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Locally Analytic Schemes for Diffusion Filtering of Images

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joint work with

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Goals of This Talk

- ◆ Focus on diffusion PDEs with applications in image denoising and enhancement
 - singular diffusion processes including total variation (TV) flow, balanced forward-backward (BFB) diffusion
 - anisotropic diffusion processes like edge-enhancing diffusion (EED), coherence-enhancing diffusion (CED)
- ◆ Introduce a class of numerical methods for diffusion PDEs providing
 - favourable stability properties
 - excellent preservation of important image structures
 - simple implementation
- ◆ Compare with wavelet shrinkage
- ◆ Derive new shrinkage rules for wavelet shrinkage inspired by diffusion, which achieve particularly good rotation invariance

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Related Numerical Work

- ◆ **Locally analytic schemes:**

Richardson et al. 1993

- ◆ **Alternative schemes for TV denoising:**

Dibos/Koepfler 2000, Combettes 2002, Chambolle 2004, Fu et al. 2005, Yin et al. 2005

- ◆ **Rotationally invariant schemes for anisotropic diffusion:**

Weickert/Scharr 2002, Wang 2004

- ◆ **Anisotropic/geometric wavelet concepts:**

Donoho 2000, Candés/Donoho 2000, Do/Vetterli 2003, Le Pennec/Mallat 2004

Related Work (2)

Related Work on Wavelet–Diffusion Connections

- ◆ **Continuous connections between wavelets and PDEs:**

Chambolle et al. 1998, Cohen et al. 2000, Bao/Krim 2001, Chambolle/Lucier 2001, Meyer 2001, Shen 2003, Bredies et al. 2004

- ◆ **Discrete connections:**

Coifman/Sowa 2001

- ◆ **Hybrid methods combining wavelet shrinkage with TV/PDE approaches:**

Chan/Zhou 2000, Durand/Froment 2001, Malgouyres 2001, Candes/Guo 2002, Durand/Nikolova 2003, Liu/Ruan 2007

Analytic Solutions for Related Processes

- ◆ **Analytic solutions for TV regularisation:**

Strong 1997, Mammen/van de Geer 1997, Grassmaier 2005

- ◆ **Equivalence results between TV diffusion and TV regularisation:**

Pollak et al. 2005

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Outline

1. Diffusion Filters and Wavelet Shrinkage
2. One-Dimensional Filters
3. Isotropic Two-Dimensional Filters
4. Anisotropic Two-Dimensional Filters
5. Higher-Dimensional and Multi-Channel Images
6. Summary and Outlook

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Diffusion Processes as Image Filters

Diffusion processes can be used to denoise images.

◆ Linear diffusion

$$u_t = \Delta u = \operatorname{div}(\nabla u)$$

Simplest method, however, undesired blurring of structures.

◆ Nonlinear isotropic diffusion

$$u_t = \operatorname{div}(g(|\nabla u_\sigma|)\nabla u)$$

with decreasing nonnegative diffusivity function g and $u_\sigma := K_\sigma * u$ (Gaussian convolution).

Allows adaptivity to structures, e.g. preservation of edges.

◆ Nonlinear anisotropic diffusion

$$u_t = \operatorname{div}(D(\nabla u_\sigma)\nabla u)$$

with diffusion tensor D depending on image gradients.

Can be designed, e.g., to enhance edges, or line-like structures, etc.

Diffusion Examples

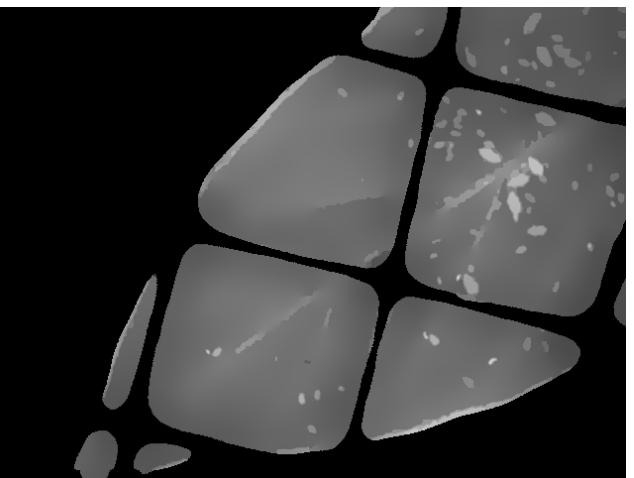
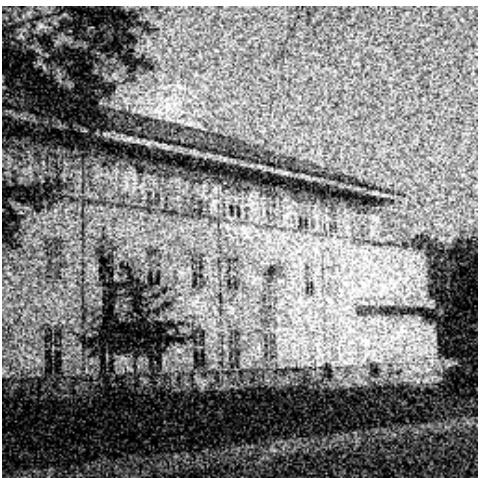


Image denoising with nonlinear diffusion methods. **Top left:** Photograph degraded by noise, 256×256 pixels; **Bottom left:** Denoised by total variation flow, a nonlinear isotropic diffusion method. **Top centre:** Photograph of a plant leave with aphids. **Bottom centre:** Nonlinear isotropic diffusion can be used in the detection of the aphids. Here, diffusion is applied to the green channel; the grid has been masked out using information from the blue channel. **Top right:** Fingerprint image, 100×100 pixels. **Bottom right:** Image enhanced by coherence-enhancing diffusion, an anisotropic diffusion method.

Total Variation (TV) Diffusion

- ◆ Simplify continuous image $f(x)$ under the diffusion equation

$$u_t = \operatorname{div} (g(|\nabla u|) \nabla u)$$

with homogeneous Neumann boundary conditions and diffusivity $g(|\nabla u|) = \frac{1}{|\nabla u|}$

- ◆ **Nice in theory:**

- Adaptive diffusivity $1/|\nabla u|$ becomes 0 at ideal edges and ∞ in ideal flat regions
- No parameters besides diffusion time t
- Flattens every signal in finite time (Andreu et al. 2001)
- Preserves shape of some objects (Bellettini et al. 2002)

- ◆ **Difficult in practice:**

- Unbounded diffusivity requires infinitesimally small time steps or creates very ill-conditioned linear systems
- Usually approximated by a model with bounded diffusivity, like

$$u_t = \operatorname{div} \left(\frac{1}{\sqrt{\varepsilon^2 + |\nabla u|^2}} \nabla u \right)$$

Wavelet Shrinkage

Wavelet Shrinkage

Fast denoising technique using localised orthonormal basis functions
(Donoho/Johnstone 1994).

◆ **Analysis step:** Represent signal/image with respect to some wavelet basis.

- Scaling coefficient c_0 , referring to coarse-scale scaling function φ_0
- Wavelet coefficients $d_{i,j}$, belonging to wavelet basis functions $\psi_{i,j}$

◆ **Shrinkage step:** Apply a **shrinkage function** S_θ to move the wavelet coefficients towards zero:

$$\tilde{d}_{i,j} := S_\theta(d_{i,j}) , \quad |S_\theta(d)| < |d|$$

- Removes small coefficients which are assumed to represent noise

◆ **Synthesis step:** Reconstruct filtered signal from scaling coefficient c_0 and modified wavelet coefficients $\tilde{d}_{i,j}$



