Minimally Stochastic Approach to Singular Diffusion Equations

Bernhard Burgeth[1], Joachim Weickert[1], and Sibel Tari[2]

[1] Saarland University, Germany

[2] METU, Turkey

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Motivation

- Popular in nonlinear diffusion filtering: Singular diffusivities (TV, BFB)
  - + good: parameter free models
  - – bad: numerically challenging ($\longrightarrow$ regularisation)
- TV-denoising without regularisation:
  - Guichard & Malgouyres, 1998
  - Dibos & Koepfler, 2000
  - Chambolle, 2004
  - K. Chen & R. Chan, 2005
  - $\cdots\cdots$
- **Goal** of this talk:

  Stochastic approach to singular diffusion filtering with **intrinsic** regularisation, inspired by Ramjan & Ramakrishnan, 1999

Outline:

- Nonlinear Filtering: TV and BFB Diffusion
- Deterministic Two-Pixel Scheme
  - Derivation in 1D: Two Pixel-Module
  - Extension to 2D
- Stochastic Two-Pixel Scheme
  - Stochastic Two-Pixel Module
  - Extension to 2D
- Experimental Results
- Summary

Basic Setting of Nonlinear Diffusion Filters

- Given: Initial data $f : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$
- f is evolved according to the PDE with Neumann boundary conditions:

$$
\begin{aligned}
\partial_t u &= \text{div}\,(g(\|\nabla u\|) \cdot \nabla u) \quad \text{on} \quad \Omega \times (0,\infty)\,, \\
u(x,0) &= f(x) \qquad \text{for all } x \in \Omega\,, \\
\partial_n u(y,t) &= 0 \qquad \text{for all } y \in \partial\Omega \times (0,\infty)\,,
\end{aligned}
$$

  where ∂_n denotes outward normal derivative
- Creates more simplified variants $u(\cdot,t)$ of f the larger t
- g is nonnegative and decreasing function of $|\nabla u|$
- Edge preservation and intra region smoothing possible via appropriate g

- In early diffusion filtering: g is bounded (Perona-Malik, Charbonier, Weickert...)
- Prominent **unbounded** diffusivities:

Total Variation (TV)
(Andreu et al., 1998):

$$g(|\nabla u|) = \frac{1}{|\nabla u|}$$

Properties:

+ finite extinction time
+ shape-preserving qualities
+ equivalence results to TV regularisation (1D) available
+ no tuning parameter

Balanced Forward-Backward (BFB)
(Keeling & Stollberger, 2002):

$$g(|\nabla u|) = \frac{1}{|\nabla u|^2}$$

Properties:

+ edge-enhancement
+ no tuning parameter

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22

- Unbounded diffusivities: Severe **numerical intricacies**
  - explicit schemes: time step size $\tau \sim \frac{1}{\text{bound on } g}$
  - (semi-)implicit schemes:
    condition number of system matrix large for large bounds on g
  - slow convergence
  - potential amplification of numerical errors
- **Regularisation** by $g_\varepsilon(|\nabla u|) = \frac{1}{|\nabla u|+\varepsilon}$ leads to blurring effects

Deterministic Two-Pixel Scheme

- Given: Initial two-pixel signal $f = (f_1, f_2)$
- Apply space-discrete TV/BFB diffusion with Neumann boundary condition:

$$\begin{aligned} \dot{u}_1 &= \frac{g_{1+\frac{1}{2}}}{h^2}(u_2 - u_1) \\ \dot{u}_2 &= -\frac{g_{1+\frac{1}{2}}}{h^2}(u_2 - u_1) \end{aligned}$$

  with initial conditions $u_i(0) = f_i$, $i = 1, 2$

- Discrete approximations g_1, g_2 give

$$g_{1+\frac{1}{2}} := \frac{g_1 + g_2}{2}$$

- $g_{1+\frac{1}{2}}$ assumed constant in time

- Decoupling of (dynamical) system of ODEs:

$$\begin{pmatrix} w_1(t) \\ v_1(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

- Linear first order ODE-solutions

$$\begin{aligned} w_1(t) &= \exp\left(-\frac{2}{h^2}\, g_{1+\frac{1}{2}} \cdot t\right) \left(u_2(0) - u_1(0)\right) \\ v_1(t) &= u_2(0) + u_1(0) \end{aligned}$$

