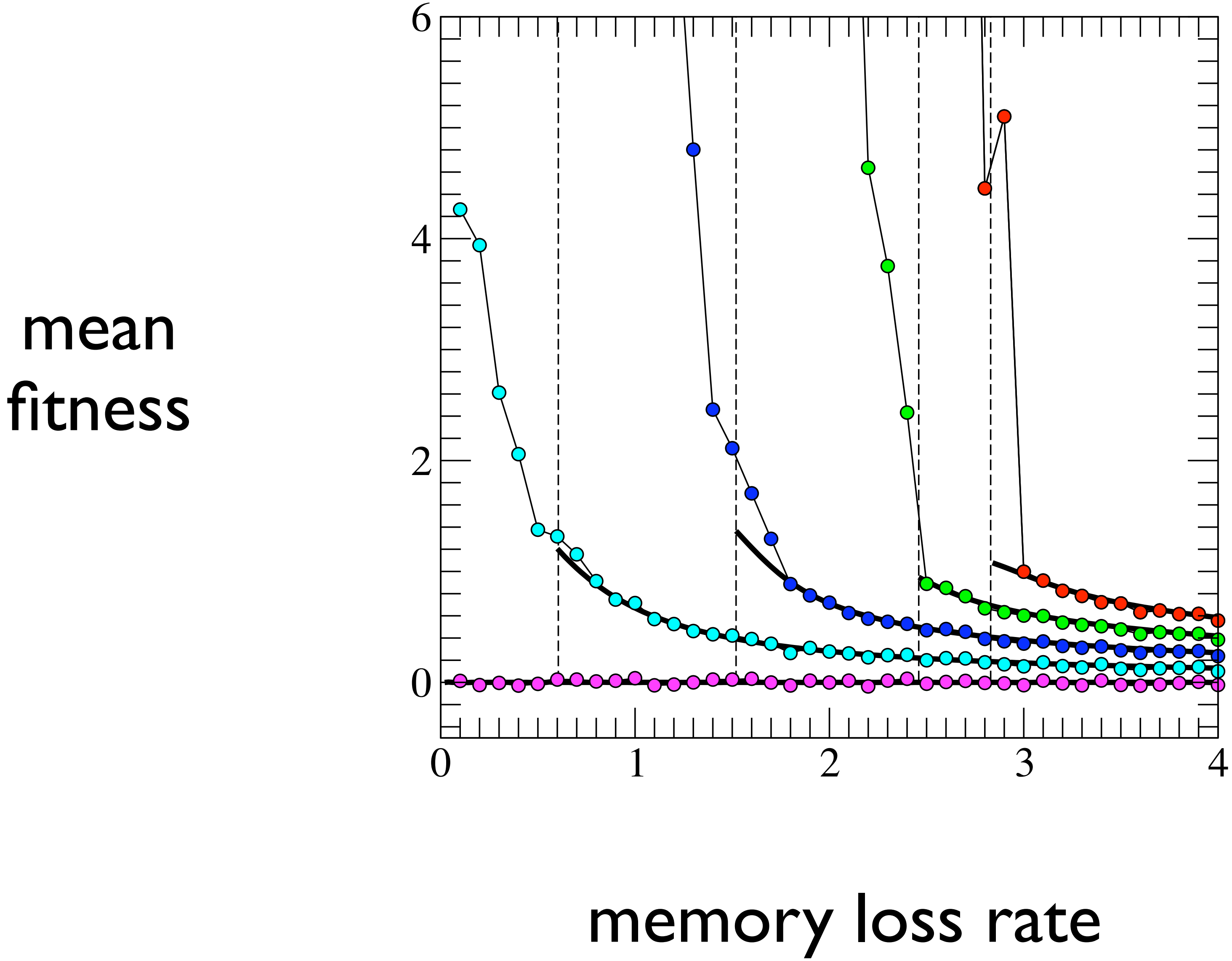
curves are for increasing Γ from left to right fraction of pure strategies played