Wavelet Shrinkage

Wavelet Shrinkage

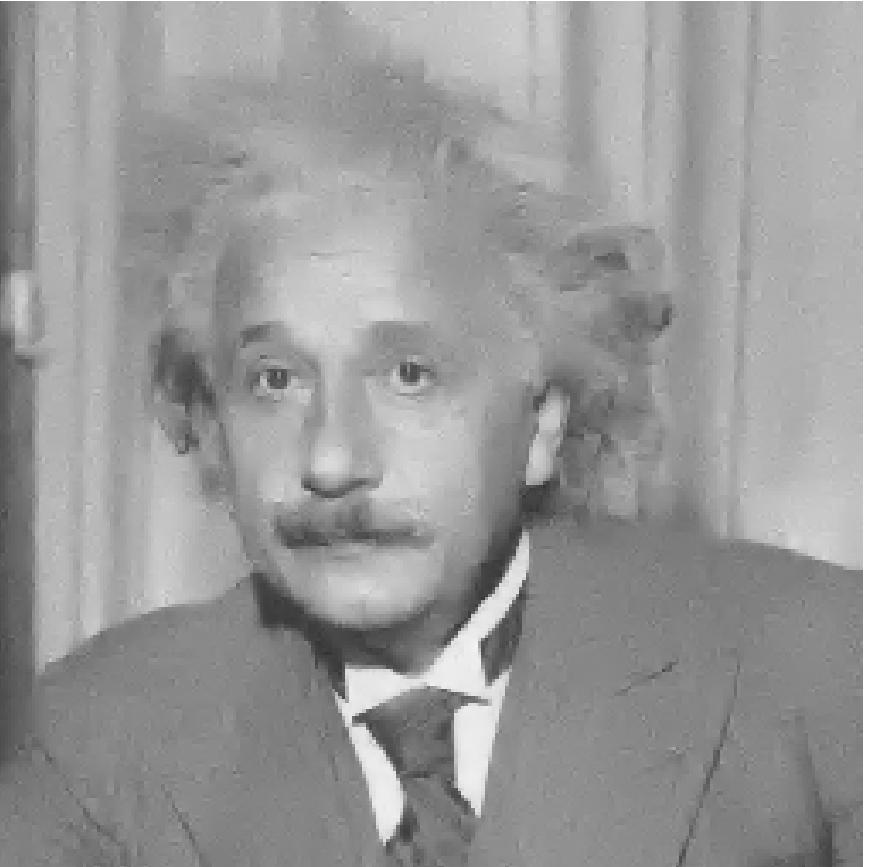
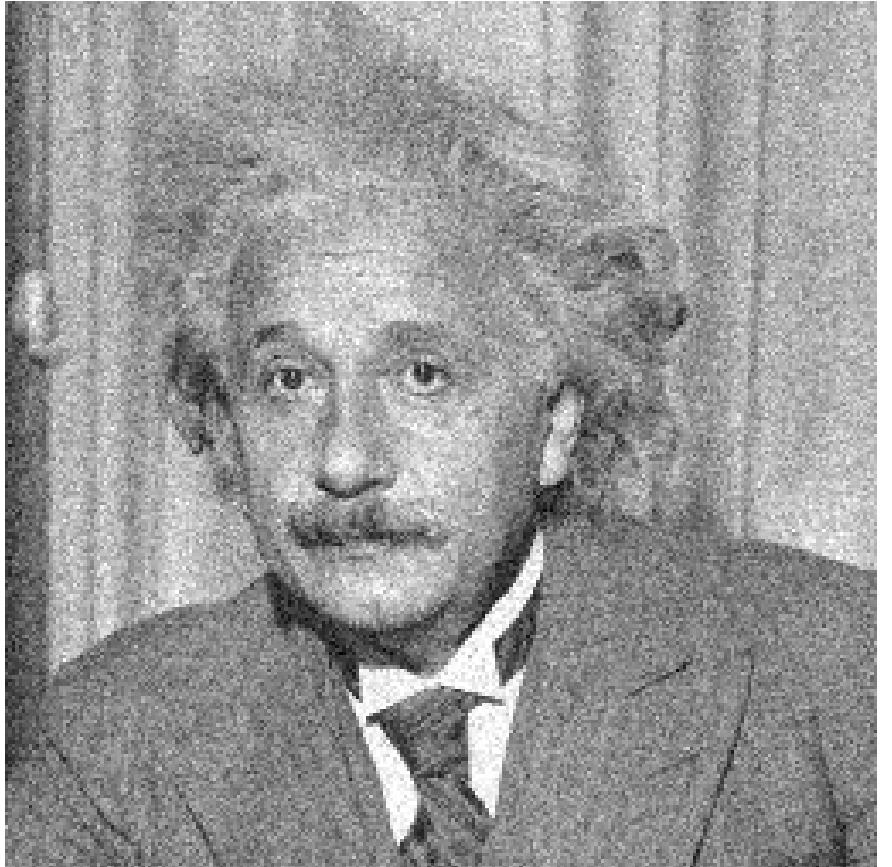


Image denoising with wavelet shrinkage. **Left:** Image of Einstein (256×256 pixels), degraded by Gaussian noise with $\sigma = 17.4$. **Right:** Result after using shift invariant soft wavelet shrinkage with Haar wavelets and threshold $\theta = 30$.

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TV Diffusion on Two Pixels

- ◆ In 1D, total variation flow simplifies to

$$u_t = \partial_x \left(\frac{u_x}{|u_x|} \right) .$$

- ◆ **Space discretisation** with mesh size 1 gives:

$$\dot{u}_0 = \operatorname{sgn}(u_1 - u_0), \quad \dot{u}_1 = -\operatorname{sgn}(u_1 - u_0)$$

with initial conditions $u_0(0) = f_0$ and $u_1(0) = f_1$.

- ◆ **Analytic solution**

$$u_0(t) = \begin{cases} f_0 + t \operatorname{sgn}(f_1 - f_0), & t < |f_1 - f_0|/2, \\ (f_0 + f_1)/2 & \text{else.} \end{cases}$$

$$u_1(t) = \begin{cases} f_1 - t \operatorname{sgn}(f_1 - f_0), & t < |f_1 - f_0|/2, \\ (f_0 + f_1)/2 & \text{else.} \end{cases}$$

- ◆ **Finite extinction time** is obvious in the two-pixel model
- ◆ **No numerical problems** with singular diffusivities

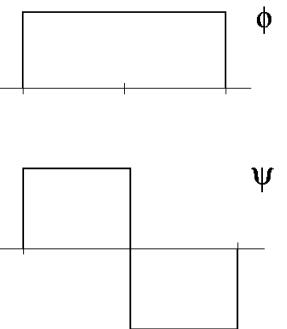
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The Two-Pixel Case: Haar Wavelet Soft Shrinkage

◆ Analysis step of signal (f_0, f_1) in the Haar basis:

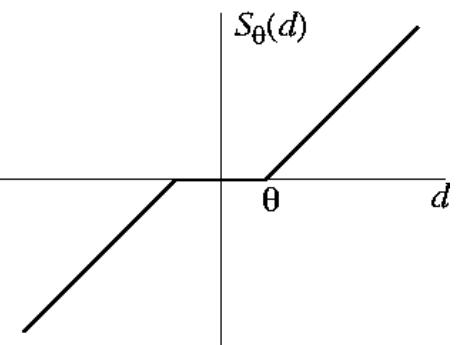
scaling coefficient for $\phi = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$: $c = \frac{f_0 + f_1}{\sqrt{2}}$

wavelet coefficient for $\psi = (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$: $d = \frac{f_0 - f_1}{\sqrt{2}}$



◆ Soft thresholding of the wavelet coefficient:

$$S_\theta(d) = \begin{cases} d - \theta \operatorname{sgn} d, & |d| > \theta, \\ 0, & |d| \leq \theta. \end{cases}$$



◆ Synthesis step:

coefficients c and $S_\theta(d)$ yield filtered signal (u_0, u_1) with

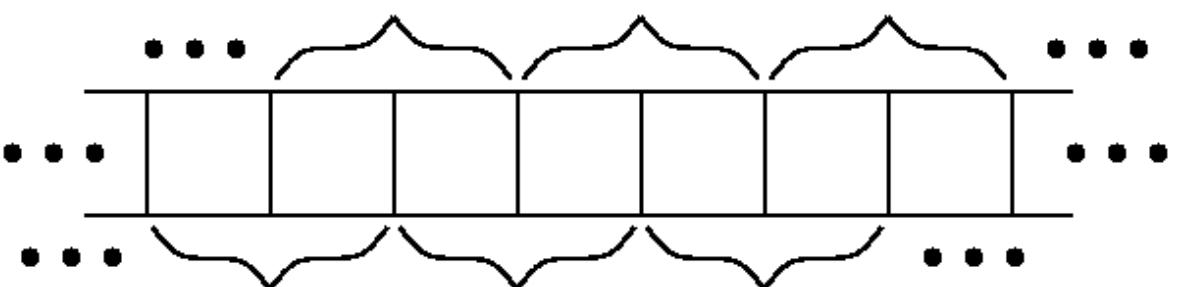
$$u_0(\theta) = \begin{cases} f_0 + \frac{\theta}{\sqrt{2}} \operatorname{sgn} (f_1 - f_0), & \theta < |f_1 - f_0|/\sqrt{2}, \\ (f_0 + f_1)/2 & \text{else.} \end{cases}$$

$$u_1(\theta) = \begin{cases} f_1 - \frac{\theta}{\sqrt{2}} \operatorname{sgn} (f_1 - f_0), & \theta < |f_1 - f_0|/\sqrt{2}, \\ (f_0 + f_1)/2 & \text{else.} \end{cases}$$

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N-Pixel Case: Shift-Invariant Haar Wavelet Shrinkage

- ◆ Perform wavelet decomposition on the finest scale only.
- ◆ Haar wavelets create natural two-pixel pairings.
- ◆ However, the wavelet shrinkage is not shift invariant.
- ◆ Remedy to create translation invariance:
cycle spinning (Coifman/Donoho 1995)
 - Perform shrinkage on the original signal
 - Shift signal by 1 pixel, perform shrinkage, shift back
 - Average both filtered signals



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N-Pixel Case: TV Diffusion

A Locally Analytic Scheme (LAS) for TV Diffusion in 1D:

- ◆ Use analytic solution of the two-pixel model for constructing a numerical scheme for TV diffusion
- ◆ Construct a numerical scheme that is equivalent to shift invariant Haar wavelet shrinkage on a single scale:
 - Perform TV diffusion on all even–odd pairs (u_{2i}, u_{2i+1})
 - Perform TV diffusion on all odd–even pairs (u_{2i-1}, u_{2i})
 - Average both results
- ◆ This gives the following numerical scheme:

$$\frac{u_i^{k+1} - u_i^k}{\tau} = \frac{1}{h} \operatorname{sgn} (u_{i+1}^k - u_i^k) \min \left(1, \frac{h}{4\tau} |u_{i+1}^k - u_i^k| \right)$$

$$- \frac{1}{h} \operatorname{sgn} (u_i^k - u_{i-1}^k) \min \left(1, \frac{h}{4\tau} |u_i^k - u_{i-1}^k| \right).$$