- Yields explicit analytical solution

$$\begin{aligned} u_1(t) &= u_1(0) + \frac{1}{2}\left(1 - \exp\left(-\frac{2t}{h^2} g_{1+\frac{1}{2}}\right)\right) \left(u_2(0) - u_1(0)\right) \\ u_2(t) &= u_2(0) - \frac{1}{2}\left(1 - \exp\left(-\frac{2t}{h^2} g_{1+\frac{1}{2}}\right)\right) \left(u_2(0) - u_1(0)\right) \end{aligned}$$

- Reasoning applied to **pairs of pixels** of a one-dimensional signal, **time-discrete** version, k-th iteration, time step τ:

$$\begin{aligned} u_i^{k+1} &= u_i^k + \frac{1}{2}\left(1 - \exp\left(-\frac{2\tau}{h^2} g_{i+\frac{1}{2}}\right)\right)(u_{i+1}^k - u_i^k) \\ u_{i+1}^{k+1}(t) &= u_{i+1}^k - \frac{1}{2}\left(1 - \exp\left(-\frac{2\tau}{h^2} g_{i+\frac{1}{2}}\right)\right)(u_{i+1}^k - u_i^k) \end{aligned}$$

- Ensures only interaction between neighbouring pixels u_i^k and u_{i+1}^k

Problem: How to transport information to other pixels ?

- **Solution:** Consider signal and a **shifted version**, **apply two-pixel scheme** with time step 2τ, and **average** (AOS):

$$
\begin{aligned}
u_i^{k+1} = u_i^k \quad &+ \quad \frac{1}{4}\left(1-\exp\left(-\frac{4\tau}{h^2}g_{i+\frac{1}{2}}\right)\right)(u_{i+1}^k - u_i^k) \\
&- \quad \frac{1}{4}\left(1-\exp\left(-\frac{4\tau}{h^2}g_{i-\frac{1}{2}}\right)\right)(u_i^k - u_{i-1}^k)
\end{aligned}
$$

- **Note:** Taylor expansion w.r.t. τ of the above **two-pixel module** gives naive, unstable scheme

$$
\begin{aligned}
u_i^{k+1} = u_i^k \quad &+ \quad \frac{\tau}{h^2}g_{i+\frac{1}{2}}(u_{i+1}^k - u_i^k) \\
&- \quad \frac{\tau}{h^2}g_{i-\frac{1}{2}}(u_i^k - u_{i-1}^k)
\end{aligned}
$$

- The exponential scheme **remains stable** for large values of g or τ, however, **looses consistency** (degenerates to averaging process).

How about extensions to 2D-setting ?

Two-Dimensional Case

- Obtained by similar reasoning in 2D:

$$
\begin{aligned}
u_{i,j}^{k+1} = u_{i,j}^{k} \quad &+ \quad \frac{1}{8}\left(1-\exp\left(-\frac{8\tau}{h^2}\,g_{i+\frac{1}{2},j}\right)\right)\left(u_{i+1,j}^{k}-u_{i,j}^{k}\right) \\
&+ \quad \frac{1}{8}\left(1-\exp\left(-\frac{8\tau}{h^2}\,g_{i-\frac{1}{2},j}\right)\right)\left(u_{i-1,j}^{k}-u_{i,j}^{k}\right) \\
&+ \quad \frac{1}{8}\left(1-\exp\left(-\frac{8\tau}{h^2}\,g_{i,j+\frac{1}{2}}\right)\right)\left(u_{i,j+1}^{k}-u_{i,j}^{k}\right) \\
&+ \quad \frac{1}{8}\left(1-\exp\left(-\frac{8\tau}{h^2}\,g_{i,j-\frac{1}{2}}\right)\right)\left(u_{i,j-1}^{k}-u_{i,j}^{k}\right)
\end{aligned}
$$

- This scheme
  - is explicit, unconditionally stable
  - is applicable to PDEs with degenerated diffusivities
  - becomes conditionally consistent for large $\tau \cdot g_{i+\frac{1}{2}}$
- Generalisations to 4-pixels schemes possible (Welk et al., 2005)

Minimally Stochastic Approach:

- Introduce randomness by **stochastic rounding** in the exponential two-pixel module
- Stochastic rounding function $SR : \mathbb{R} \longrightarrow \mathbb{N}$ with integer-part-function $[x]$

$$SR(x) = \begin{cases} [x] & \text{with probability} \quad 1 - |[x] - x| \\ [x] + 1 & \text{with probability} \quad |[x] - x| \end{cases}$$