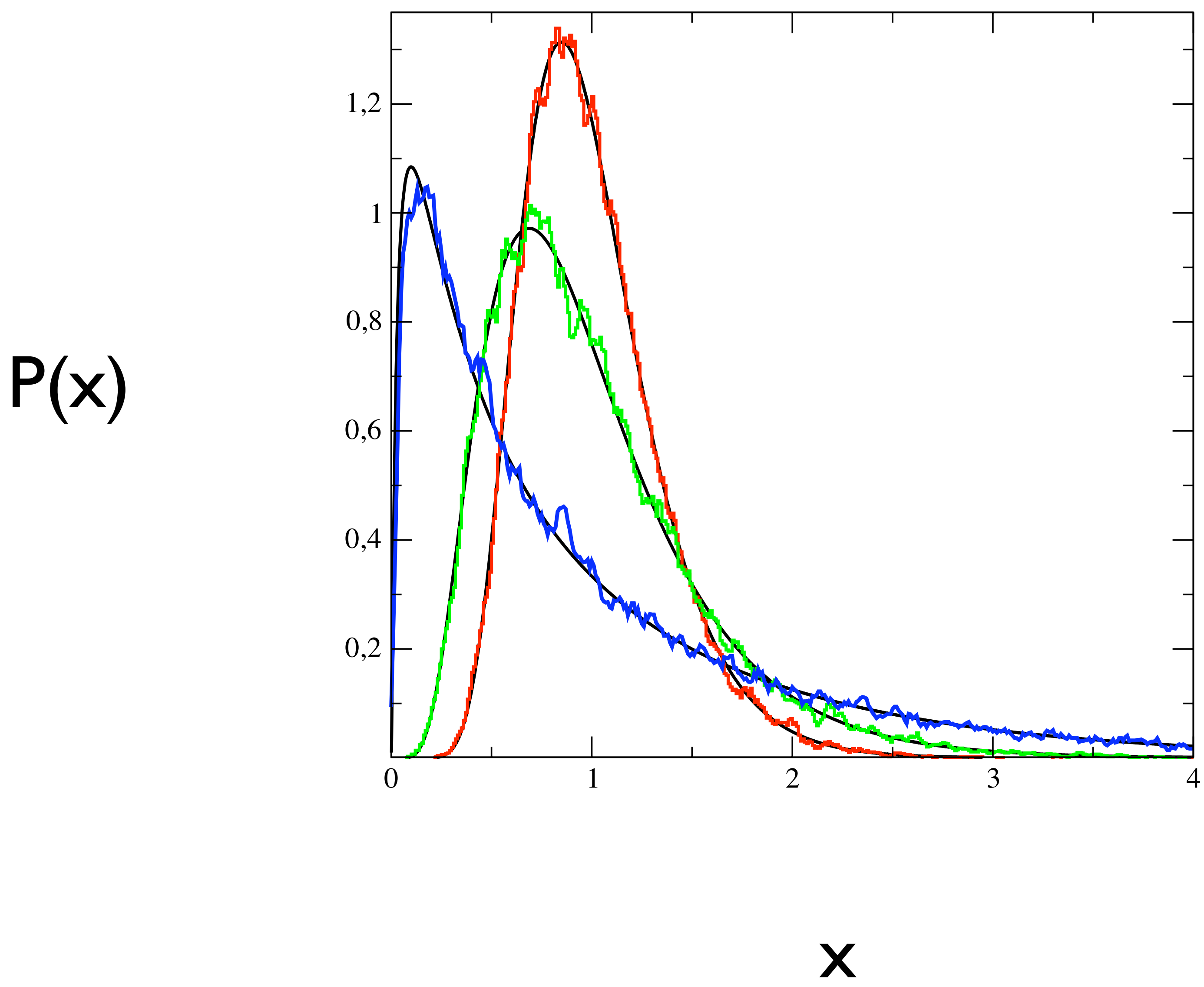


memory loss rate

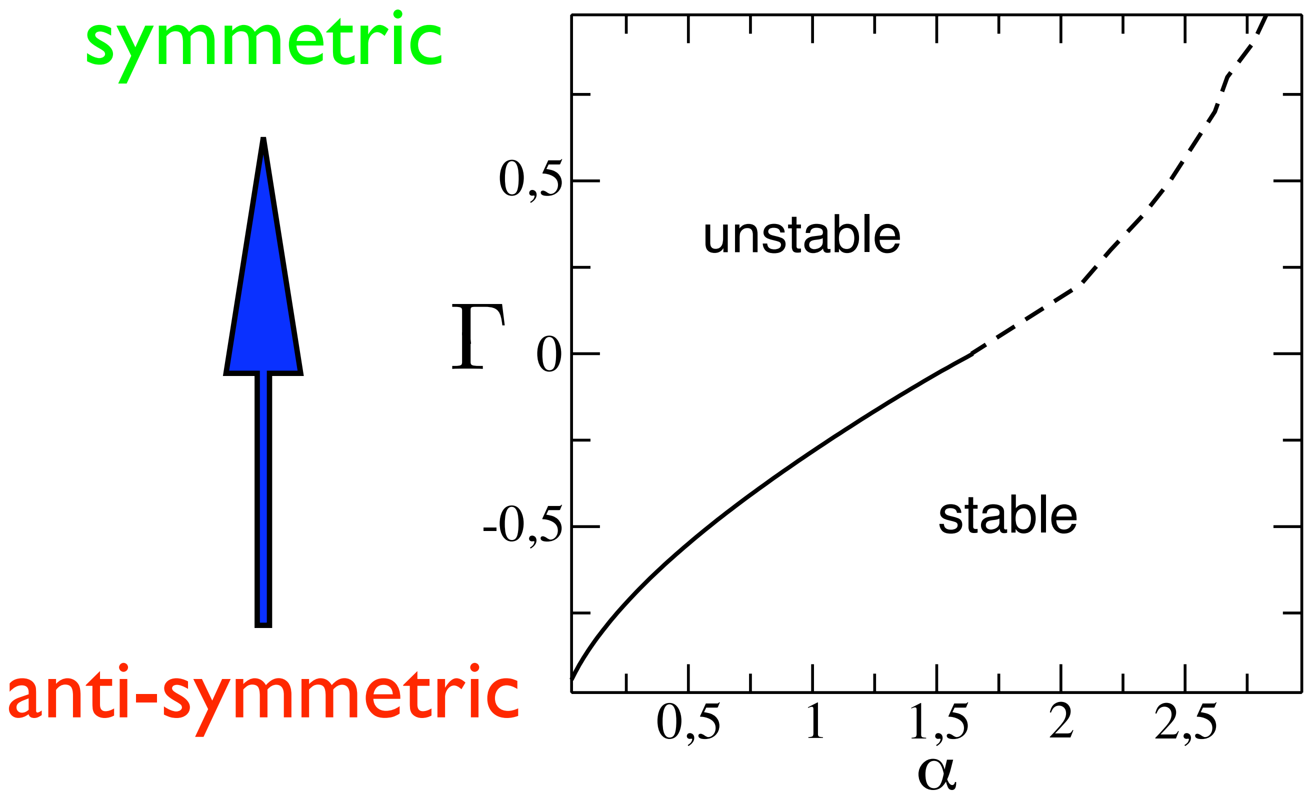
uniform use of all pure strategies

condensated phase, only fraction of strategies used

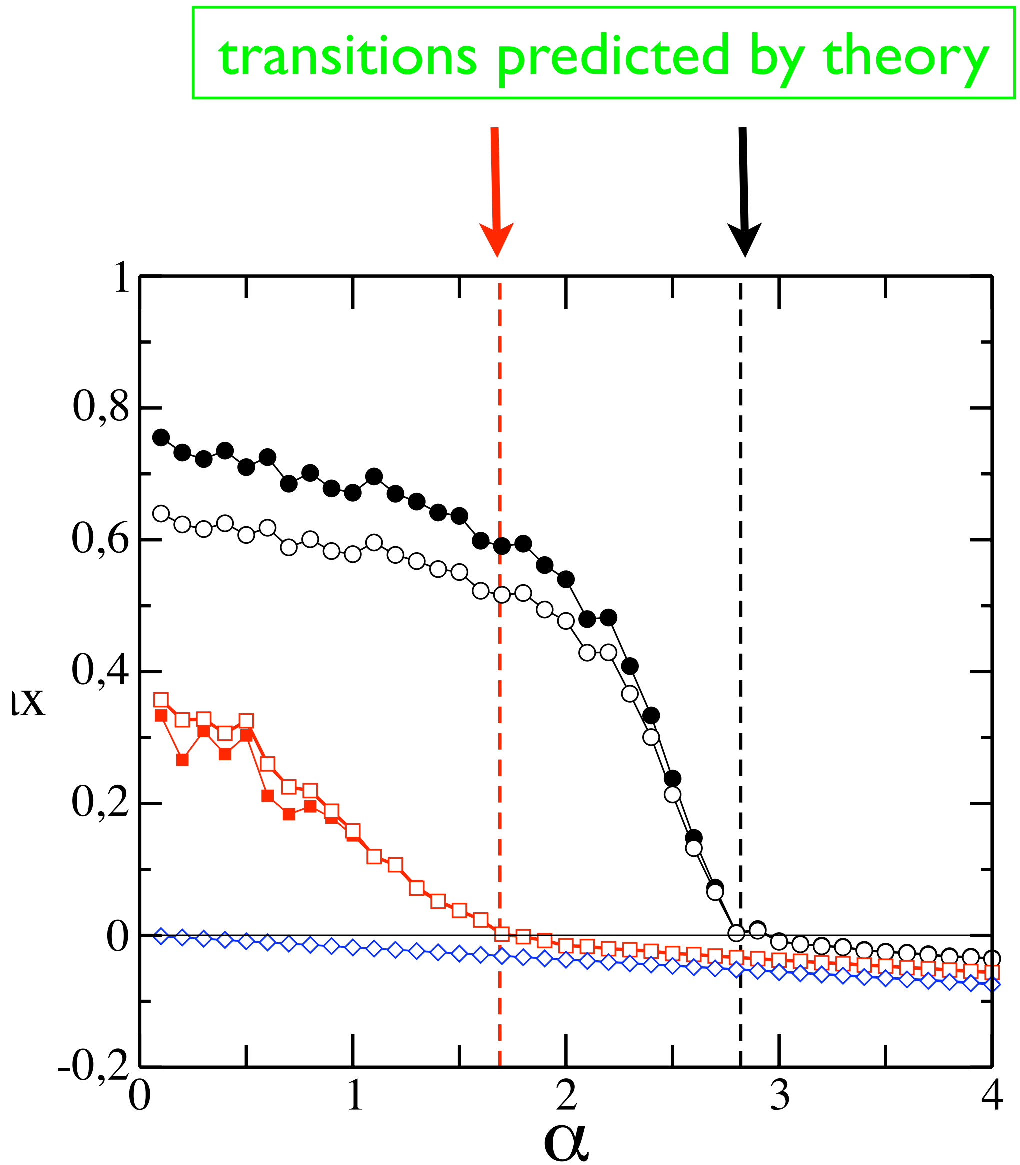

mean
fitness
6
4
2
0
0
1
2
3
4
memory loss rate


P(x)
1,2
1
0,8
0,6
0,4
0,2
0
1
2
3
4
x

Phase diagram from generating functional analysis



memory loss rate