N -Pixel Case: TV Diffusion

- ◆ Explicit scheme
- ◆ **Absolute stability:** satisfies extremum principle

$$\min_j f_j \leq u_i^{k+1} \leq \max_j f_j$$

- ◆ **Conditional consistency:** $O(\tau + h^2)$ approximation to the continuous TV diffusion for

$$\tau \leq \frac{h}{4} \min(|u_{i+1}^k - u_i^k|, |u_i^k - u_{i-1}^k|).$$

for large τ : approximates a linear diffusion process

- ◆ Avoids regularised TV diffusivity, as in

$$u_t = \partial_x \left(\frac{1}{\sqrt{\varepsilon^2 + u_x^2}} u_x \right)$$

- ◆ Competitive performance

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Nonlinear Isotropic Diffusion in Two Dimensions

Specific Discretisation

- ◆ Consider regular grid with spatial mesh size 1 in both directions.
- ◆ Discretise gradient magnitudes $|\nabla u|$ and thus diffusivity g in *cell midpoints* $(i + \frac{1}{2}, j + \frac{1}{2})$, i, j integers, yielding values $g_{i+\frac{1}{2},j+\frac{1}{2}}$.
- ◆ The nonlinear diffusion process

$$u_t = \operatorname{div}(g(\cdot) \nabla u)$$

can then be discretised as

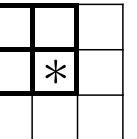
$$\begin{aligned}\dot{u}_{i,j} = & \frac{1}{4} g_{i-\frac{1}{2},j-\frac{1}{2}} \cdot (u_{i-1,j-1} + u_{i-1,j} + u_{i,j-1} - 3u_{i,j}) \\ & + \frac{1}{4} g_{i+\frac{1}{2},j-\frac{1}{2}} \cdot (u_{i+1,j-1} + u_{i+1,j} + u_{i,j-1} - 3u_{i,j}) \\ & + \frac{1}{4} g_{i-\frac{1}{2},j+\frac{1}{2}} \cdot (u_{i-1,j+1} + u_{i-1,j} + u_{i,j+1} - 3u_{i,j}) \\ & + \frac{1}{4} g_{i+\frac{1}{2},j+\frac{1}{2}} \cdot (u_{i+1,j+1} + u_{i+1,j} + u_{i,j+1} - 3u_{i,j})\end{aligned}$$

Nonlinear Isotropic Diffusion in Two Dimensions

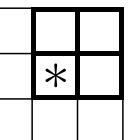
Specific Discretisation

- ◆ Discretisation decomposes into contributions from four 2×2 -pixel patches:

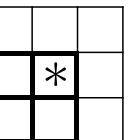
$$\dot{u}_{i,j} = \frac{1}{4} g_{i-\frac{1}{2},j-\frac{1}{2}} \cdot (u_{i-1,j-1} + u_{i-1,j} + u_{i,j-1} - 3u_{i,j})$$



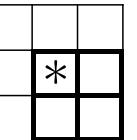
$$+ \frac{1}{4} g_{i+\frac{1}{2},j-\frac{1}{2}} \cdot (u_{i+1,j-1} + u_{i+1,j} + u_{i,j-1} - 3u_{i,j})$$



$$+ \frac{1}{4} g_{i-\frac{1}{2},j+\frac{1}{2}} \cdot (u_{i-1,j+1} + u_{i-1,j} + u_{i,j+1} - 3u_{i,j})$$



$$+ \frac{1}{4} g_{i+\frac{1}{2},j+\frac{1}{2}} \cdot (u_{i+1,j+1} + u_{i+1,j} + u_{i,j+1} - 3u_{i,j})$$



- ◆ If $g = g(|\nabla u|^2)$ (no pre-smoothing) and $g = \Psi'$, this space-discrete evolution equation is also obtained as gradient descent for the energy

$$E(u) = \sum_{i,j} \Psi(|\nabla u|^2)_{i+\frac{1}{2},j+\frac{1}{2}}$$

where $(|\nabla u|^2)_{i+\frac{1}{2},j+\frac{1}{2}}$ is a suitable discretisation of $|\nabla u|$ at location $(i + \frac{1}{2}, j + \frac{1}{2})$.

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Nonlinear Isotropic Diffusion in Two Dimensions

Dynamical System on an Isolated Four-Pixel Cell

Decomposition gives rise to splitting of the space-discrete system into dynamical systems on four-pixel cells:

$$\dot{u}_{1,1} = g_{\frac{3}{2}, \frac{3}{2}} \cdot (-3u_{1,1} + u_{2,1} + u_{1,2} + u_{2,2})$$

$$\dot{u}_{2,1} = g_{\frac{3}{2}, \frac{3}{2}} \cdot (u_{1,1} - 3u_{2,1} + u_{1,2} + u_{2,2})$$

$$\dot{u}_{1,2} = g_{\frac{3}{2}, \frac{3}{2}} \cdot (u_{1,1} + u_{2,1} - 3u_{1,2} + u_{2,2})$$

$$\dot{u}_{2,2} = g_{\frac{3}{2}, \frac{3}{2}} \cdot (u_{1,1} + u_{2,1} + u_{1,2} - 3u_{2,2})$$

Idea: Approximate these four-pixel systems separately and average results.

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Singular Isotropic Diffusion on Four Pixels

Analytic Solution of Singular Diffusion for Four-Pixel Images

- ◆ Consider the singular diffusivities

$$g(|\nabla u|) = \frac{1}{|\nabla u|^p},$$

e.g.

- TV diffusion for $p = 1$
- balanced forward–backward (BFB) diffusion for $p = 2$.
- ◆ Let 2×2 pixel image $(f_{i,j})$ with $i, j \in \{1, 2\}$ be given as initial condition.
- ◆ **Analytic solution** of the four-pixel system evolves towards $\mu := \frac{f_{1,1} + f_{2,1} + f_{1,2} + f_{2,2}}{4}$:

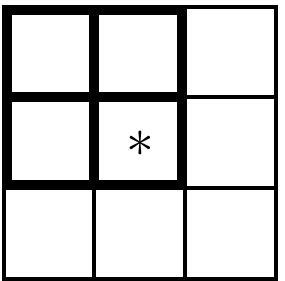
$$u_{i,j}(t) = \begin{cases} \mu + \sqrt[p]{1 - \frac{4pt}{\delta(f)^p}} (f_{i,j} - \mu), & 0 \leq t < \frac{\delta(f)^p}{4p}, \\ \mu, & t \geq \frac{\delta(f)^p}{4p}. \end{cases}$$

Singular Isotropic Diffusion in Two Dimensions

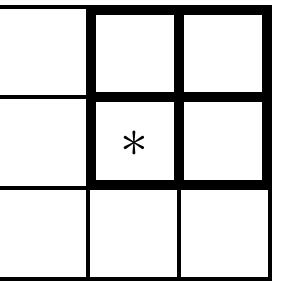
A Locally Analytic Scheme (LAS) for Singular Isotropic Diffusion in 2D

- ◆ For each time step:

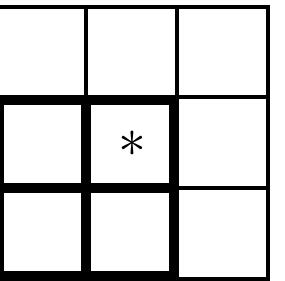
- ▶ Consider the four 2×2 -cells containing some pixel (i, j)



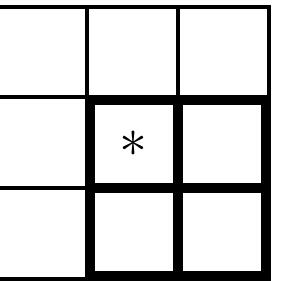
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- ▶ Compute their analytic solutions

- ▶ Average the four results for (i, j)

