On-Line Learning

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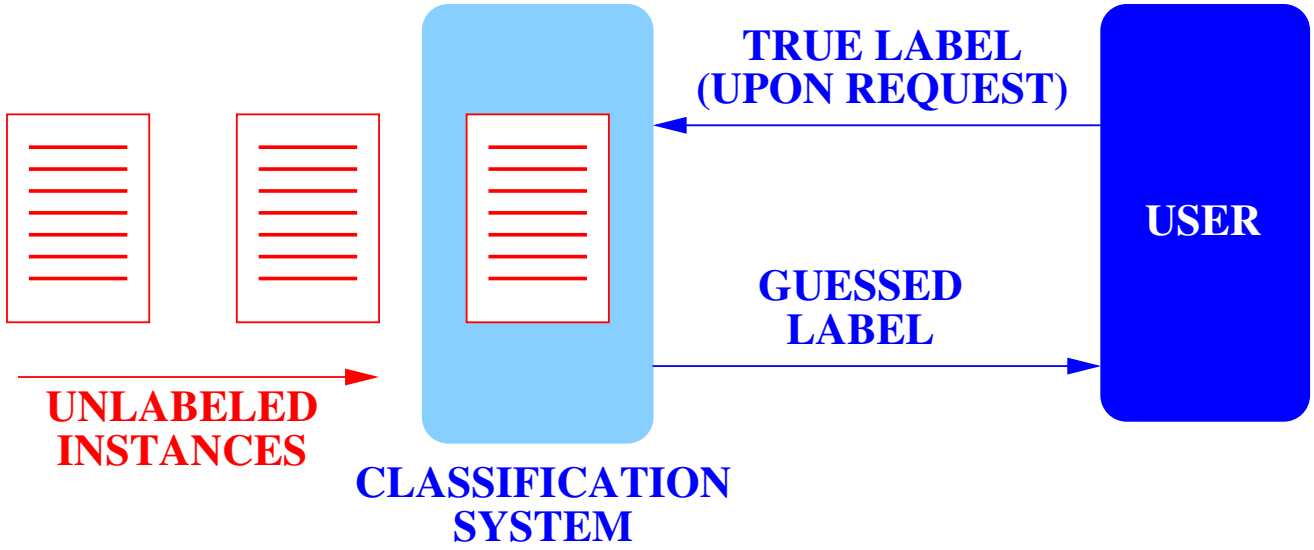
Summary

Summary

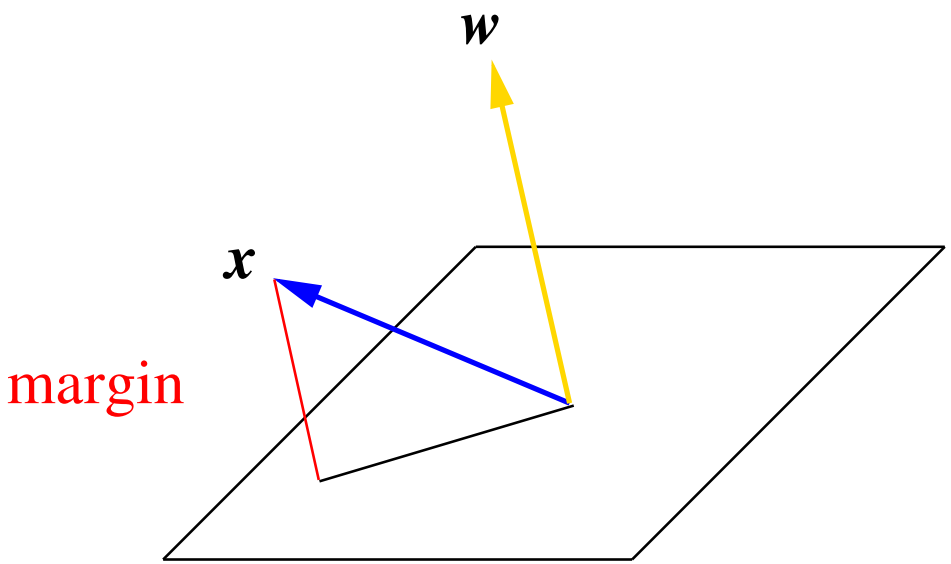
On-line classification



Linear classifiers

- Stream of data instances encoded as vectors $x_1, x_2, \cdots \in \mathbb{R}^d$
- A binary label $y_t \in \{-1, 1\}$ associated to each x_t
- A linear classifier $w_{t-1} \in \mathbb{R}^d$ predicts label y_t of x_t with

$$\widehat{y}_t = \text{SGN}(w_{t-1}^\top x_t) \qquad w_{t-1} \in \mathbb{R}^d$$



On-line learning protocol

- Learning proceeds in rounds
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3. Observe $y_t \in \{-1, +1\}$
4. Update $\boldsymbol{w}_{t-1} \to \boldsymbol{w}_t$

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- Learner observes a stream of examples $(\boldsymbol{x}_t, y_t) \quad t = 1, 2, \ldots$ and produces a sequence of classifiers $\boldsymbol{w}_0, \boldsymbol{w}_1, \ldots$
- Learner's performance measure: number of classification mistakes on the stream

Remarks

- natural on certain tasks (e.g., market forecasting)
- efficient to run (scalability) and easy to code
- kernel-based learning
- strong performance guarantees (nonstochastic)
- stochastic risk bounds easy to prove (online to batch conversions)
- active learning variants when y_t comes at a cost
- tracking abilities

Summary

Perceptron algorithm

[Rosenblatt, 1952]

- Initial weight $\boldsymbol{w}_0 = (0, \ldots, 0)$.
- Update $\boldsymbol{w}_t = \begin{cases} \boldsymbol{w}_{t-1} & \text{if } \widehat{y}_t = y_t \text{ (no mistake)} \\ \boldsymbol{w}_{t-1} + y_t \boldsymbol{x}_t & \text{otherwise} \end{cases}$

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Intuition

- Want positive margin: $\widehat{y}_t \neq y_t \quad \text{iff} \quad \underbrace{y_t \, w_{t-1}^\top x_t}_{\text{margin}} \leqslant 0$
- Effect of Perceptron update on margin:

$$y_t w_t^\top x_t = y_t \left(w_{t-1} + y_t x_t\right)^\top x_t = y_t w_{t-1}^\top x_t + \|x_t\|^2$$

- Margin has increased by $\|x_t\|^2$ (non-corrective update)

Linear separability

Linearly separable dataset

$(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_T, y_T)$

$$\exists \mathbf{u} \in \mathbb{R}^d \quad y_t \mathbf{u}^\top \mathbf{x}_t > 0 \quad t = 1, \ldots, T$$

Note: $\|\mathbf{u}\|$ does not matter

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- Does the Perceptron find any such $\mathbf{u}$ if cycled on the stream?
- What if stream is not linearly separable?

Relative loss bound

$$\sum_t m_t(\boldsymbol{w}_{t-1}) \leqslant \inf_{\boldsymbol{u}\in\mathbb{R}^d} \sum_t m_t(\boldsymbol{u}) + \text{smaller terms}$$

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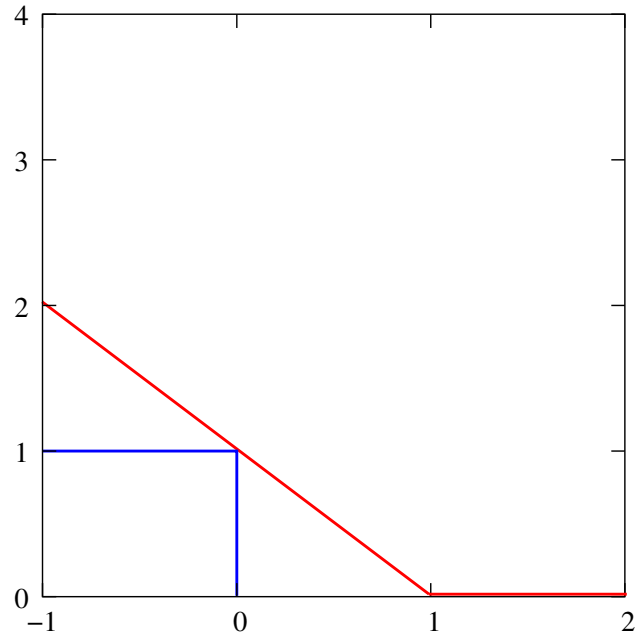
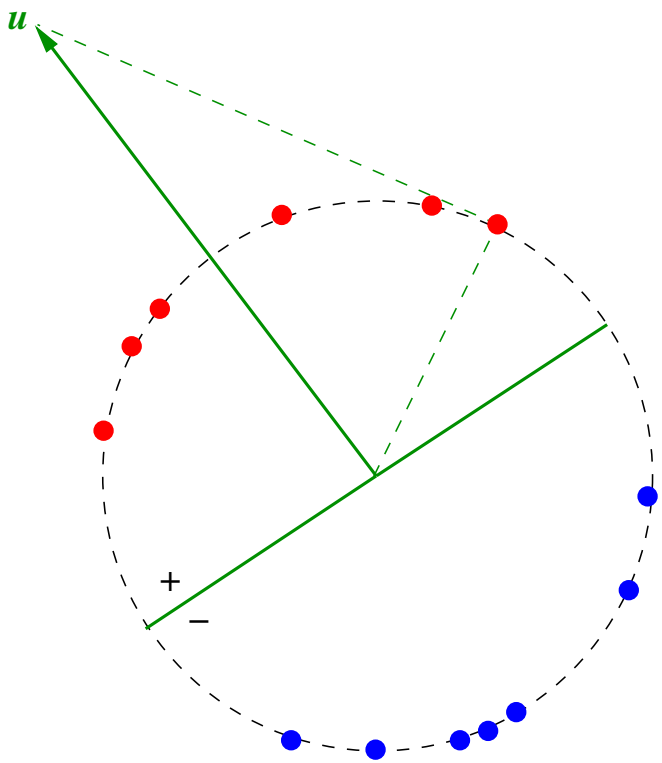