- Example:

$$SR(2,7) = \begin{cases} 2 & \text{with probability} \quad 0,3 \\ 3 & \text{with probability} \quad 0,7 \end{cases}$$

- **Randomised two-pixel module** define by

$$u_m^{k+1} = u_m^k \quad + \quad \omega$$

$$u_n^{k+1} = u_n^k \quad - \quad \omega$$

with $\omega := SR\left\{\left(1 - \exp\left(-\frac{2\tau}{h^2}(g_n^k + g_m^k)\right)\right)\frac{u_n^k - u_m^k}{2}\right\}$

- If (f_i) is integer valued then scheme requires **integer arithmetic** only.
- Diffusivities are unbounded, $g_m^k = \infty$ or $g_n^k = \infty$, we set $\omega = SR\left(\frac{u_n^k - u_m^k}{2}\right)$
- Implementational **modification**:

$$\omega = \begin{cases} SR\left\{\left(1 - \exp\left(-\frac{4\tau}{h^2}\left(\frac{1}{g_n^k} + \frac{1}{g_m^k}\right)^{-1}\right)\right)\frac{u_n^k - u_m^k}{2}\right\} & \text{for } \frac{1}{g_m^k}, \frac{1}{g_n^k} > 0\,, \\ SR\left(\frac{u_n^k - u_m^k}{2}\right) & \text{for } \frac{1}{g_m^k}, \frac{1}{g_n^k} = 0 \end{cases}$$

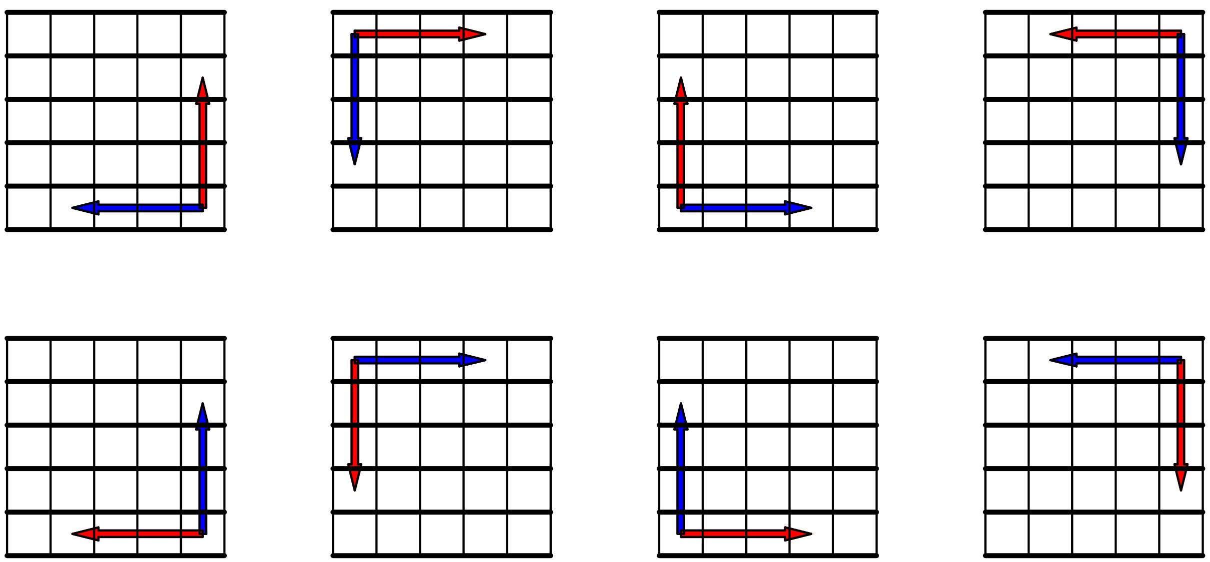
How to apply the two-pixel scheme to the whole image domain ?

- Apply the stochastic two-pixel module to **overlapping** pairs of pixels
- **Average** the results (as in the deterministic case)

But where to start ?