◆ **No regularisation** of singular diffusivities

◆ **Absolutely stable:** satisfies extremum principle

◆ **Conditionally consistent**

◆ **Good sharpness at edges**, even for large time steps

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Example: TV Diffusion



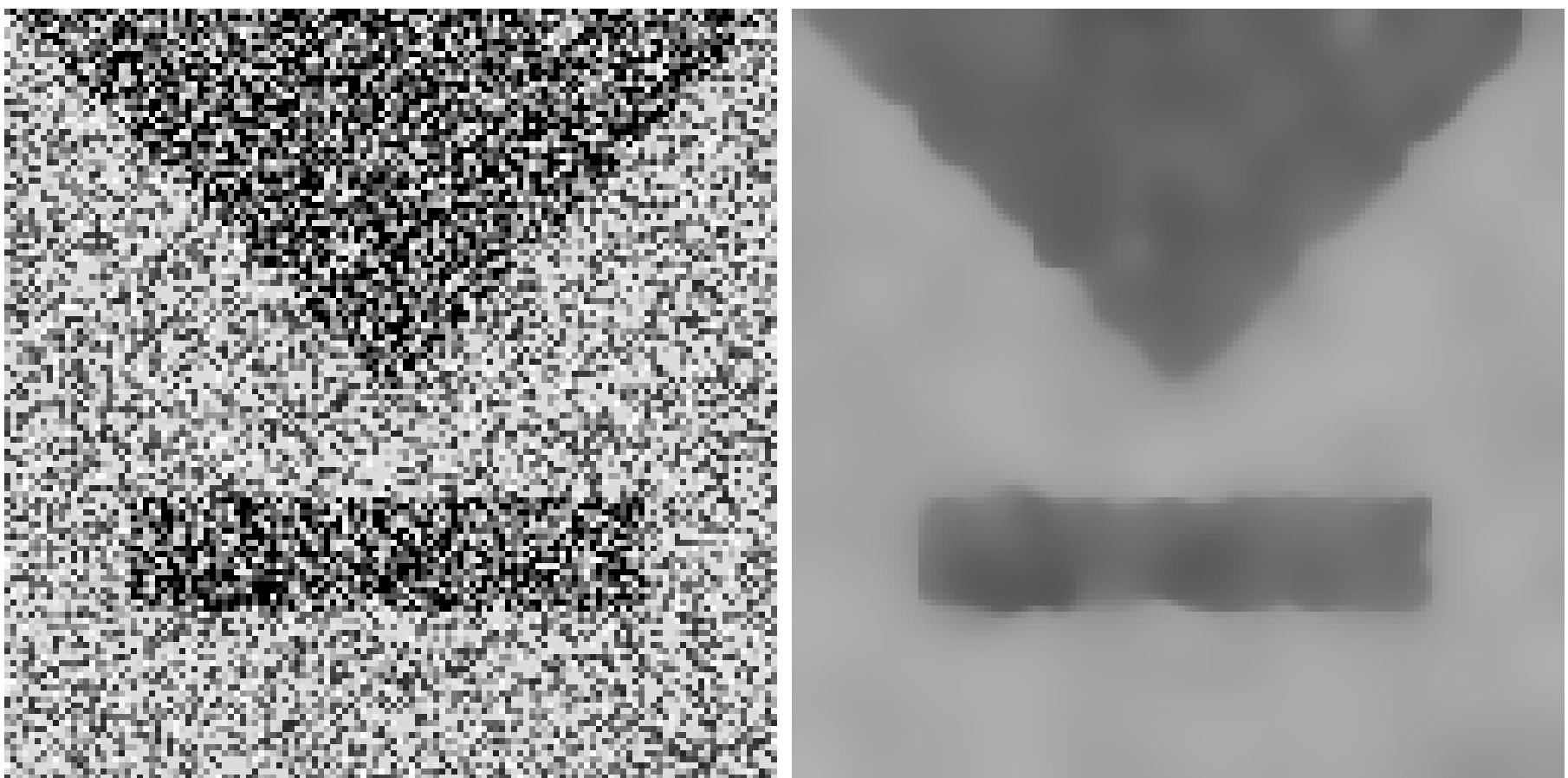
Comparison of schemes for TV diffusion. **Left:** Original image, 93×93 pixels. **Middle:** Standard explicit scheme for regularised TV diffusion ($\varepsilon = 0.01$, $\tau = 0.0025$, 40000 iterations). **Right:** Same with LAS scheme without regularisation ($\tau = 0.1$, 1000 iterations). Note that 40 times larger time steps are used.

Example: BFB Diffusion



Comparison of schemes for BFB diffusion. **Left:** Original image, 93×93 pixels. **Middle:** Standard explicit scheme for regularised BFB diffusion ($\varepsilon = 0.1$, $\tau = 0.0025$, 160000 iterations). **Right:** Same with LAS scheme without regularisation ($\tau = 0.1$, 4000 iterations). Note that 40 times larger time steps are used.

Example: TV Diffusion

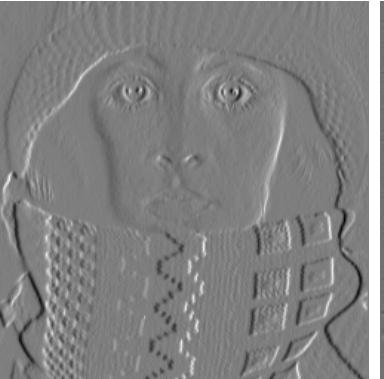
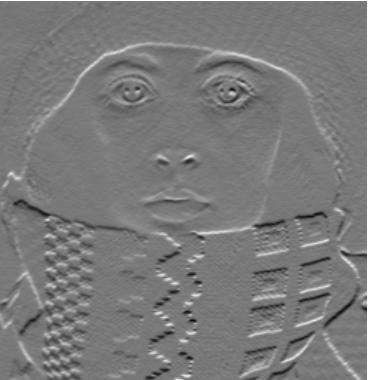


Left: Original image, 128×128 pixels. **Right:** Filtered by LAS scheme for TV diffusion ($\tau = 1, 100$ iterations).

Two-Dimensional Haar Wavelets

- ◆ Haar wavelet shrinking at the finest scale performs a tiling of the discrete image into four-pixel subimages. The overlapping tiles correspond to 2D cycle spinning.
- ◆ Transformation of a 2×2 subimage $U = (u_{i,j})$ comes down to

$$W = H U H, \quad H := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$



(a) **Left:** original image. (b) **Right:** first level of a Haar wavelet decomposition. From left to right and top to bottom: $w_{1,1}$, $w_{1,2}$, $w_{2,1}$, $w_{2,2}$.
 For cycle-spinning, four differently shifted copies need to be considered.

Diffusion-Inspired Wavelet Shrinkage

Interpretation of Isotropic LAS in Terms of Haar Wavelet Shrinkage

- ◆ TV diffusion ($p = 1$) at time θ shrinks wavelet channels $w_{1,2}$, $w_{2,1}$, $w_{2,2}$ via

$$S_\theta(w_{i,j}) := \begin{cases} w_{i,j} - \frac{\theta}{\gamma(W)} \operatorname{sgn}(w_{i,j}), & \gamma(W) \geq \theta, \\ 0 & \text{otherwise} \end{cases}$$

with $\gamma(W) := \sqrt{w_{2,1}^2 + w_{1,2}^2 + w_{2,2}^2} = \delta(u)$.

- ◆ Diffusion-inspired, *coupled* shrinkage rule
- ◆ Translation invariance is guaranteed by cycle spinning.
- ◆ PDE approximation secures also outstanding rotational invariance.

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Outline

1. Diffusion Filters and Wavelet Shrinkage
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6. Summary and Outlook

Anisotropic Diffusion

Tensor-Driven Anisotropic Diffusion in 2D

- ◆ Consider the diffusion PDE

$$u_t = \operatorname{div} (D(J) \nabla u)$$

with a *matrix-valued* diffusion tensor $D(J)$.

- ◆ The *structure tensor* J (Förstner/Gülch 1987) is given by

$$J = J_\varrho(\nabla u_\sigma) := K_\varrho * (\nabla u_\sigma \nabla u_\sigma^T)$$

where $u_\sigma = K_\sigma * u$ and K_σ is a Gaussian with standard deviation σ .

- ◆ Specifying the eigenvectors and eigenvalues of $D(J)$ allows to steer the process in the desired direction (Weickert 1998).

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Anisotropic Diffusion

Example: Coherence-Enhancing Diffusion (CED)

- ◆ Emphasises flow-like character of image structures, useful e.g. in seismic images, fingerprint images, flow measurements
- ◆ Let $J = \mu_1 \mathbf{e}_1 \mathbf{e}_1^T + \mu_2 \mathbf{e}_2 \mathbf{e}_2^T$ with $\mu_1 \geq \mu_2$ be the eigendecomposition of J .
- ◆ Then $D(J) := \lambda_1 \mathbf{e}_1 \mathbf{e}_1^T + \lambda_2 \mathbf{e}_2 \mathbf{e}_2^T$ with eigenvalues

$$\lambda_1 := \alpha,$$

$$\lambda_2 := \begin{cases} \varepsilon, & \text{if } \mu_1 = \mu_2, \\ \varepsilon + (1 - \varepsilon) \exp\left(\frac{-C}{(\mu_1 - \mu_2)^2}\right) & \text{else,} \end{cases}$$

where $\varepsilon > 0$ is a small regularisation parameter and C a contrast parameter

- ◆ Performs essentially 1-D diffusion in the flow direction \mathbf{e}_2 , that increases with the anisotropy $(\mu_1 - \mu_2)^2$

Problem: CED is highly sensitive to directional errors and numerical dissipation. Can 4-pixel ideas help? Is there a wavelet interpretation?

Anisotropic Diffusion on Four Pixels

Four-Pixel Dynamical Systems for Anisotropic Diffusion

- ◆ Discretise the structure tensor and the diffusion tensor in *cell midpoints* $(i + \frac{1}{2}, j + \frac{1}{2})$
- ◆ A *specific local space discretisation* of the anisotropic diffusion PDE on a 3×3 stencil can be decomposed into four processes on 2×2 -pixel cells
- ◆ **Four-pixel dynamical system** in cell $\{1, 2\} \times \{1, 2\}$:

$$\dot{u}_{1,1} = q_\alpha(u_{2,1} - u_{1,1}) + r_\alpha(u_{1,2} - u_{1,1}) + s_\alpha(u_{2,2} - u_{1,1})$$

$$\dot{u}_{2,1} = q_\alpha(u_{1,1} - u_{2,1}) + r_\alpha(u_{2,2} - u_{2,1}) + s_\alpha(u_{1,2} - u_{2,1})$$

$$\dot{u}_{1,2} = q_\alpha(u_{2,2} - u_{1,2}) + r_\alpha(u_{1,1} - u_{1,2}) + s_\alpha(u_{2,1} - u_{1,2})$$

$$\dot{u}_{2,2} = q_\alpha(u_{1,2} - u_{2,2}) + r_\alpha(u_{2,1} - u_{2,2}) + s_\alpha(u_{1,1} - u_{2,2}) ,$$

where $q_\alpha, r_\alpha, s_\alpha$ depend on the matrix entries of D and a parameter $\alpha \in [0, 1]$.

