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Connections between random Boolean networks and their annealed model

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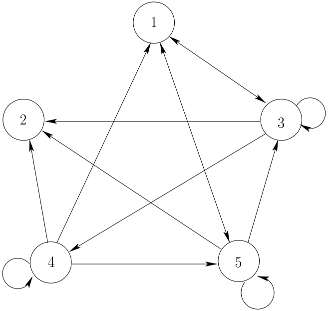
Outline

Boolean networks



	f_1	f_2	f_3	f_4	f_5
000	0	0	0	0	0
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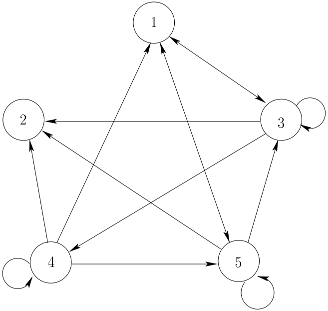
010	1	1	1	0	0
011	1	0	0	1	0
100	0	0	1	0	0
101	1	1	1	1	0
110	1	1	0	1	0
111	1	1	1	1	1

- A Boolean network consists of N interconnected nodes i each capable of storing a binary value.
- Each node i has K input edges $j_{i_1}, \ldots, j_{i_K}$.
- To each node a Boolean function f_i is assigned.

Boolean networks



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- Define $s_i(t)$ as the value stored by i at time t.

Then:

$$s_i(t+1) = f(s_{i_1}(t), \ldots, s_{i_K}(t))$$

.

NK networks

NK networks: Random Boolean networks with N nodes, where:

- for each node a Boolean function is chosen among all equally likely functions with K arguments,
- for each function the K arguments are chosen among $\binom{N}{K}$ equally likely possibilities,
- finally a random initial state is chosen.

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By numerical simulations S. Kauffman found that if $K \leq 2$, the networks show *ordered* behaviour:

- Large proportion of weak nodes.
- Large proportion of frozen nodes.
- Small attractor cycles.

Contrary if $K > 2$ the networks are *disordered* or *chaotic*.

The ensemble: RBN(K, P)

Here we consider the ensemble RBN(K, P):

- for each node a Boolean function f with K arguments is chosen as follows: each of the 2^K positions in the truth table of f is set to 1 with probability P.
- for each function the K arguments are chosen among $\binom{N}{K}$ equally likely possibilities,
- finally a random initial state is chosen.

Derridas annealed approximation

[Derrida & Pomeau 1986] introduced the *annealed model*.
In contrast to the classical model (the so called *quenched model*) the functions and connections are chosen at random at each time step.

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Theorem

Ordered behaviour, which means

$$\frac{\mathbb{E}(d_H(\mathbf{s}_1(t), \mathbf{s}_2(t)))}{N} \to 0,$$

if and only if

$$2KP(1-P) \leq 1.$$

Lynchs analysis of the quenched model

Consider a network (with N nodes) with an arbitrary state.

- A node G is t-weak if a perturbation of G vanishes in t steps.

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Theorem

Ordered behaviour, which means

$$\lim_{N\to\infty} Pr(G \textit{ is } \alpha \log N\text{-}\textit{weak}) = 1,$$

if and only if

$$2KP(1-P) \leq 1.$$

(α is a constant depending on K only)

Motivation

Question

- What is the connection between the two models?

l-Sensitivity

Definition

- The *l-sensitivity* $s_f^l(\mathbf{w})$ of a function f with argument $\mathbf{w} \in \mathbb{F}_2^K$ is the number of vectors $\mathbf{x}$ in Hamming distance l to $\mathbf{w}$, for which $f(\mathbf{w}) \neq f(\mathbf{x})$.
- The *average l-sensitivity* s_f^l is the average of $s_f^l(\mathbf{w})$ of all $\mathbf{w}$.

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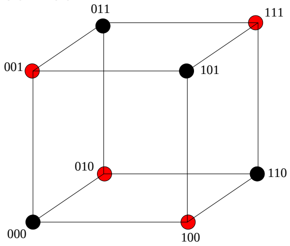
Example:

$f((w_1, w_2, w_3)) = w_1 \oplus w_2 \oplus w_3$

Argument space of f, $f = 1$ marked red:



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Example:
$f((w_1, w_2, w_3)) = w_1 \oplus w_2 \oplus w_3$
for all $\mathbf{w}$:

$$s_f^1(\mathbf{w}) = 3 \quad \text{and}$$
$$s_f^2(\mathbf{w}) = 0$$

.