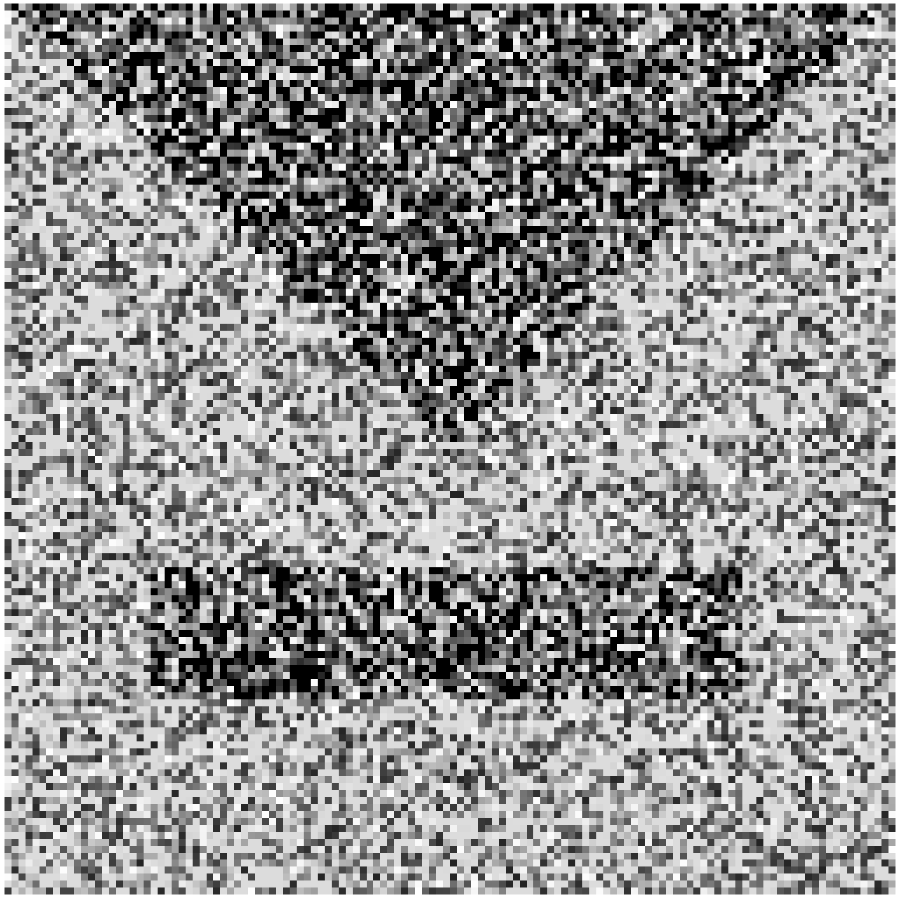
How to apply the two-pixel scheme to the whole image domain ?

- Apply the stochastic two-pixel module to **overlapping** pairs of pixels
- **Average** the results (as in the deterministic case)
- Choose **randomly** one of the starting points (indicated below)
  - proceed first in blue direction
  - then in red direction



- Random selection of starting position for each cycle avoids bias
- Minimally stochastic scheme obeys **maximum-minimum principle** but
- Allows for one-pixel fluctuations