- ◆ If $\sigma = 0$, there exists again a gradient descent interpretation.

Anisotropic Diffusion on Four Pixels

Semi-Analytic Solution of the Four-Pixel System

- ◆ The four-pixel system cannot be solved analytically.
 - However, when fixing $D = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$, an analytic solution can be found.
 - ◆ Introducing new variables $w_{i,j}$ by $W := HUH$ with $H := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ gives
- $$\dot{w}_{1,1} = 0 ,$$
- $$\dot{w}_{2,1} = -4a w_{2,1} - 4c w_{1,2} ,$$
- $$\dot{w}_{1,2} = -4c w_{2,1} - 4b w_{1,2} ,$$
- $$\dot{w}_{2,2} = -4\alpha(a + b)w_{2,2} .$$
- ◆ With the eigendecomposition $D = \lambda_1 \mathbf{e}_1 \mathbf{e}_1^T + \lambda_2 \mathbf{e}_2 \mathbf{e}_2^T$ and $\mathbf{w} := (w_{2,1}, w_{1,2})^T$ we obtain the **analytic solution**

$$\begin{pmatrix} w_{2,1}(t) \\ w_{1,2}(t) \end{pmatrix} = e^{-4\lambda_1 t} \left(\mathbf{e}_1^T \begin{pmatrix} w_{2,1}(0) \\ w_{1,2}(0) \end{pmatrix} \right) \mathbf{e}_1 + e^{-4\lambda_2 t} \left(\mathbf{e}_2^T \begin{pmatrix} w_{2,1}(0) \\ w_{1,2}(0) \end{pmatrix} \right) \mathbf{e}_2 ,$$

$$w_{2,2}(t) = e^{-4\alpha \text{tr}(D)t} w_{2,2}(0) .$$

- ◆ Transform back via $U(t) = H W(t) H$.

Numerical Scheme for Anisotropic Diffusion

A Locally Semi-Analytic Scheme (LSAS) for Anisotropic Diffusion

- ◆ For each time step:

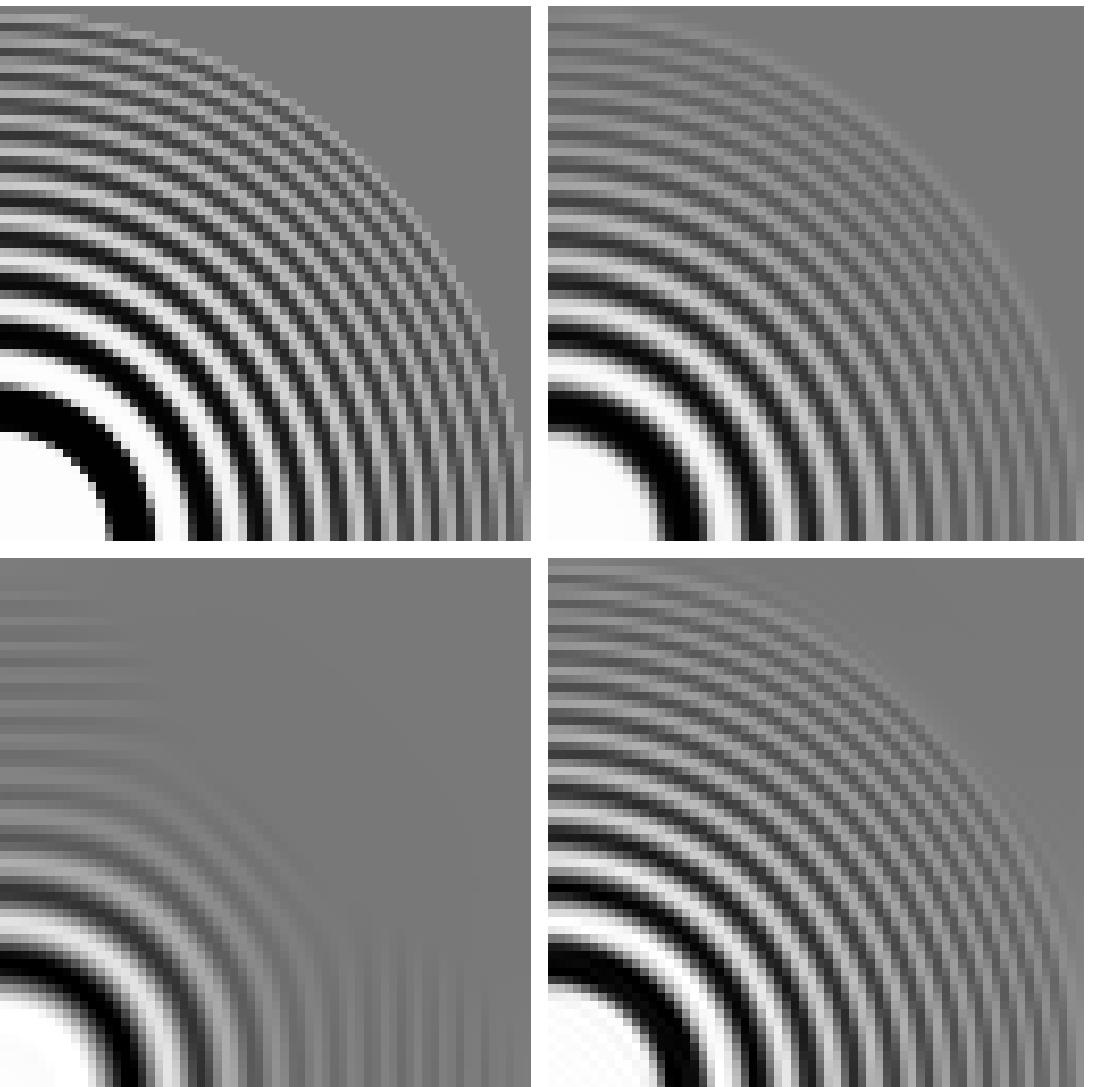
- ▶ Consider the four 2×2 -cells containing some pixel (i, j)
 - ▶ Compute their semi-analytic solutions with fixed D
 - ▶ Average the four results for pixel (i, j)

- ◆ **Absolutely stable** in the Euclidean norm
- ◆ **Conditionally consistent:**
 - $O(\tau + h^2 + \tau/h^2)$ approximation, i.e., consistent if $\tau/h^2 \rightarrow 0$ as $\tau, h \rightarrow 0$
 - for h fixed, unconditionally consistent $O(\tau)$ approximation
- ◆ **Sharp structures** with high resolution
- ◆ **No visible numerical dissipation**
- ◆ **Rotation invariance** well approximated: hardly any directional errors

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Anisotropic Diffusion Examples



Top left: One quadrant of a rotationally invariant test image, 64×64 pixels. **Top right:** Exact solution for coherence-enhancing diffusion with $\varepsilon = 0.001$, $C = 1$, $\sigma = 0.5$, $\varrho = 4$, and $t = 250$. **Bottom left:** Filtered with the nonnegativity scheme with $\tau = 1/6$, and $N = 1500$ iterations. Average absolute error: 17.99. **Bottom right:** Processed with the LSAS algorithm ($\alpha = 0$), same parameters. Average absolute error: 3.81.

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Anisotropic Diffusion Examples



Left: Fingerprint image, 100×100 pixels. **Middle:** Filtered with a nonnegativity scheme for CED with $\lambda = 1$, $\sigma = 0.5$, $\varrho = 4$, $\tau = 1/6$, and $N = 60$ iterations. **Right:** Filtered with LSAS algorithm ($\alpha = 0$), same parameters.

Anisotropic Wavelet Shrinkage

Interpretation of Anisotropic LSAS in Terms of Haar Wavelet Shrinkage

- ◆ Anisotropic diffusion at time θ shrinks wavelet channels $w_{1,2}, w_{2,1}$ via

$$S_\theta \begin{pmatrix} w_{2,1} \\ w_{1,2} \\ w_{2,2} \end{pmatrix} := \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-4\lambda_1\theta} & 0 & 0 \\ 0 & e^{-4\lambda_2\theta} & 0 \\ 0 & 0 & e^{-4\alpha(\lambda_1+\lambda_2)\theta} \end{pmatrix} \begin{pmatrix} Q^T & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_{2,1} \\ w_{1,2} \\ w_{2,2} \end{pmatrix}.$$

where $Q := (\mathbf{e}_1, \mathbf{e}_2)$ is the eigenvector matrix of D .