$$\underbrace{\mathbb{I}_{\{\mathrm{SGN}(z) \neq y\}}}_{\text{mistake ind.}} \leqslant \underbrace{(1 - y\,z)_+}_{\text{hinge loss}}$$

- Computing an hyperplane minimizing the number of misclassified examples is NP-hard
- Relax goal: the hinge loss is a convex upper bound on the mistake indicator function
- Hinge loss grows linearly if margin is smaller than 1

Norm of the separator



- $\ell_t(\mathbf{u}) = [1 - y_t \mathbf{u}^\top \mathbf{x}_t]_+$
- Now $\|\mathbf{u}\|$ matters
- If the stream is hardly separable, then $\|\mathbf{u}\|$ has to be big to keep the hinge loss at zero

Analysis of Perceptron

- Any (nonseparable) stream, any $\mathbf{u} \in \mathbb{R}^d$, $\|\mathbf{x}_t\| = 1$
- Two proof techniques:

1. evolution of $\|\mathbf{u} - \mathbf{w}_t\|^2$
2. evolution of $\mathbf{u}^\top \mathbf{w}_t$

Analysis: when a mistake occurs

$$
\begin{aligned}
\mathbf{u}^\top \mathbf{w}_t &\geqslant \mathbf{u}^\top (\mathbf{w}_{t-1} + y_t \mathbf{x}_t) \\
&= \mathbf{u}^\top \mathbf{w}_{t-1} + \underbrace{y_t \mathbf{u}^\top \mathbf{x}_t}_{1 - \left(1 - y_t \mathbf{u}^\top \mathbf{x}_t\right)} \\
&\geqslant \mathbf{u}^\top \mathbf{w}_{t-1} + 1 - \ell_t(\mathbf{u})
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Unwrapping the recurrence

$$
\mathbf{u}^\top \mathbf{w}_T \geqslant \mathbf{u}^\top \mathbf{w}_0 + \sum_t m_t(\mathbf{w}_{t-1}) - \sum_t \ell_t(\mathbf{u})
$$

When a mistake occurs (cont.)

$$
\begin{aligned}
\|\boldsymbol{w}_t\|^2 &\leqslant \|\boldsymbol{w}_{t-1} + y_t \boldsymbol{x}_t\|^2 \\
&= \|\boldsymbol{w}_{t-1}\|^2 + \|\boldsymbol{x}_t\|^2 + 2 \underbrace{y_t \boldsymbol{w}_{t-1} \boldsymbol{x}_t}_{\leqslant 0} \\
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Unwrapping the recurrence

$$\|w_T\|^2 \leqslant \|w_0\|^2 + \sum_t m_t(w_{t-1})$$

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Use $u^\top w_T \leqslant \|u\| \, \|w\|_T$ and solve for $\sum_t m_t(w_{t-1})$

The relative mistake bound

Generalized Perceptron convergence theorem

$$\sum_t m_t(\boldsymbol{w}_{t-1}) \leqslant \inf_{\boldsymbol{u}\in\mathbb{R}^d} \sum_t \ell_t(\boldsymbol{u}) + \|\boldsymbol{u}\|^2 + \|\boldsymbol{u}\| \sqrt{\sum_t \ell_t(\boldsymbol{u})}$$

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Remarks

- Oracle inequality for individual sequences
- If cycled over a linear separable $(x_1, y_1), \ldots, (x_T, y_T)$ converges to some linear separator after at most $\|u^*\|^2$ updates, where u^* is the SVM hyperplane $\min \|u\|^2 \quad \text{s.t.} \quad y_t u^\top x_t \geqslant 1 \quad \forall t$
- This implies that there always exists a linear separator expressible as a linear combination of at most $\|u^*\|^2$ supports

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- $y_t \left(\boldsymbol{w}_{t-1} + \eta_t y_t \boldsymbol{x}_t\right)^\top \boldsymbol{x}_t \geqslant 1$ iff $\eta_t = \dfrac{1}{\|\boldsymbol{x}_t\|^2}\left(1 - y_t \boldsymbol{w}_{t-1}^\top \boldsymbol{x}_t\right)$

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- This is also the solution of the first-order SVM

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- Geometrically, we project $\boldsymbol{w}_{t-1}$ onto the hyperplane

$$\left\{\boldsymbol{w} \in \mathbb{R}^d : y_t \boldsymbol{w}^\top \boldsymbol{x}_t = 1\right\}$$

Analysis for linearly separable streams

Recall

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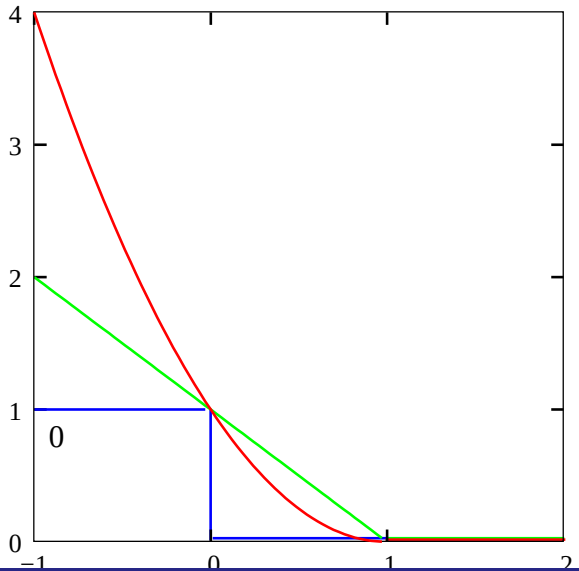
&= y_t (w_t - w_{t-1})^\top x_t \\
&\leqslant \|w_t - w_{t-1}\| \, \|x_t\|
\end{aligned}$$

Thus we get

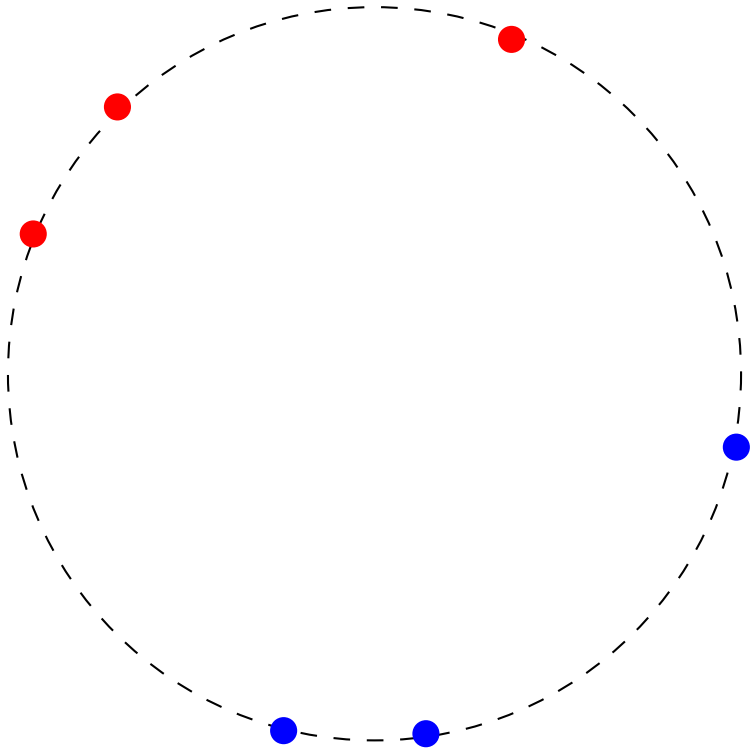
$$\ell_t(w_{t-1})^2 \leqslant \|w_t - w_{t-1}\|^2 \, \|x_t\|^2$$

Analysis (cont.)