Diffusion filtering, experimental setup



128×128 test image
70% uniform noise

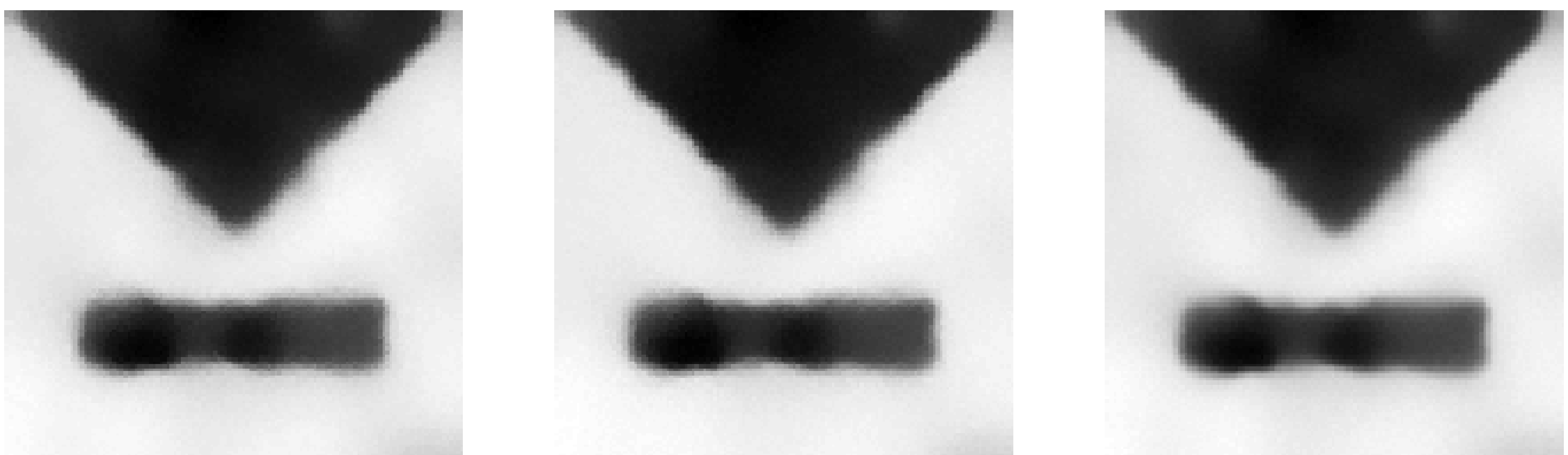
256×256 test image
no additional noise

Experimental Results (1)