- ◆ Diffusion-inspired, *anisotropic* shrinkage rule with channel coupling

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Multi-Channel Images

Multi-Channel Images

- ◆ Consider multi-channel images, e.g.
 - colour images
 - vector fields
 - matrix-valued images
- ◆ The LAS and LSAS can easily be extended to these image types:
 - Divergence forms are transferred channel-wise
 - Diffusivities/diffusion tensors are channel-coupled

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Three-Dimensional Images

Singular Isotropic Diffusion in 3D

- ◆ All the locally analytic and locally semi-analytic schemes transfer easily to 3D images. Eight-voxel “bricks” take the role of four-pixel cells.
- ◆ Analytic solution of singular isotropic diffusion

$$u_t = \operatorname{div} \left(\frac{\nabla u}{|\nabla u|^p} \right)$$

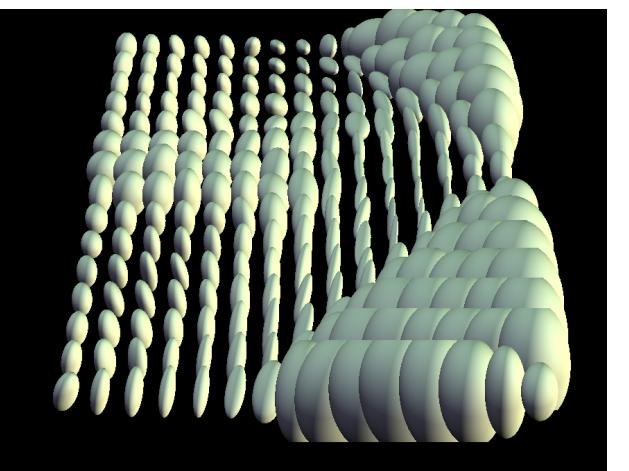
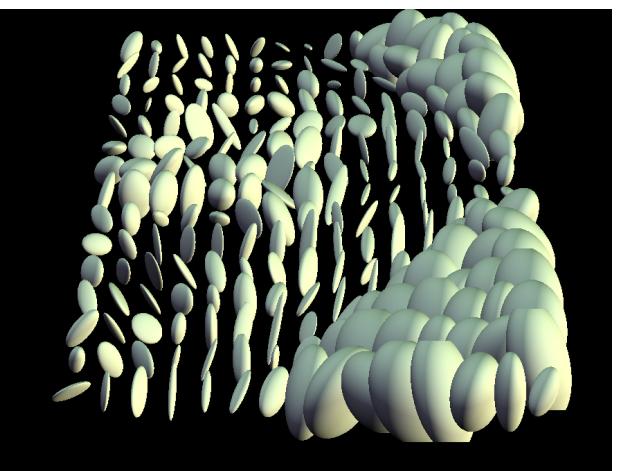
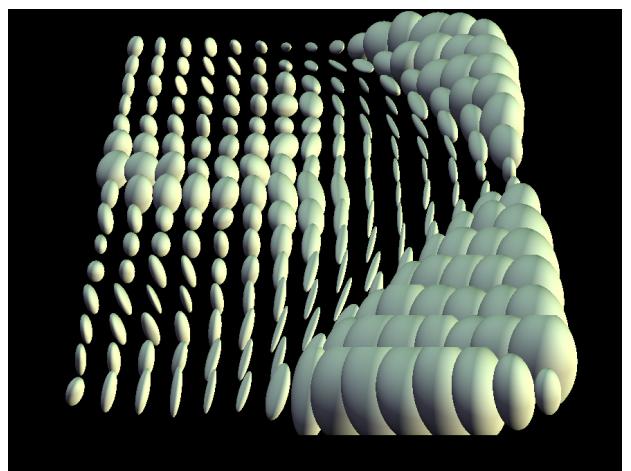
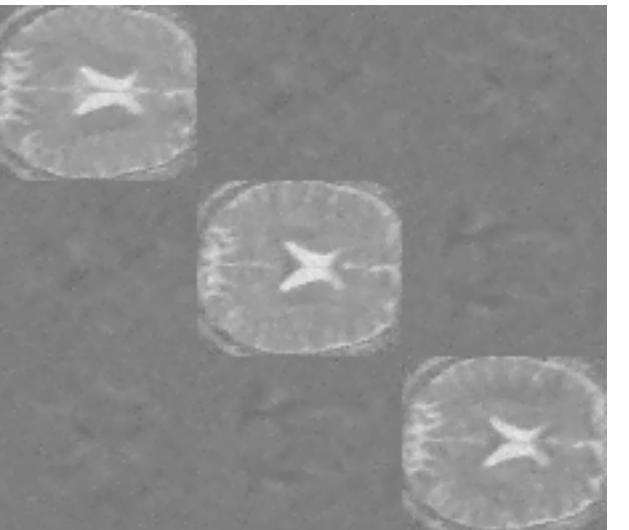
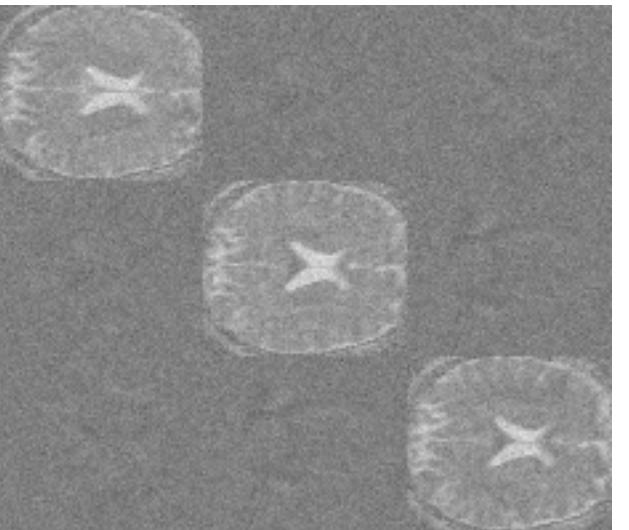
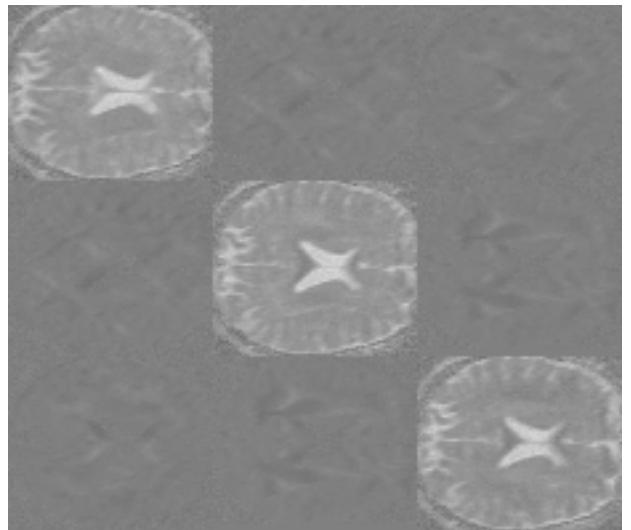
on a $2 \times 2 \times 2$ -voxel cell:

$$u_{i,j,k}(t) = \begin{cases} \mu + \sqrt[p]{1 - \frac{4pt}{\delta(f)^p}} (f_{i,j,k} - \mu), & 0 \leq t < \frac{\delta(f)^p}{4p}, \\ \mu, & t \geq \frac{\delta(f)^p}{4p}, \end{cases} \quad i, j, k = 1, 2,$$

where $\delta(f)$ approximates $|\nabla u|$ in $(\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$.

- ◆ LAS can be constructed like in 2D, with same properties
- ◆ Haar wavelet interpretation carries over

Total Variation Diffusion: DT-MRI Example



Top, left to right: DT-MRI image; degraded by 30 % uniform noise; filtered by multi-channel, 3D total variation diffusion implemented by eight-voxel scheme (*DT-MRI data: O. Gruber, I. Henseler, Saarland University Hospital, Homburg; filtering: M. Eisemann*). **Bottom:** Details from top row images visualised by ellipsoids

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Three-Dimensional Images

Tensor-Driven Anisotropic Diffusion in 3D

- ◆ Consider anisotropic diffusion PDE on a 3D regular grid with mesh size 1 in x , y , and z directions
- ◆ Discretise structure tensor and diffusion tensor in cell midpoints $(i + \frac{1}{2}, j + \frac{1}{2}, k + \frac{1}{2})$
- ◆ Then one can state a discretisation of the PDE that allows a decomposition into separate dynamical systems on $2 \times 2 \times 2$ -voxel cells:

$$\dot{\mathbf{u}} = F(D)\mathbf{u}$$

with $\mathbf{u} := (u_{1,1,1}, u_{1,1,2}, \dots, u_{2,2,2})^T$ and an 8×8 -matrix $F(D)$