- Fix shorter separator $\mathbf{u}$ with zero hinge loss (SVM solution)
- By the law of cosines $\|\boldsymbol{w}_t - \boldsymbol{w}_{t-1}\|^2 \leqslant \|\boldsymbol{w}_{t-1} - \mathbf{u}\|^2 - \|\boldsymbol{w}_t - \mathbf{u}\|^2$
- Thus $\sum_t m_t(\boldsymbol{w}_{t-1})^2 \leqslant \sum_t \ell_t(\boldsymbol{w}_{t-1})^2 \leqslant \|\mathbf{u}\|^2 \left(\max_t \|\mathbf{x}_t\|\right)^2$
- This is a stronger result than Perceptron convergence theorem, but it does not imply convergence to SVM solution



The cone of consistent hyperplanes



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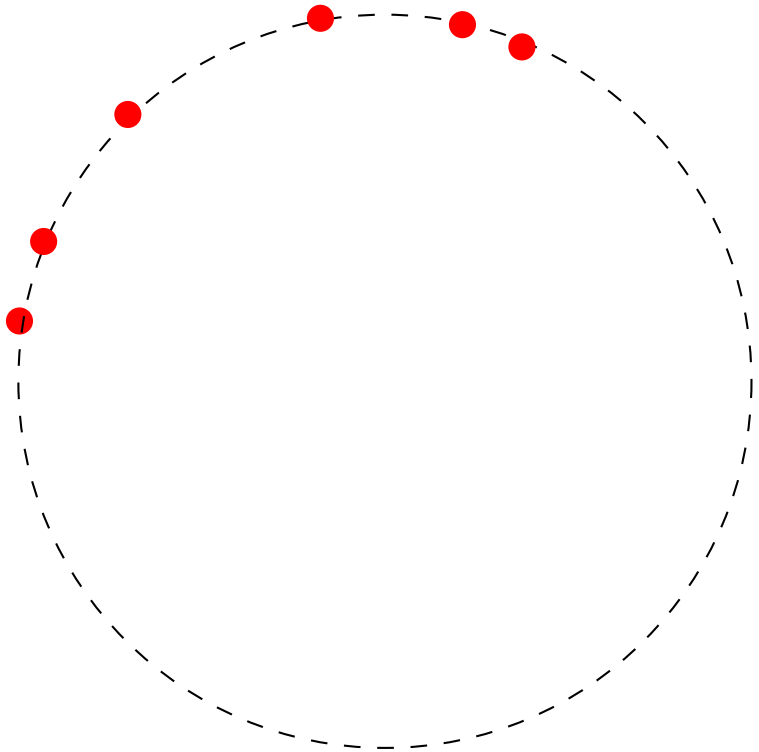

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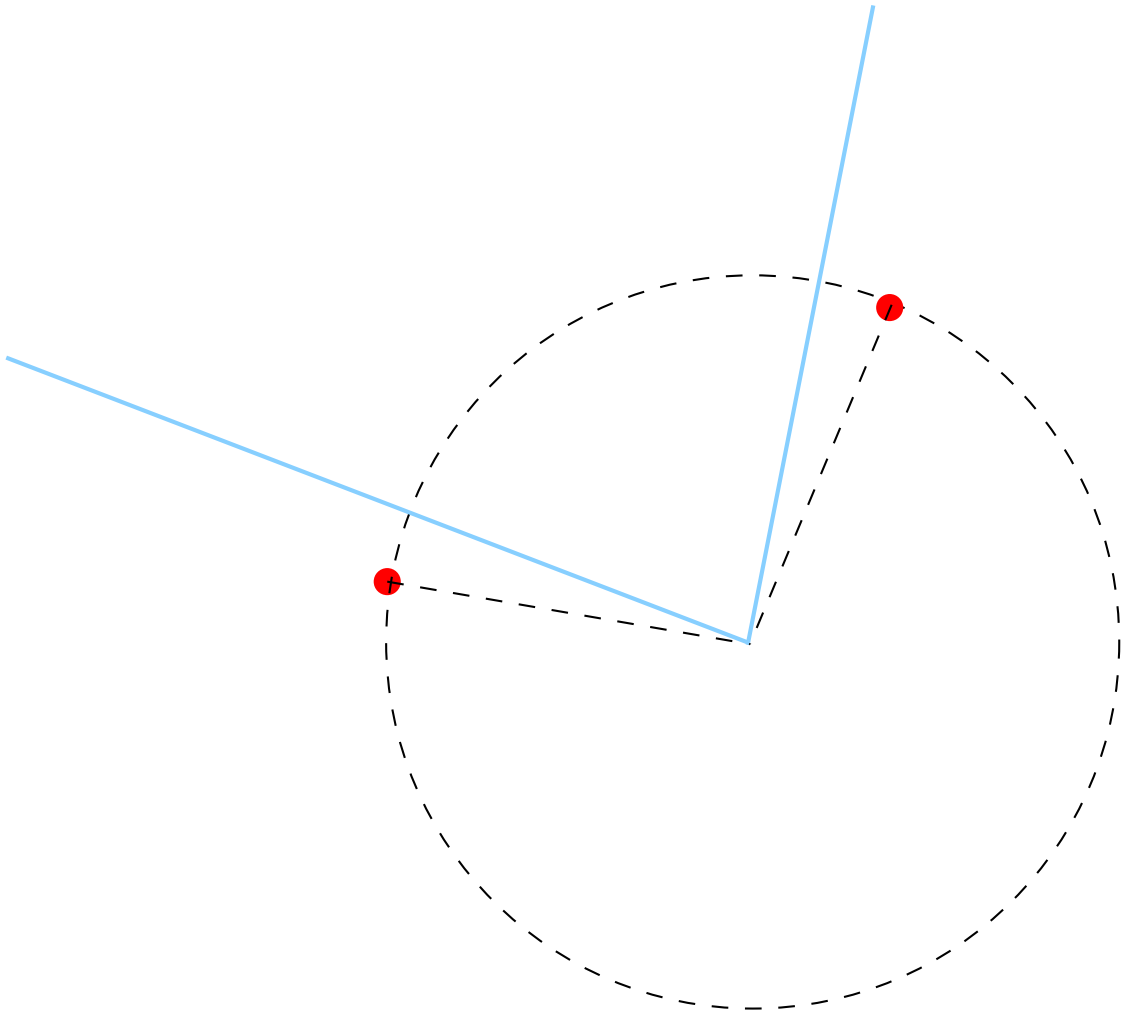

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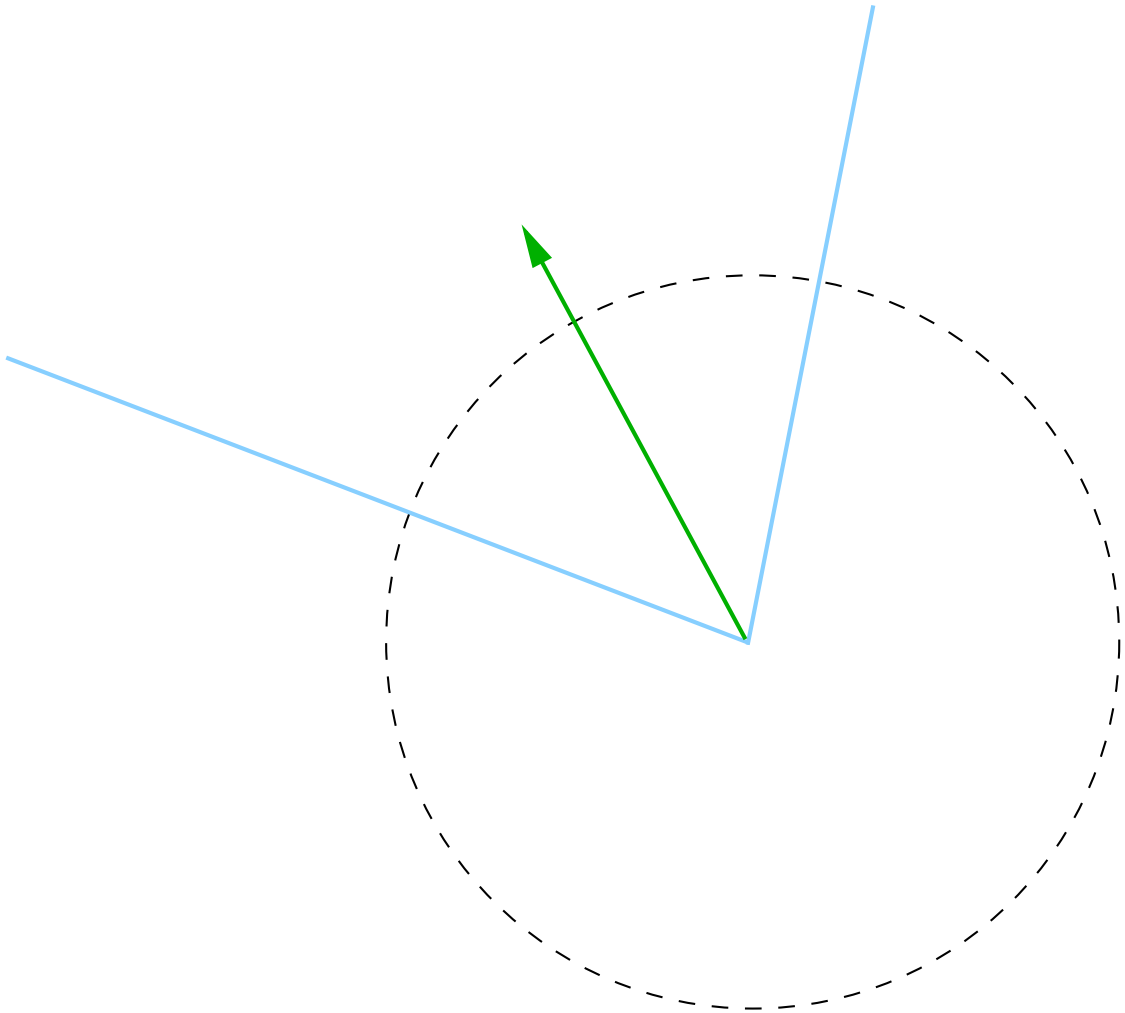