TV diffusion filtering, stopping time: 100

Top: Deterministic exponential scheme

$\tau = 0.01$ $\tau = 0.1$ $\tau = 1$

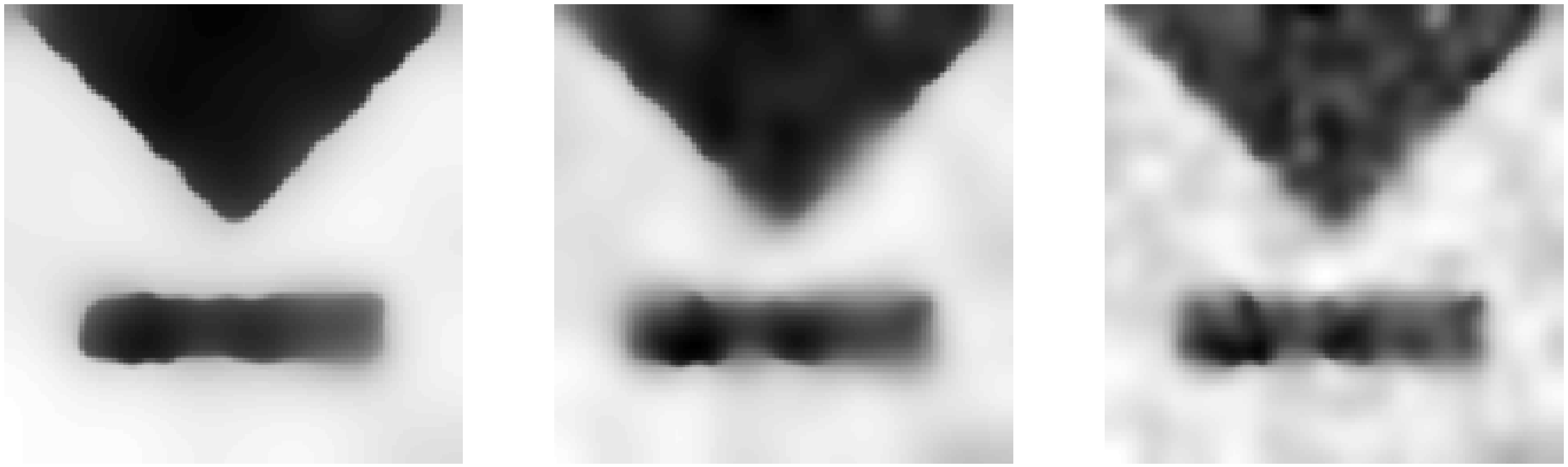


Bottom: Minimally stochastic scheme

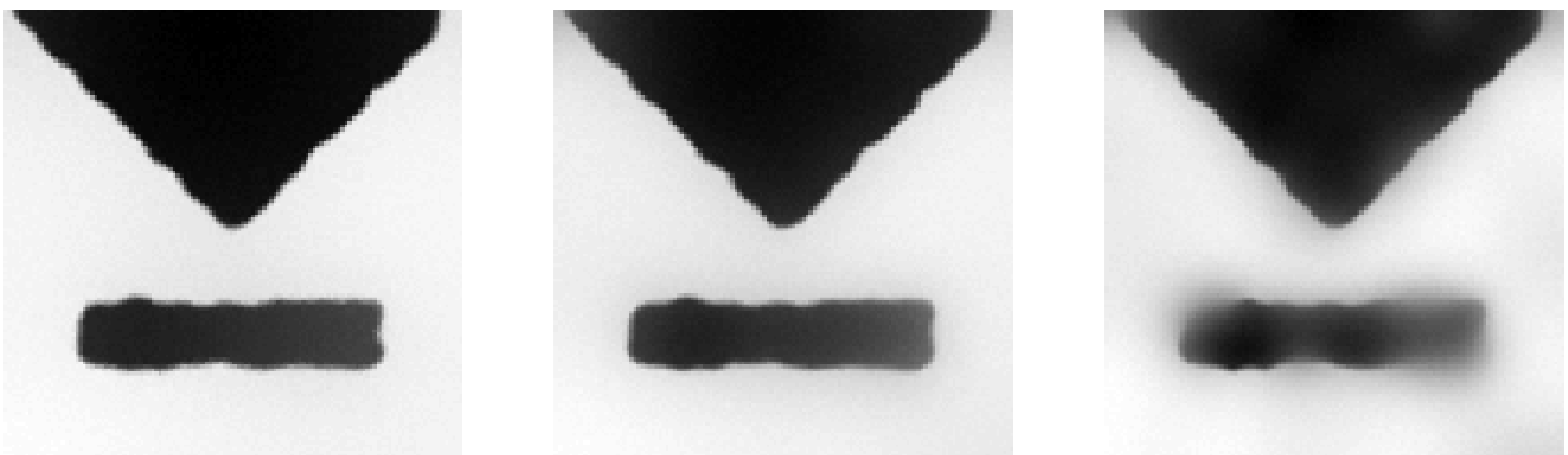
Experimental Results (2)