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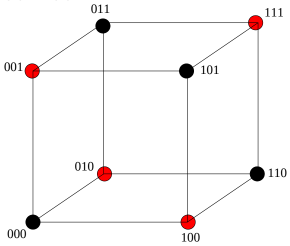

Expectation of l-Sensitivity for random function

Suppose that Boolean functions are chosen at random. The probability of choosing a function f is given by p_f: The *expectation* of the l-sensitivity is given by

$$\mathbb{E}\left(s_f^l(\mathbf{w})\right) = \sum_f p_f s_f^l(\mathbf{w}).$$

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$$\mathbb{E}\left(s_f^l\right) = \sum_f p_f s_f^l.$$

For RBN(K, P) it turns out that

Lemma

$$\textit{for all } \mathbf{w} : \mathbb{E}\left(s_f^l(\mathbf{w})\right) = const. = \mathbb{E}\left(s_f^l\right).$$

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Lemma

$$\mathbb{E}\left(s_f^l\right) = \frac{\binom{K}{l}}{K}\mathbb{E}(s_f^1)$$

Lynchs order parameter

Suppose

- the probability for a function f is given by p_f and
- the *mean activity* is independent of time and given by a.

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Definition (Lynch)

$$\lambda = \sum_f p_f \sum_{\mathbf{w} \in \mathbb{F}_2^K} s_f(\mathbf{w}) a^{w_H(\mathbf{w})} (1-a)^{K_i - w_H(\mathbf{w})},$$

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in general

Theorem (Lynch)

Ordered behaviour if and only if

$$\lambda \leq 1.$$

Lynchs order parameter

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for RBN(K,p):

Theorem

$$\lambda = \mathbb{E}(s_f^1)$$

hence ordered behaviour if and only if

$$\mathbb{E}(s_f^1) \leq 1.$$

Annealed analysis I

Consider two instances of the same random network (with N nodes) starting from two different initial states (s_1,s_2). Define the fractional overlap $a(t) = 1 - \frac{\mathbb{E}(d_H(\mathbf{s}_1(t),\mathbf{s}_2(t)))}{N}$.

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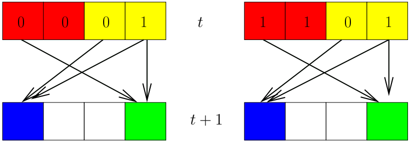
At time t: Define a set of nodes A_t which store the same value in both instances (marked as yellow below).



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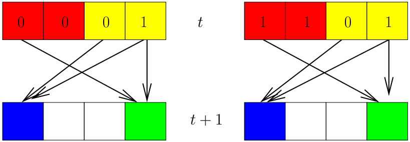

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Next time step: there are nodes (blue) that receive their input only from A_t. We expect $Na(t)^K$ blue nodes and $N(1 - a(t)^K)$ other nodes at time $t + 1$, the latter having probability P_d of being different. Therefore:

$$a(t + 1) = a(t)^K + (1 - P_d)(1 - a(t)^K).$$

Annealed analysis II

Suppose that s_1 and s_2 are randomly chosen but different and f is a random function.

$$P_d = Pr\left(f(\mathbf{s}_1) \neq f(\mathbf{s}_2) \,|\, \mathbf{s}_1 \neq \mathbf{s}_2\right)$$

Annealed analysis II

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For RBN(K, P) it turns out that

$$P_d = \frac{\mathbb{E}(s_f^1)}{K}.$$

Annealed analysis II

Therefore $a(t)$ evolves according a one-dimensional map

$$a(t+1) = A(a(t))$$

where

$$A(x) = 1 + P_d(x^K - 1) = 1 + \mathbb{E}(s_f^1)(x^K - 1).$$

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Theorem

Stable fixed point $x_0 = 1$ (total overlap, hence ordered behaviour) if and only if

$$\mathbb{E}(s_f^1) \leq 1.$$

Summery and comments

Due to the simple form of the expectation of the l-sensitivity:

- For RBN(K, P) the phase of both models, the annealed and the quenched, is determined by the expectation of the average sensitivity (order 1).
- This is also true for other ensembles (see paper).

Summery and comments

Due to the simple form of the expectation of the l-sensitivity:

- For RBN(K, P) the phase of both models, the annealed and the quenched, is determined by the expectation of the average sensitivity (order 1).
- This is also true for other ensembles (see paper).

Note:
It can be shown, that similar results hold, if the probability of a function is only dependent on the number of 1 in the truth table (not yet published).

Thank you

for your attention!