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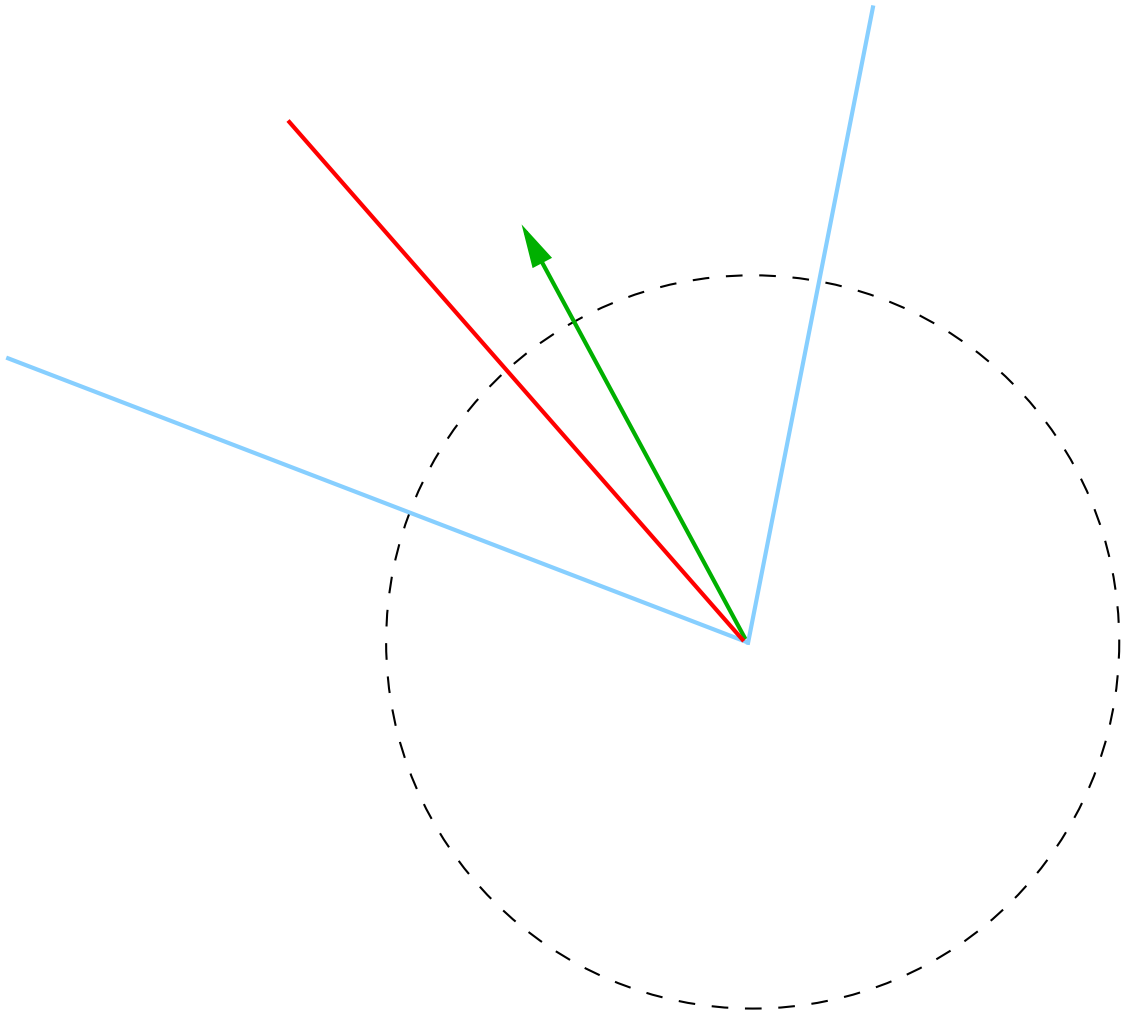

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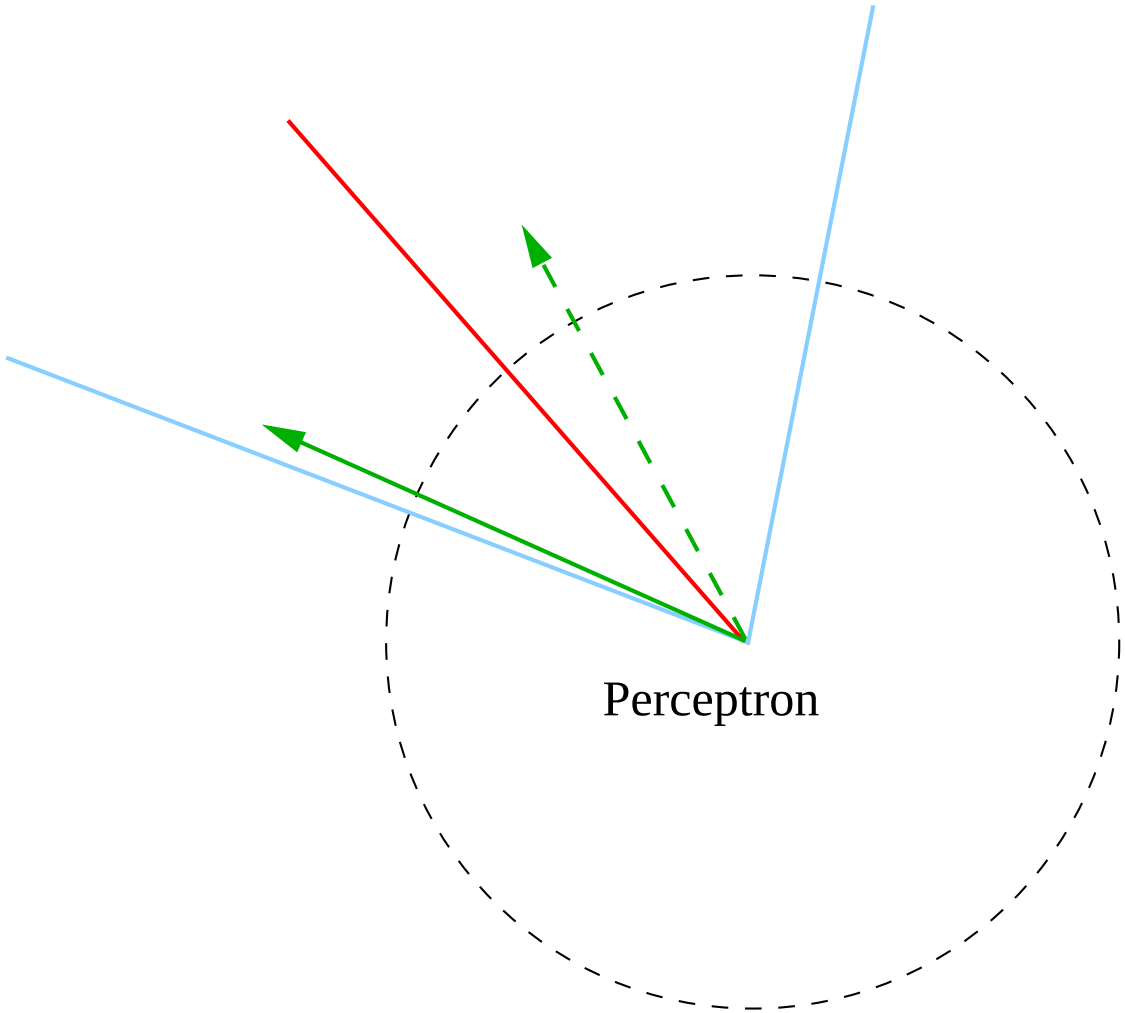