BFB diffusion filtering, stopping time: 3000

Top: Deterministic exponential scheme



$\tau = 3$ $\tau = 10$ $\tau = 30$

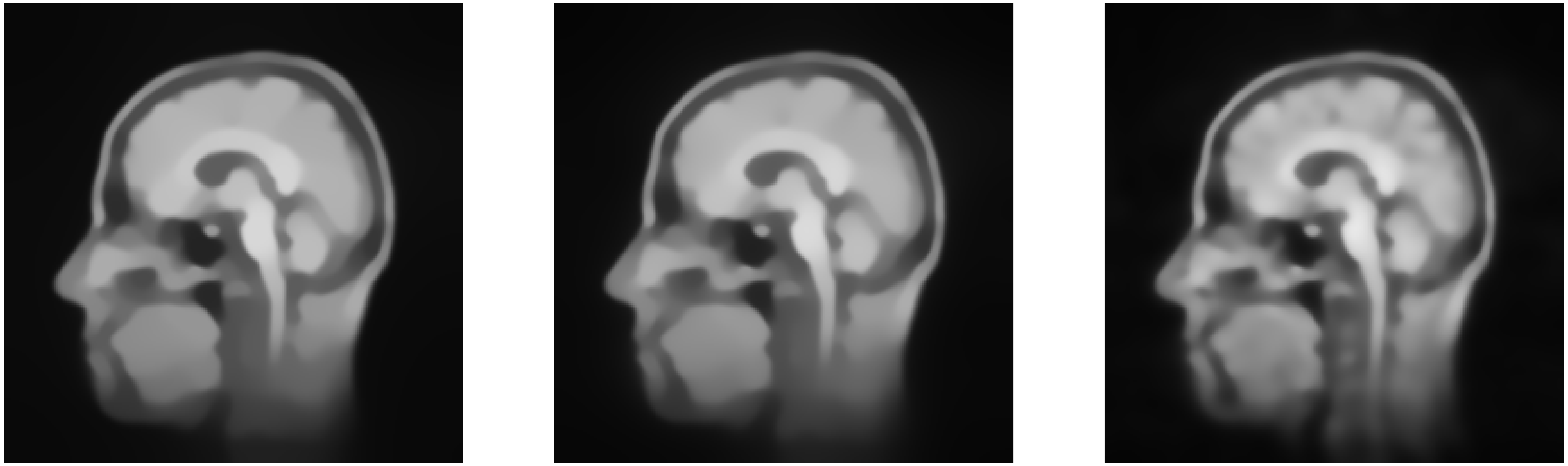


Bottom: Minimally stochastic scheme

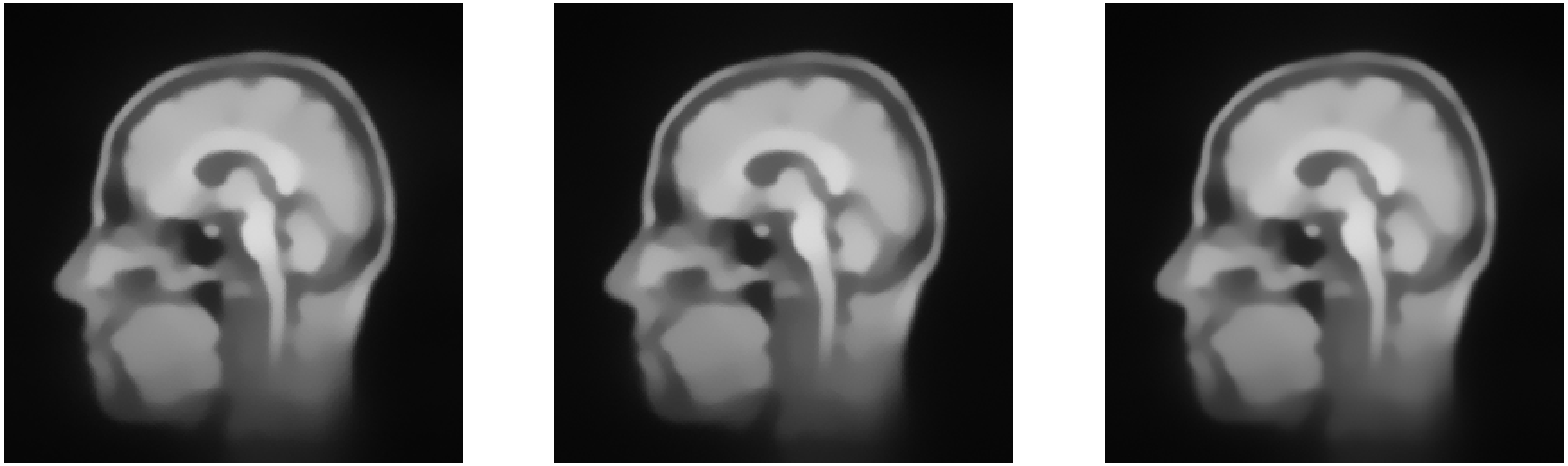
Experimental Results (3)