Largest Lyapunov exponent

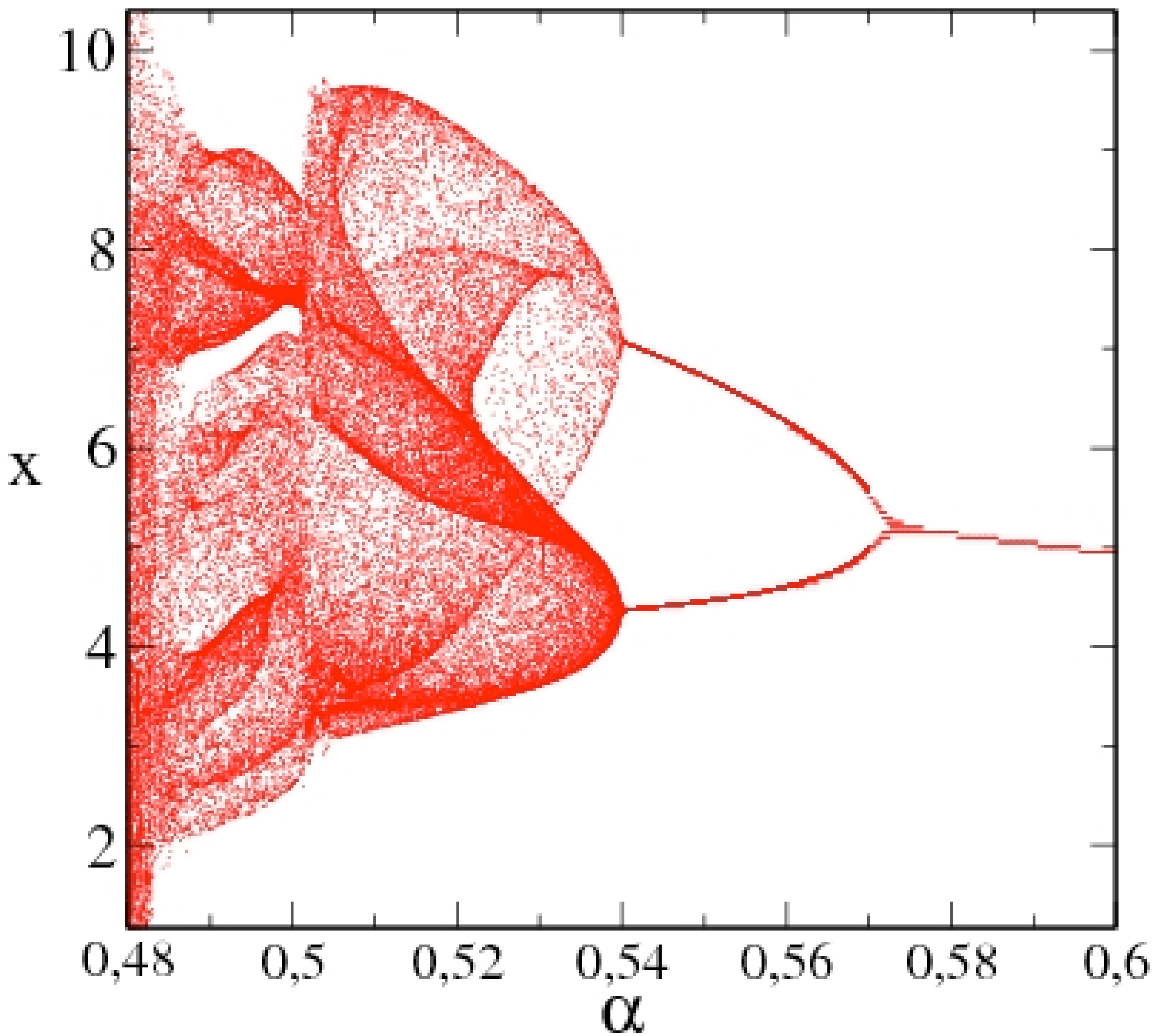
memory loss rate

○ symmetric payoffs

□ uncorrelated payoffs

◇ anti-symmetric payoffs

Bifurcation diagram at $\Gamma = -\frac{1}{2}$



Conclusions

- learning of random games exhibits complex dynamical features
- non-convergence to NE appears to be present in generic large games
- 2 regimes separated by transition, solvable by stat mechanics methods

low accuracy of learning (small memory)

- fixed point regime
- all strategies approximately equally often played
- not sensitive to initial conditions

large accuracy of learning (long memory)

- dynamics remains volatile
- condensation on few strategies
- sensitive to initial conditions, potentially chaotic

See also

Two-population replicator dynamics and number of Nash equilibria in random matrix games
Tobias Galla, Europhysics Letters 78 (2007) 20005

Random replicators with asymmetric couplings
Tobias Galla, J. Phys. A: Math. Gen. 39 (2006) 3853-3869

Dynamics of random replicators with Hebbian interactions
Tobias Galla J. Stat. Mech. (2005) P11005

Statistical mechanics and stability of a model eco-system
Yoshimi Yoshino, Tobias Galla, Kei Tokita, arXiv:0705.1523v1