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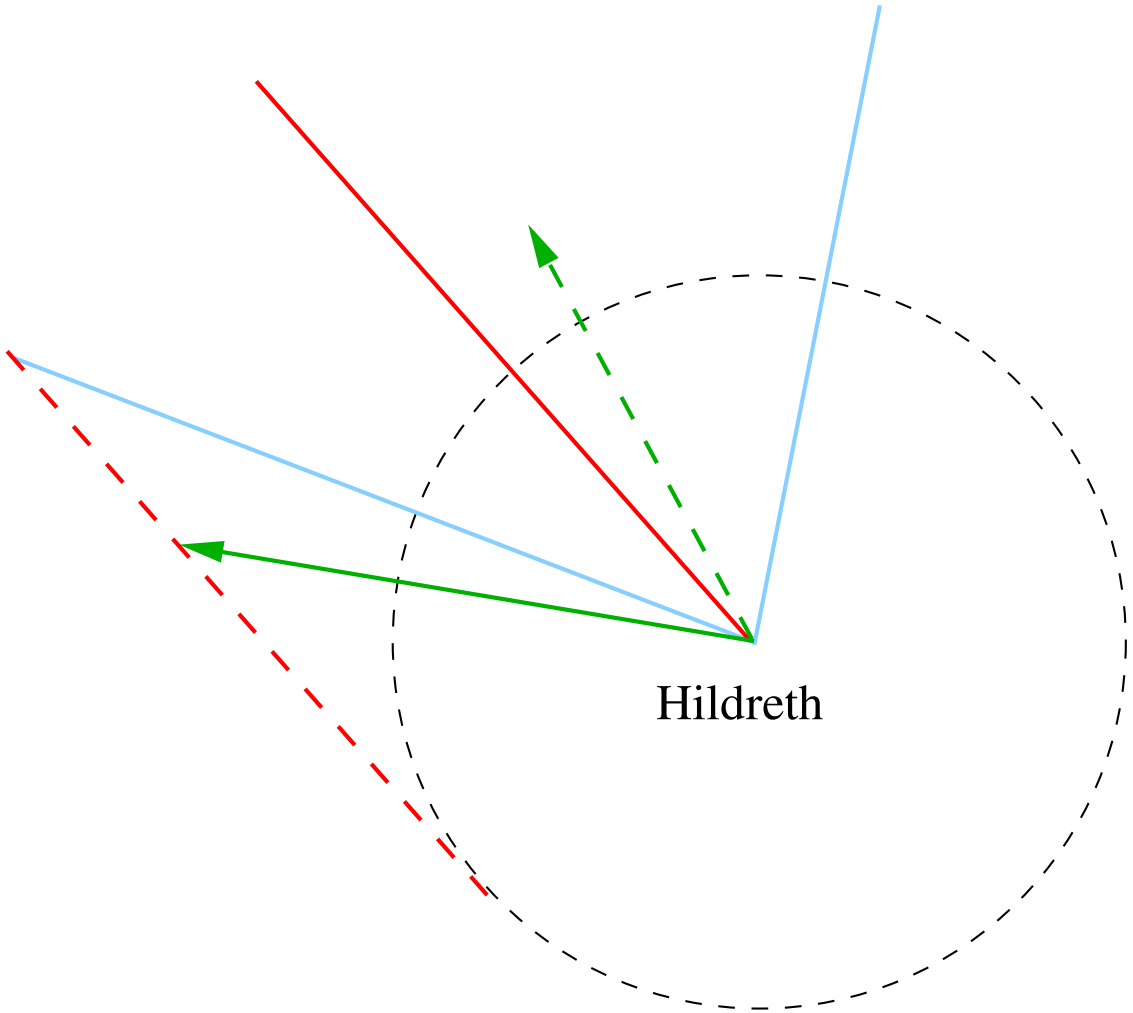

Mistake bounds for various updates

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BOUND	ALGORITHM	UPDATE TIME
U^2	Perceptron	$O(d)$
U^2	Hildreth	$O(d)$
$dU \ln U$	2nd order Perceptron	$O(d^2)$
$d^2 \ln U$	Ellipsoid	$O(d^3)$
$d \ln U$	Volumetric center (U known)	$O(d^{3.5})$
$d \ln U$	Geometric center (U known)	$O(d^4)$

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Note

- U^2 optimal for fixed U
- $d \ln U$ optimal for fixed U and d

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Passive-aggressive algorithm [Crammer et al. 2006]

- General update form is $\boldsymbol{w}_t = \boldsymbol{w}_{t-1} + \eta_t y_t \boldsymbol{x}_t$

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SVM and passive-aggressive

SVM primal objective

$$\text{SVMOPT} = \min_{\mathbf{u}} \left(\frac{1}{2} \|\mathbf{u}\|^2 + C \sum_{t=1}^{T} \xi_t \right) \quad \text{s.t.} \quad y_t \mathbf{u}^\top \mathbf{x}_t \geqslant 1 - \xi_t \ \forall t$$

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SVM dual Lagrange function

$$D_{\text{svm}}(\alpha_1, \ldots, \alpha_T) = -\frac{1}{2} \left\| \sum_t \alpha_t \mathbf{x}_t \right\|^2 + \sum_t \alpha_t \qquad 0 \leqslant \alpha_t \leqslant C \quad \forall t$$

SVM and passive-aggressive

SVM primal objective

$$\text{SVMOPT} = \min_{\mathbf{u}} \left(\frac{1}{2} \|\mathbf{u}\|^2 + C \sum_{t=1}^{T} \xi_t \right) \quad \text{s.t.} \quad y_t \mathbf{u}^\top \mathbf{x}_t \geqslant 1 - \xi_t \ \forall t$$

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Duality

For all $\alpha'_1, \ldots, \alpha'_T \geqslant 0$

$$D_{\text{svm}}(\alpha'_1, \ldots, \alpha'_T) \leqslant \sup_{\boldsymbol{\alpha}} D_{\text{svm}}(\boldsymbol{\alpha}) = \text{SVMOPT}$$

SVM and passive-aggressive (cont.)

Following [Shalev-Shwartz and Singer, 2006]

$$
\begin{aligned}
D_{\mathrm{svm}}(0,\ldots,0) &= 0 \\
D_{\mathrm{svm}}(\alpha_1,\ldots,\alpha_T) &= \sum_{t=1}^{T} \Big(D_{\mathrm{svm}}(\alpha_1,\ldots,\alpha_t,0,\ldots,0) \\
&\qquad - D_{\mathrm{svm}}(\alpha_1,\ldots,\alpha_{t-1},0,\ldots,0) \Big)
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D_{\text{svm}}(\alpha_1,\ldots,\alpha_t,0,\ldots,0) &- D_{\text{svm}}(\alpha_1,\ldots,\alpha_{t-1},0,\ldots,0) \\
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\end{aligned}
$$

where $w_{t-1} = \sum_{s=1}^{t-1} \alpha_s y_s x_s$

SVM and passive-aggressive (cont.)

PA-I primal objective

$$\min_{w} \left(\frac{1}{2} \|w - w_{t-1}\|^2 + C\xi_t \right) \quad \text{s.t.} \quad y_t w^\top x_t \geqslant 1 - \xi_t$$

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PA-I learning rate is maximizer of dual objective

$$\eta_t = \operatorname{argmin} \left\{ C, \frac{\ell_t(w_{t-1})}{\|x_t\|^2} \right\} = \operatorname*{argmax}_{0 \leqslant \alpha \leqslant C} D_t(\alpha) \stackrel{\text{def}}{=} \alpha_t$$

Mistake bounds for PA-I

We note that

$$D_t(\alpha_t) = D_{\text{svm}}(\alpha_1, \ldots, \alpha_t, 0, \ldots, 0) - D_{\text{svm}}(\alpha_1, \ldots, \alpha_{t-1}, 0, \ldots, 0)$$

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$$\sum_{t=1}^{T} D_t(\alpha_t) \leqslant \text{SVMOPT} = \min_{\mathbf{u}} \left(\frac{1}{2} \|\mathbf{u}\|^2 + C \sum_{t=1}^{T} \ell_t(\mathbf{u}) \right)$$

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- We can use this inequality to prove mistake bounds for PA-I
- Proof idea: lower bound $D_t(\alpha_t)$ at mistakes

Proof of mistake bound for PA-I

Assume for simplicity $\|\mathbf{x}_t\| = 1$

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Lemma

$$y_t \mathbf{w}_{t-1}^\top \mathbf{x}_t \leqslant 0 \quad \text{implies} \quad D_t(\alpha_t) \geqslant \min\{C, \ell_t(\mathbf{w}_{t-1})\} \frac{\ell_t(\mathbf{w}_{t-1})}{2}$$