Three-Dimensional Images

Tensor-Driven Anisotropic Diffusion in 3D

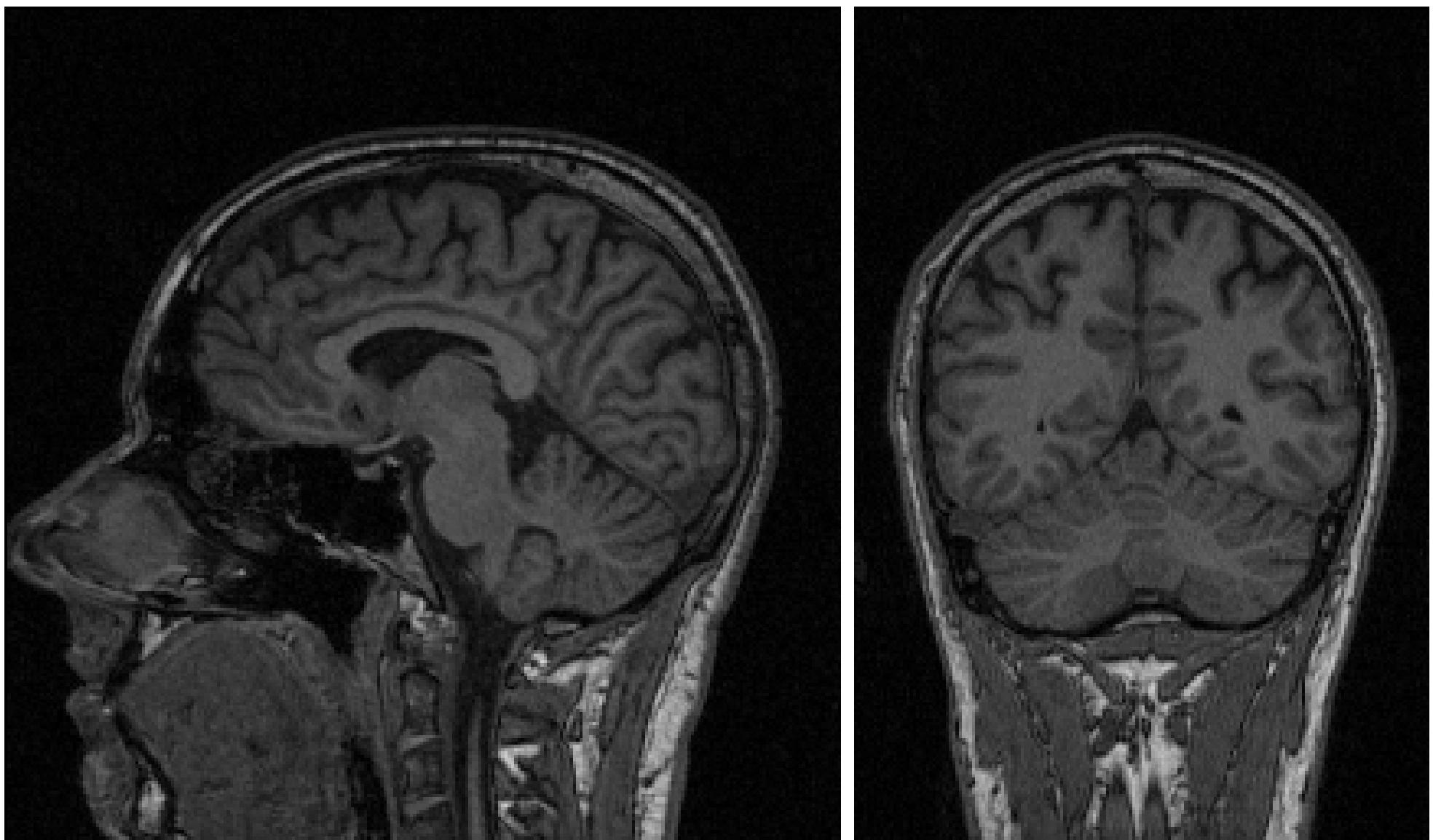
- ◆ Rewriting in 3D Haar wavelet basis $\{w_{1,1,1}, w_{1,1,2}, \dots, w_{2,2,2}\}$ yields

$$\begin{aligned} \dot{w}_{1,1,1} &= 0 & \dot{w}_{2,2,2} &= -8\alpha \operatorname{tr}(D)w_7 \\ \begin{pmatrix} \dot{w}_{2,1,1} \\ \dot{w}_{1,2,1} \\ \dot{w}_{1,1,2} \end{pmatrix} &= -8D \begin{pmatrix} w_{2,1,1} \\ w_{1,2,1} \\ w_{1,1,2} \end{pmatrix} & \begin{pmatrix} \dot{w}_{2,2,1} \\ \dot{w}_{2,1,2} \\ \dot{w}_{1,2,2} \end{pmatrix} &= -8\alpha \tilde{D} \begin{pmatrix} w_{2,2,1} \\ w_{2,1,2} \\ w_{1,2,2} \end{pmatrix} \end{aligned}$$

- ◆ Analytic solution in $2 \times 2 \times 2$ -voxel cell can be used to construct LSAS for 3D anisotropic diffusion, with same properties as in 2D
- ◆ Haar wavelet interpretation carries over directly
- ◆ **Question:** Which anisotropic diffusion processes are adequate in 3D?

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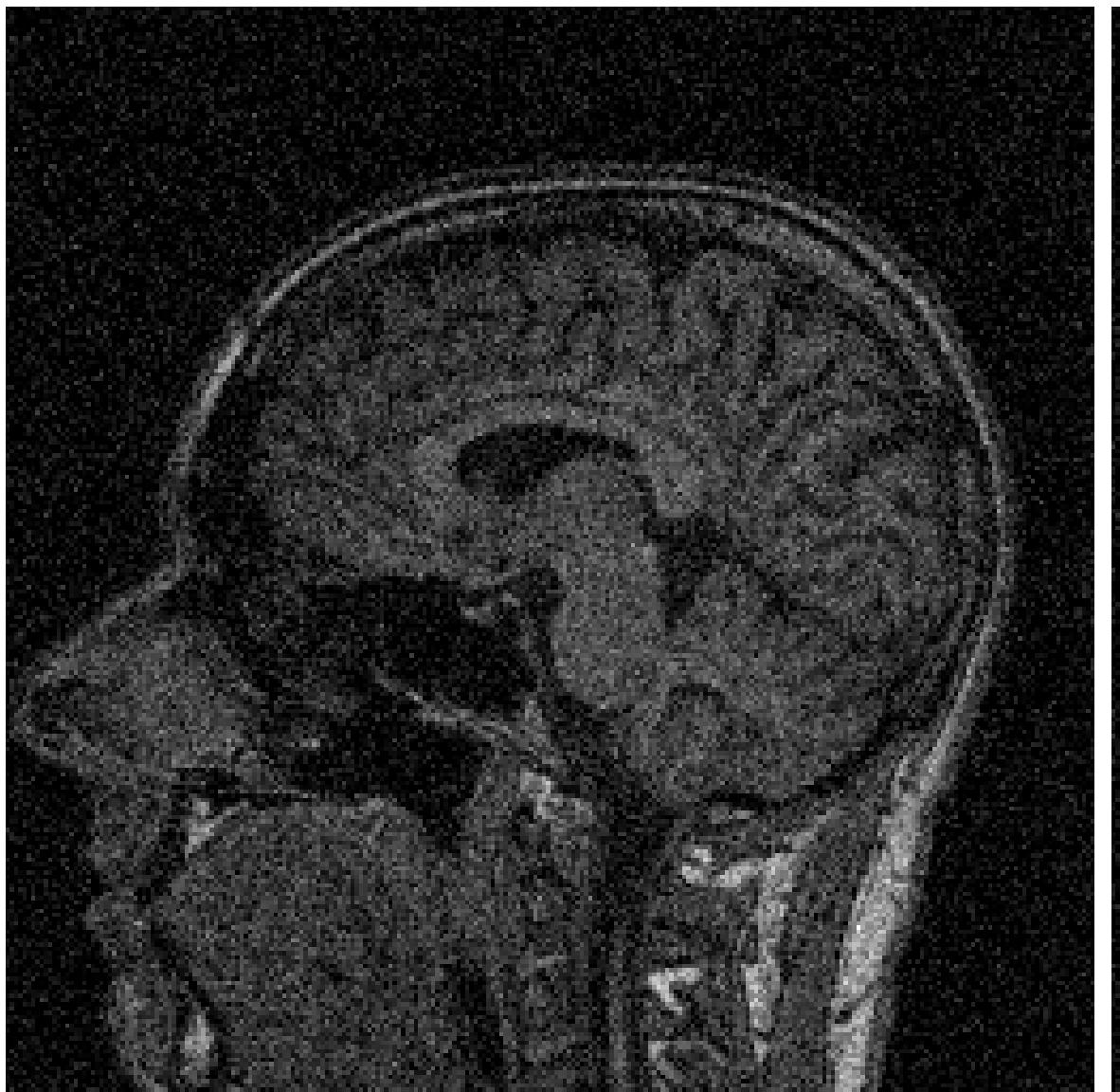
Anisotropic Diffusion in 3D



Two sections of a 3D MRI image of a human head (*Measurement: O. Gruber, I. Henseler*).