TV diffusion filtering, stopping time: 100

Top: Deterministic exponential scheme



$\tau = 0.01$ $\tau = 0.1$ $\tau = 1$

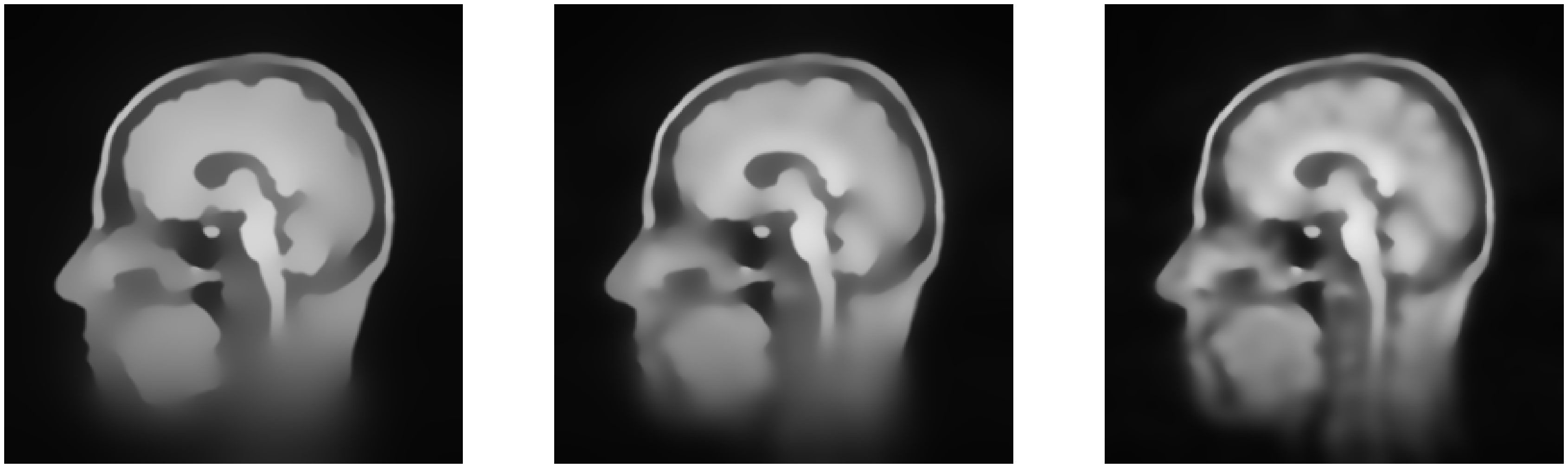


Bottom: Minimally stochastic scheme

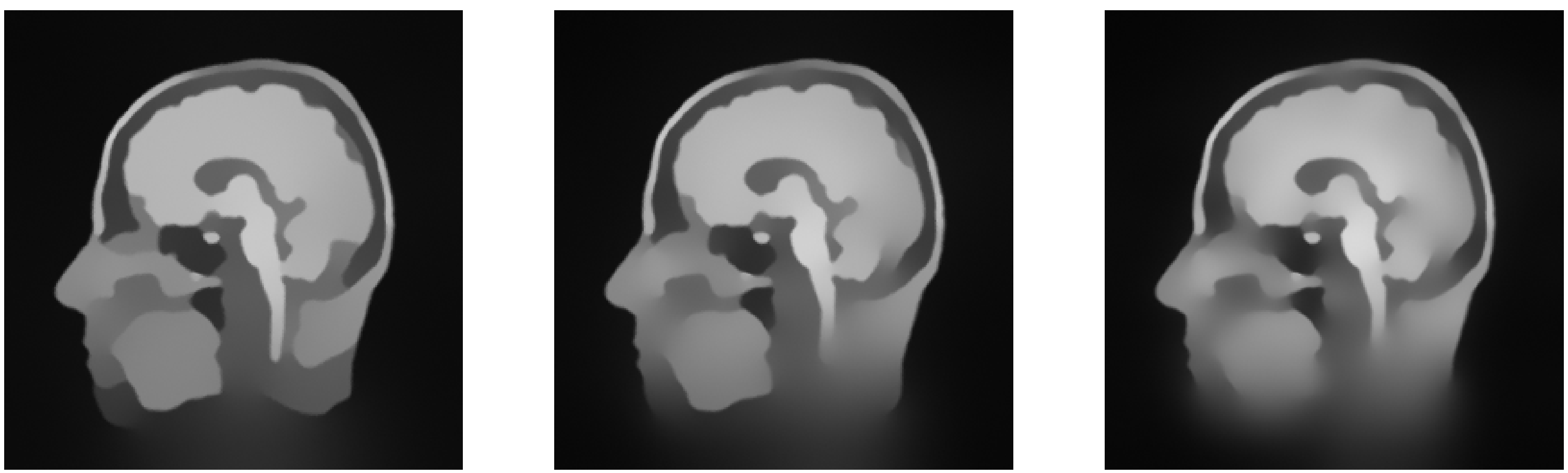
Experimental Results (4)

BFB diffusion filtering, stopping time: 3000

Top: Deterministic exponential scheme



$\tau = 3$ $\tau = 10$ $\tau = 30$



Bottom: Minimally stochastic scheme

Summary:

- Considered semi-discrete approach to singular diffusion
- Introduced two-pixel scheme
  - dynamical system allows analytical solution
  - yields stable scheme
  - extendable to 2D (in spirit of AOS)
- Stochastic rounding introduces (minimal) randomness
  - allows for **integer aritmethics** (realisable in hardware)
  - stochsticity has **regularising effect**
- Extensions to 2D (in stochastic or deterministic manner)
- Numerical examples demonstrate: Minimally stochastic approach is competitive, approx. 10 times faster than deterministic scheme.

- B. Burgeth, J. Weickert, S. Tari: Minimally stochastic schemes for singular diffusion equations. In X.-C. Tai, K.-A. Lie, T. F. Chan, S. Osher (Eds.): Image Processing Based on Partial Differential Equations, Springer, Berlin, 2006.

Thank you very much for your attention !