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This implies

$$\min\{C, 1\} m_t(w_{t-1}) \leqslant \min\{C, \underbrace{\ell_t(w_{t-1})}_{\geqslant 1}\} \underbrace{\ell_t(w_{t-1})}_{\geqslant m_t(w_{t-1})} \leqslant 2 D_t(\alpha_t)$$

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Theorem [Crammer et al., 2006]

$$\sum_{t=1}^{T} m_t(w_{t-1}) \leqslant \max\left\{\frac{1}{C}, 1\right\} \inf_{u \in \mathbb{R}^d} \left(\|u\|^2 + 2C \sum_{t=1}^{T} \ell_t(u) \right)$$

Summary

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- Feature map $\phi : \mathbb{R}^d \to \text{RKHS}$

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- Then we can learn $w = \sum_i \alpha_i \phi(x_{t_i})$ in the RKHS because

$$\begin{aligned} \text{SGN}\big(\langle w, \phi(x) \rangle\big) &= \text{SGN}\left(\sum_i y_{t_i} \langle \phi(x_{t_i}), \phi(x) \rangle \right) \\ &= \text{SGN}\left(\sum_i y_{t_i} K(x_{t_i}, x) \right) \end{aligned}$$

Kernel Perceptron

Start with empty cache $\mathcal{L}$ of examples

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Mistake bounds hold in the whole RKHS

Memory bounded learning

Can we control the rate of mistakes when at most $B < \infty$ supports are used?

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- Thus $B \geqslant U^2$ is necessary to compete against $\mathbf{u}$ of length U
- Can we compete against any $\mathbf{u}$ with $\|\mathbf{u}\| \leqslant U$ using $B = (1+\varepsilon)U^2$ supports?

A randomized perceptron

Randomized Budget Perceptron

Parameter: size B of cache for supports

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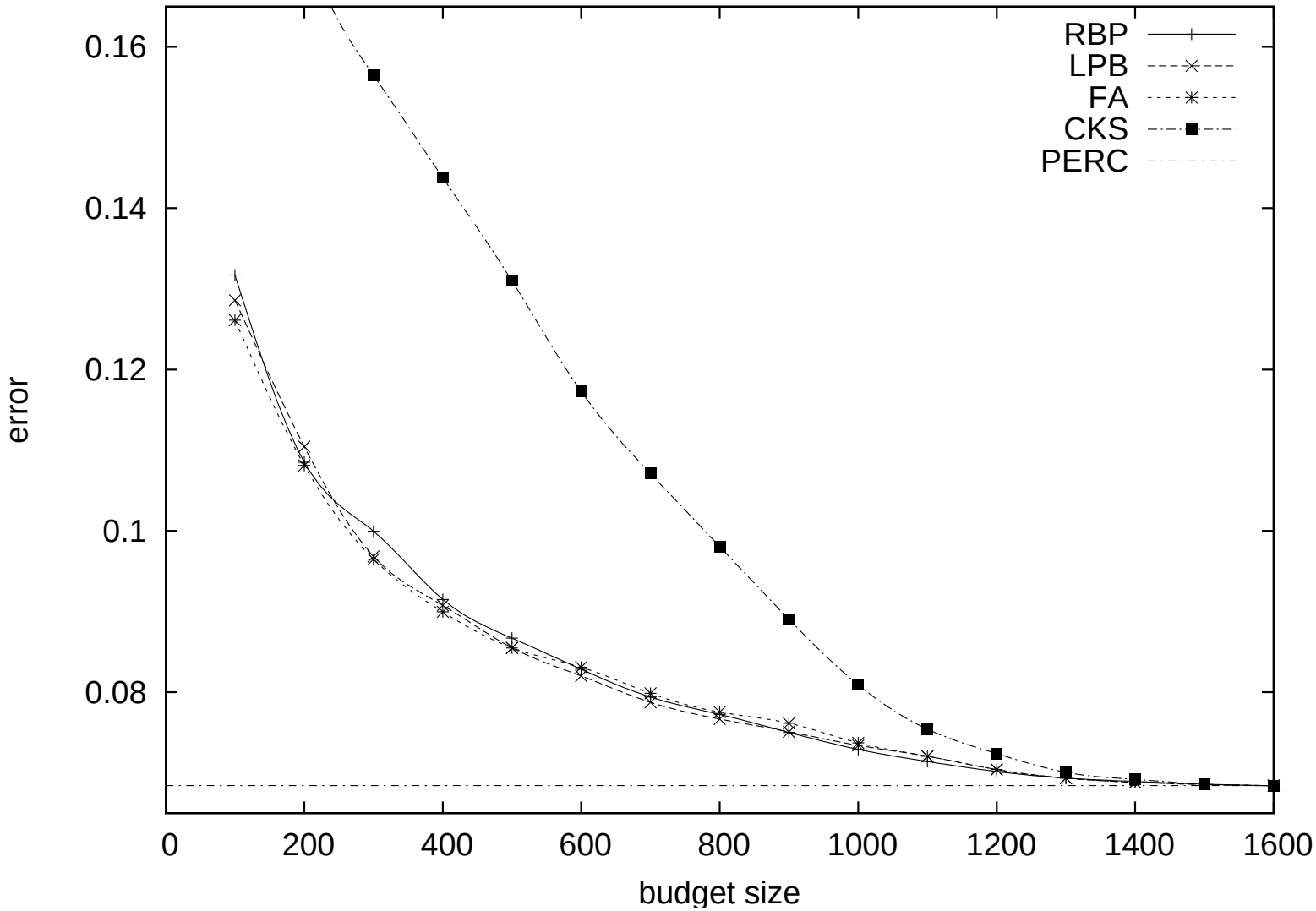
Result:

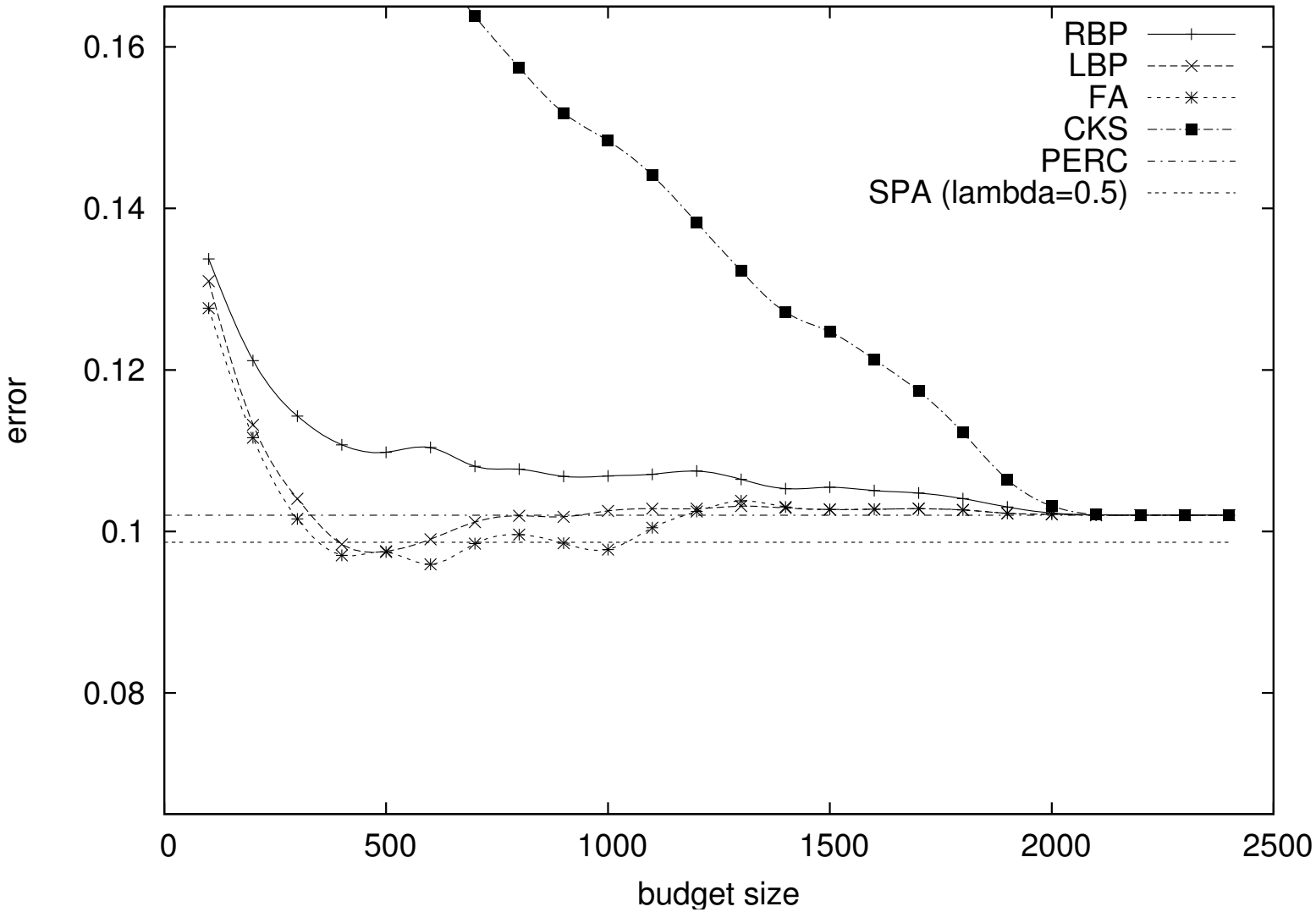
Bound on mistakes scales roughly with $1 + 1/\varepsilon$

(when using $B = (1+\varepsilon)U^2$ supports)

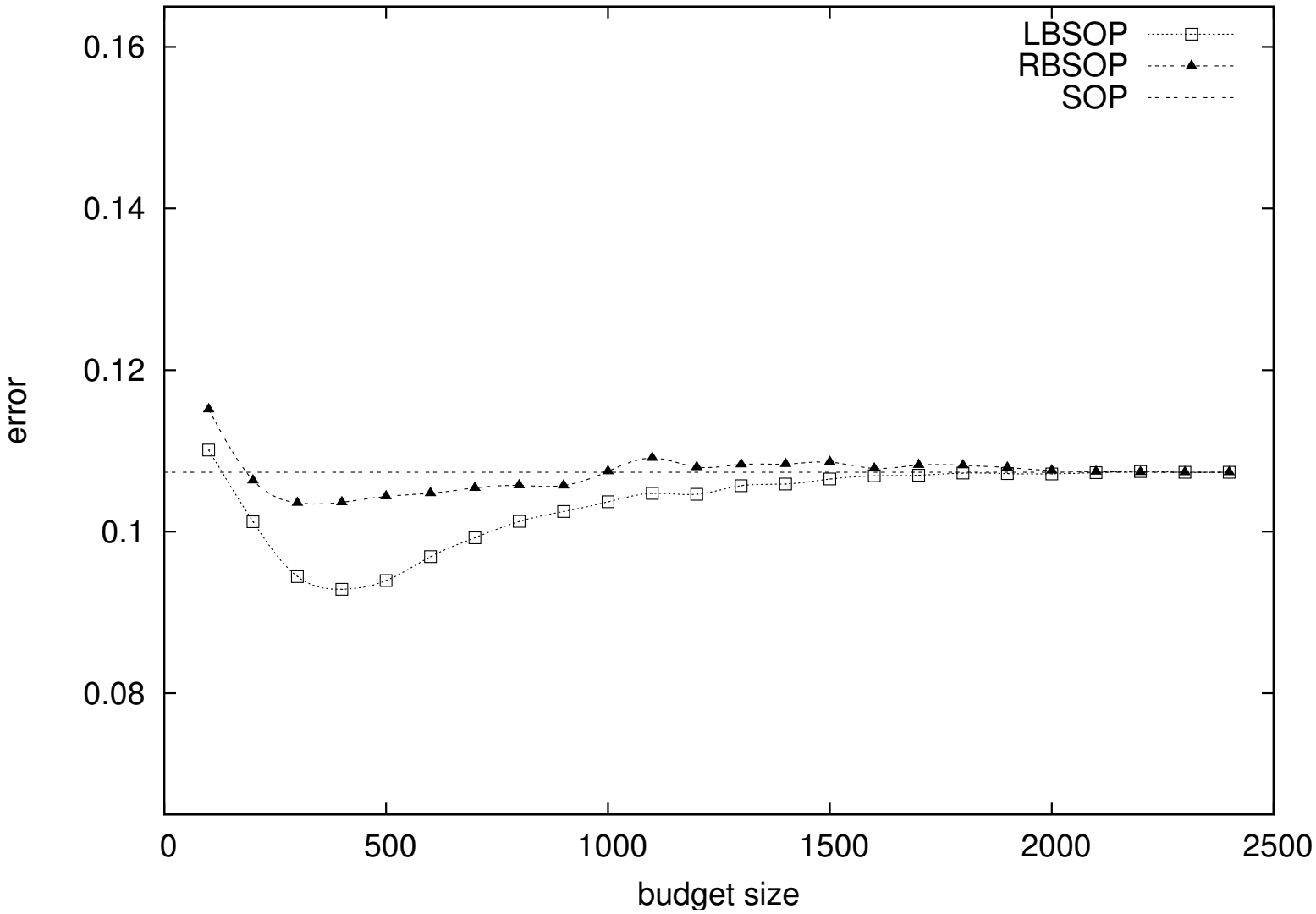
Empirical performance — stationary



Empirical performance — nonstationary



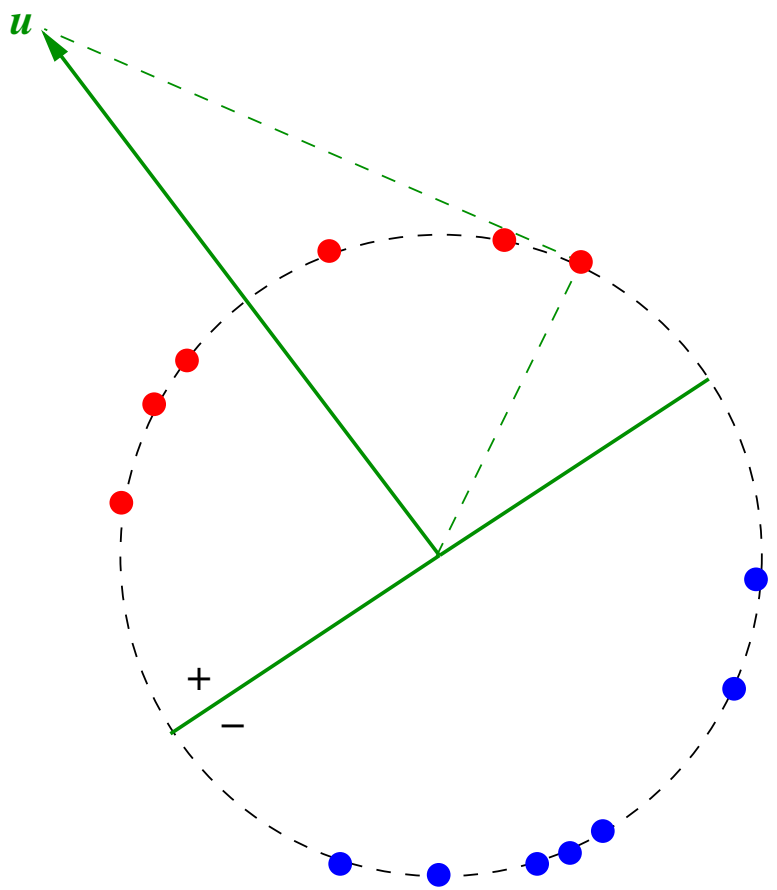
Empirical performance 2nd order — nonstationary



Summary

Online approximation of SVM hyperplane

The SVM hyperplane is the shortest $\mathbf{u}$ such that $y_t \mathbf{u}^\top \mathbf{x}_t \geqslant 1$ for all t



Online approximation of SVM hyperplane (cont.)

The ALMA algorithm

Parameter: $0 < \alpha \leqslant 1$

Set mistake counter $k = 1$

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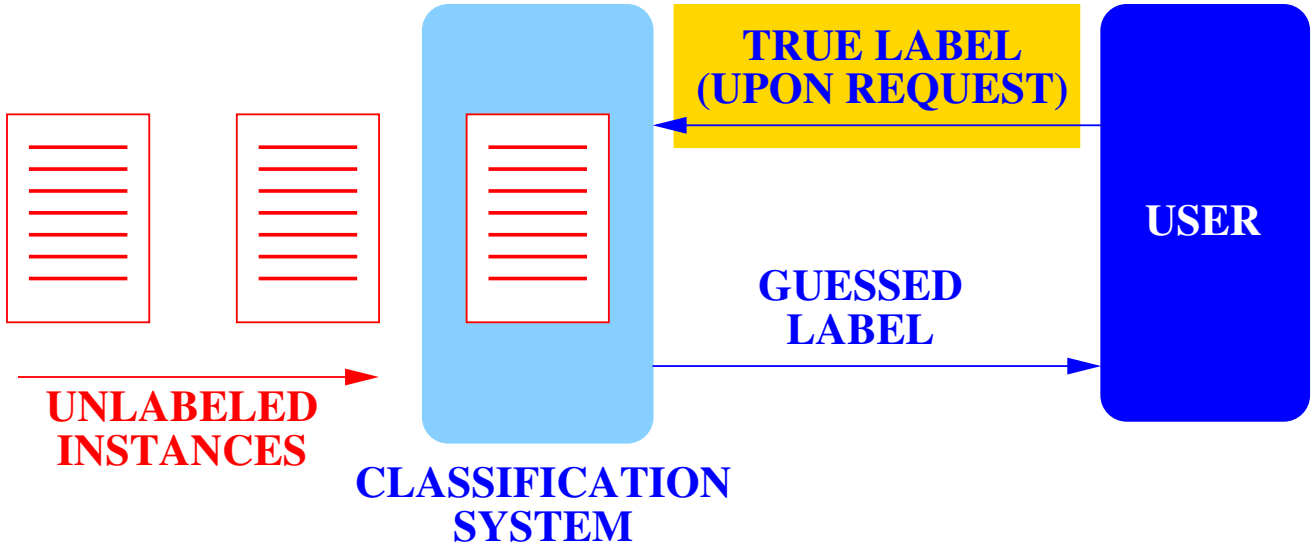
Result

Finds separating $\boldsymbol{u}$ with $\|\boldsymbol{u}\| \leqslant \|\boldsymbol{u}_{\text{SVM}}\| / (1-\alpha)$ after at most $\left(\|\boldsymbol{u}_{\text{SVM}}\| / \alpha\right)^2$ updates

Selective sampling



A selective sampling classifier

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3. If label queried then use (x_t, y_t) to update w

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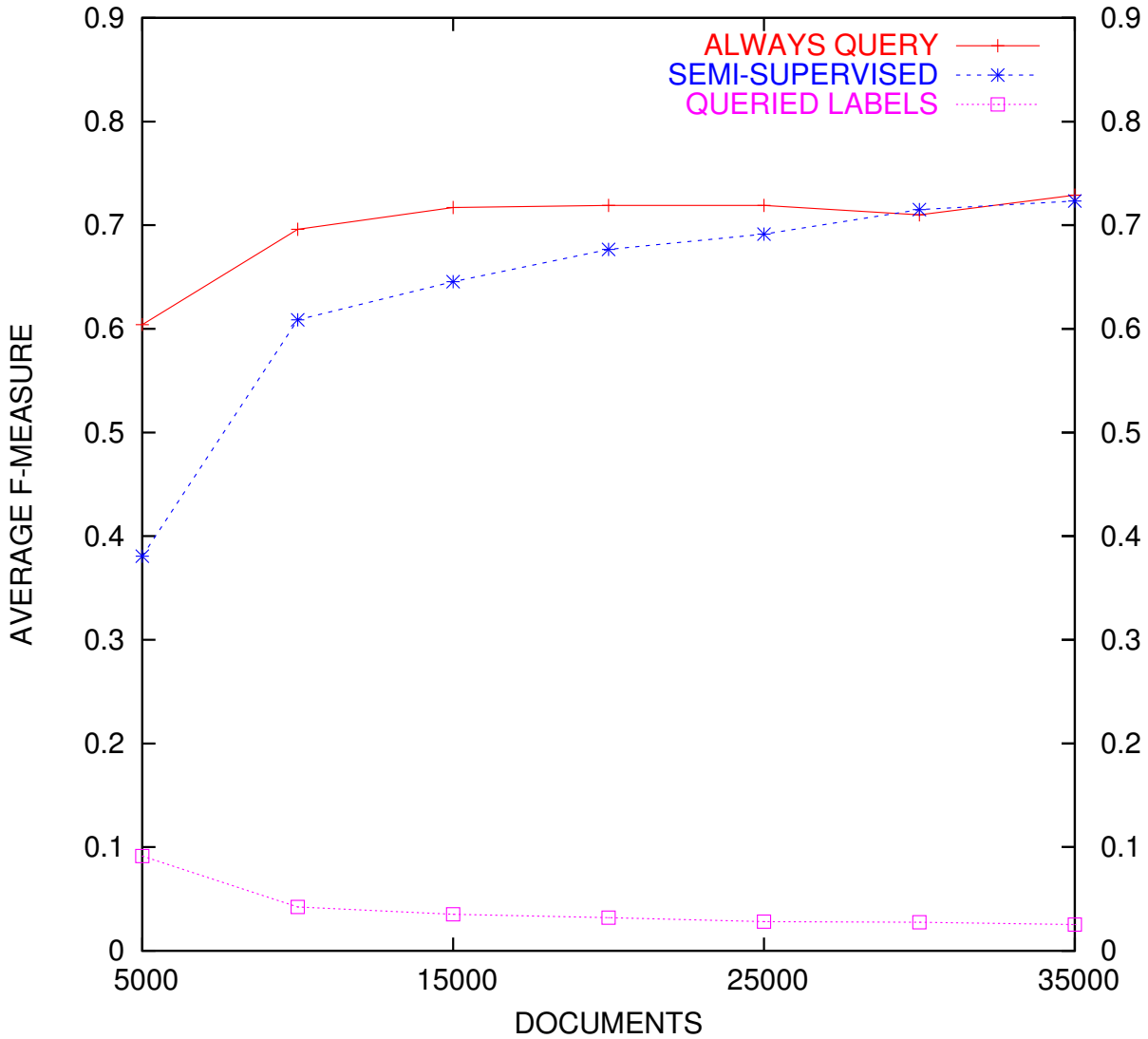
w updated with the 2nd order Perceptron update rule

Empirical performance on RCV1



Summary

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- Learning algorithm

$$(\boldsymbol{x}_1, y_1), \ldots, (\boldsymbol{x}_n, y_n) \longrightarrow \boxed{A} \longrightarrow \widehat{H} : \mathbb{R}^d \to \{-1, +1\}$$

$\widehat{H}$ is (random) hypothesis output by learner

The ensemble of hypotheses

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Goals

1. Bound the average risk of the ensemble in terms of the size of the ensemble
2. Find an element of the ensemble whose risk is close to the ensemble average

Step 1: bound the average risk

The difference

$$\mathtt{risk}(H_{t-1}) - \mathbb{I}_{\{H_{t-1}(x_t) \neq y_t\}}$$

is a martingale difference sequence because

$$\mathbb{E}\left[\mathtt{risk}(H_{t-1}) - \mathbb{I}_{\{H_{t-1}(x_t) \neq y_t\}} \,\middle|\, (x_1, y_1), \ldots, (x_{t-1}, y_{t-1})\right] = 0$$

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The associated martingale is

$$\sum_{t=1}^{n} \left(\texttt{risk}(H_{t-1}) - \mathbb{I}_{\{H_{t-1}(x_t) \neq y_t\}}\right)$$

$$\iff \underbrace{\frac{1}{n}\sum_{t=1}^{n} \texttt{risk}(H_{t-1})}_{\text{average risk}} - \underbrace{\frac{1}{n}\sum_{t=1}^{n} \mathbb{I}_{\{H_{t-1}(x_t) \neq y_t\}}}_{\text{fraction of mistakes}}$$

Bernstein's bound

If $Z_1, Z_2, \ldots$ is a martingale difference sequence with increments bounded by 1 and

$$V_n = \sum_{t=1}^{n} \mathbb{E}\left[Z_t^2 \mid Z_1, \ldots, Z_{t-1}\right]$$

then for all $S, K > 0$

$$\mathbb{P}\left(\sum_{t=1}^{n} Z_n \geqslant S, \quad V_n \leqslant K\right) \leqslant \exp\left(-\frac{S^2}{2(S/3 + K)}\right)$$

Application of Bernstein's bound

Since $0 \leqslant \mathbb{I}_{\{H(x) \neq y\}} \leqslant 1$,

$$\begin{aligned} &\textsc{var}\left[\mathbb{I}_{\{H_{t-1}(x_t), y_t\}} \,\middle|\, (x_1, y_1), \ldots, (x_{t-1}, y_{t-1})\right] \\ &\leqslant \quad \mathbb{E}\left[\mathtt{risk}(H_{t-1}) \,\middle|\, (x_1, y_1), \ldots, (x_{t-1}, y_{t-1})\right] \end{aligned}$$

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Applying Bernstein's gives

$$\frac{1}{n}\sum_{t=1}^{n} \mathtt{risk}(H_{t-1}) \quad \leqslant \quad \frac{M_n}{n} + \frac{c}{n}\left(\ln M_n + \sqrt{M_n \ln M_n}\right) \quad \text{w.h.p.}$$

Where $\dfrac{M_n}{n} = \dfrac{1}{n}\displaystyle\sum_{t=1}^{n} \mathbb{I}_{\{H_{t-1}(x_t) \neq Y_t\}}$ is the fraction of mistakes

Step 2: pick a good classifier in the ensemble

- Start from the ensemble $H_0, H_1, \ldots, H_n$
- Do the following:
  1. test each H_t on $(x_{t+1}, y_{t+1}), \ldots, (x_n, y_n)$
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Guaranteed bound

$$\texttt{risk}(\widehat{H}) \quad \leqslant \quad \frac{M_n}{n} + \frac{c}{n}\left((\ln n)^2 + \sqrt{M_n \ln n}\right) \qquad \text{w.h.p.}$$