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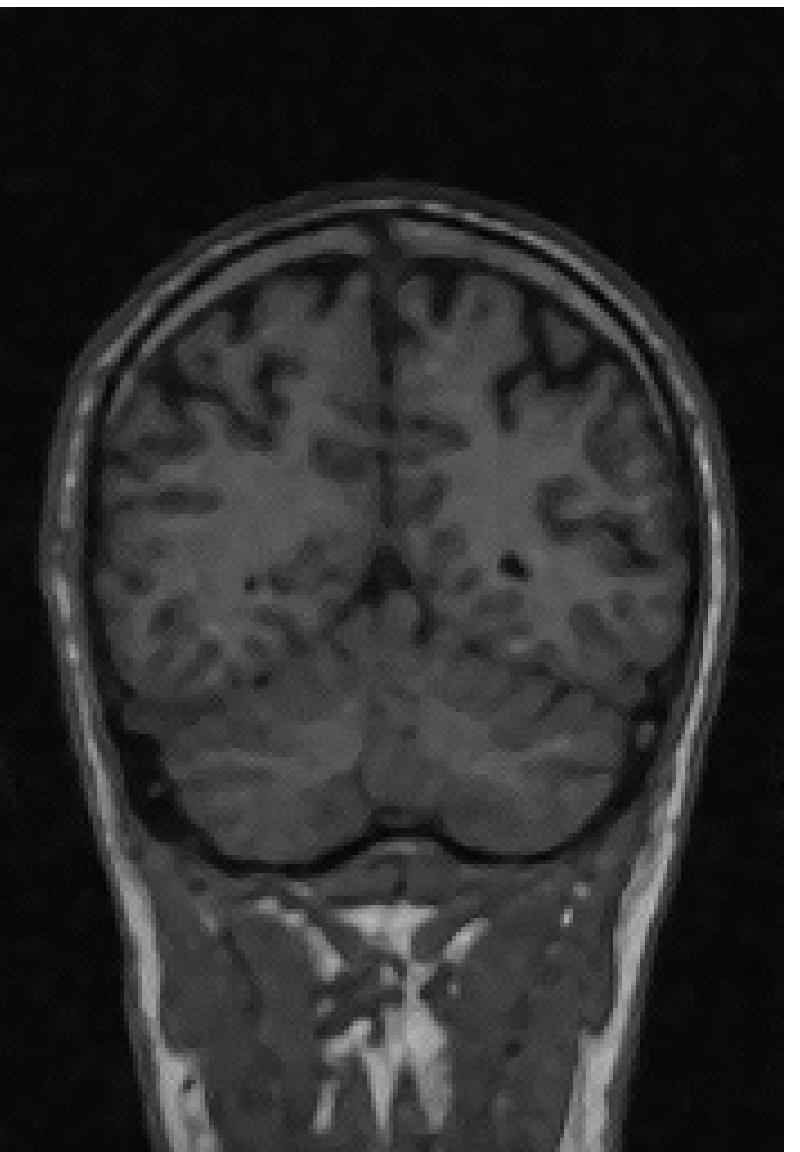
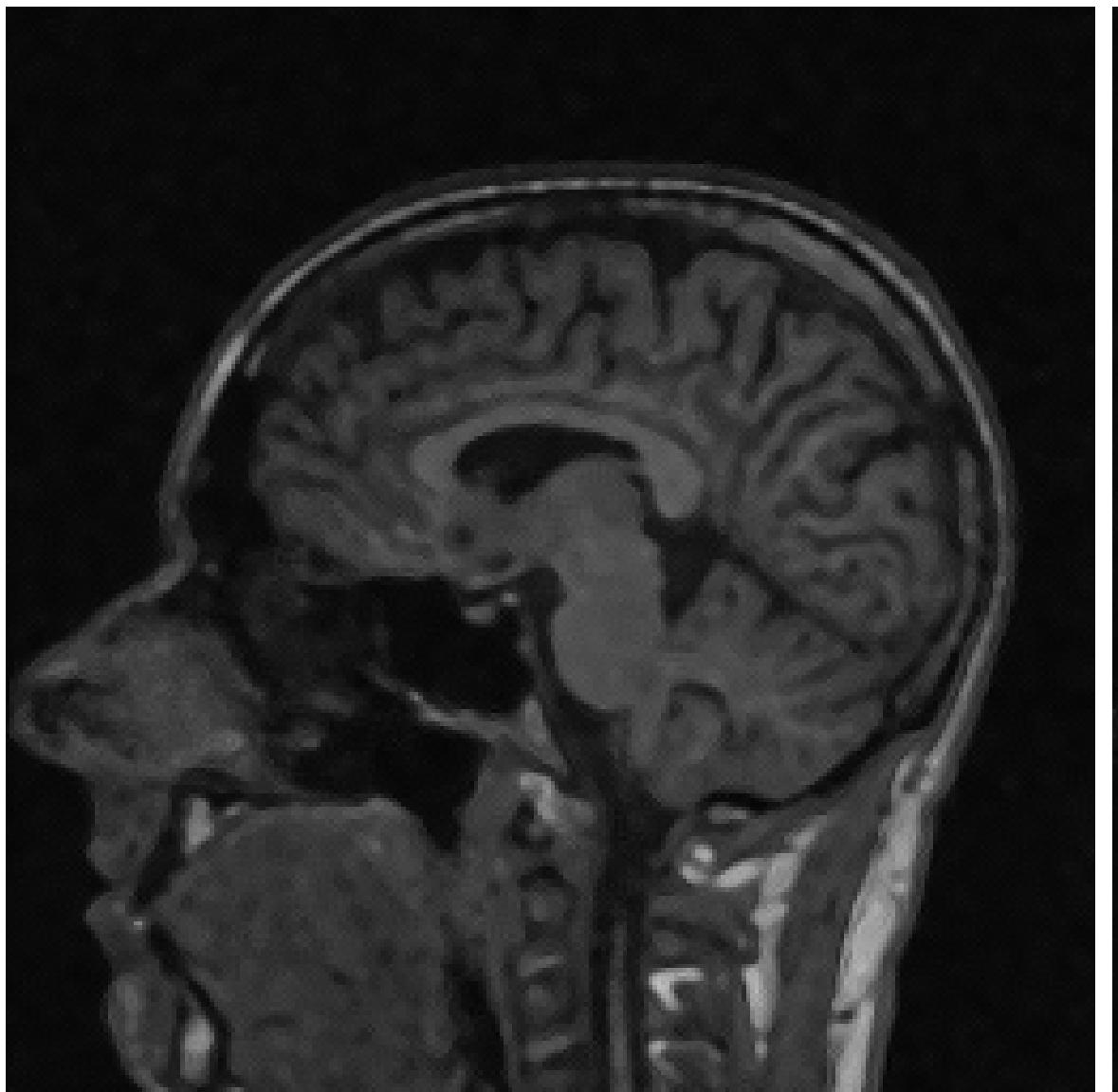
Anisotropic Diffusion in 3D



Same sections of 3D MRI image degraded by Gaussian noise (standard deviation 20 on grey-value range [0, 255])

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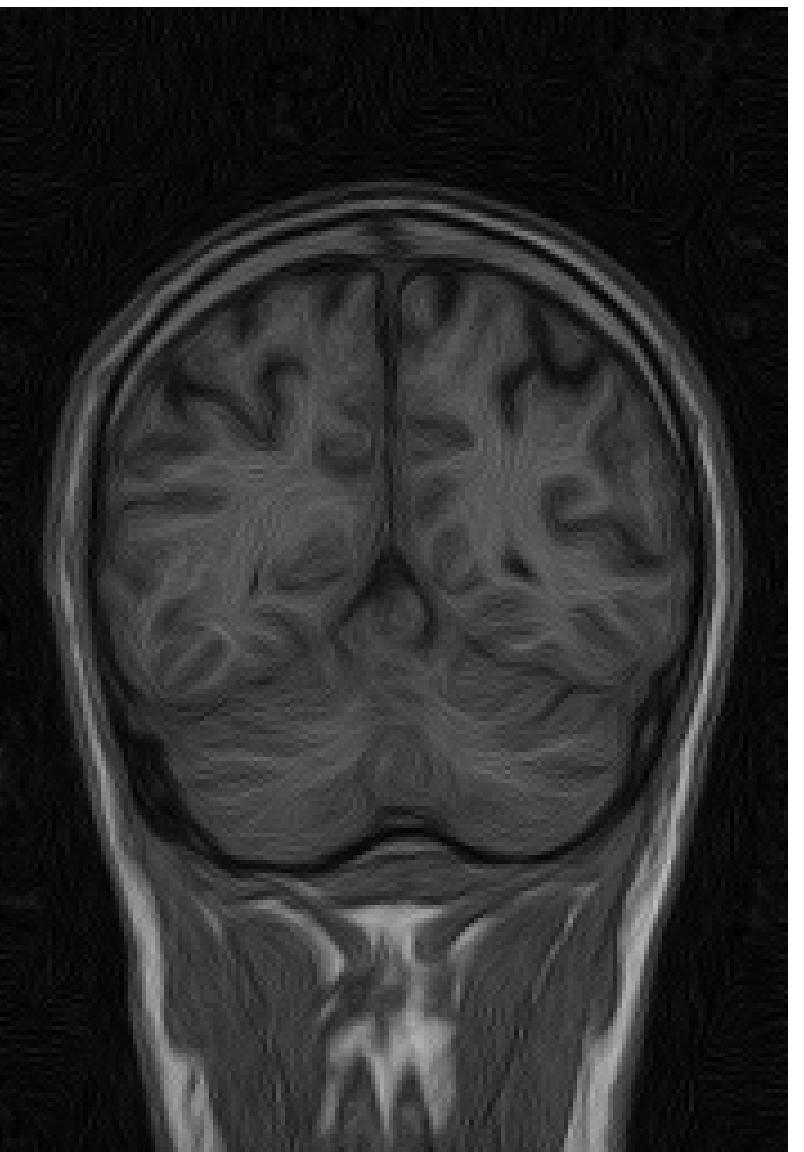
Anisotropic Diffusion in 3D



Noisy 3D data set processed by a 3D version of edge-enhancing diffusion (EED) that smoothes preferably along level surfaces.

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Anisotropic Diffusion in 3D



Noisy 3D data set processed by a 3D version of coherence-enhancing diffusion (CED).

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6. **Summary and Outlook**

Summary and Outlook

Summary

- ◆ Analytic solutions for two-pixel signals and four-pixel images can be used as building blocks for numerical schemes for PDEs
- ◆ Locally analytic/semi-analytic schemes (LAS/LSAS)
 - average local analytic solutions, related to splitting ideas
 - explicit, absolutely stable, conditionally consistent
 - no need for regularisation of singular PDEs
 - sharp discontinuities
- ◆ Wavelet Methods:
 - LAS/LSAS allow an interpretation as Haar wavelet shrinkage on finest scale with novel shrinkage rules
 - Anisotropic image filtering realised by simple Haar wavelets
 - Newly obtained shrinkage rules can be transferred to multi-level wavelet shrinkage

Ongoing Work

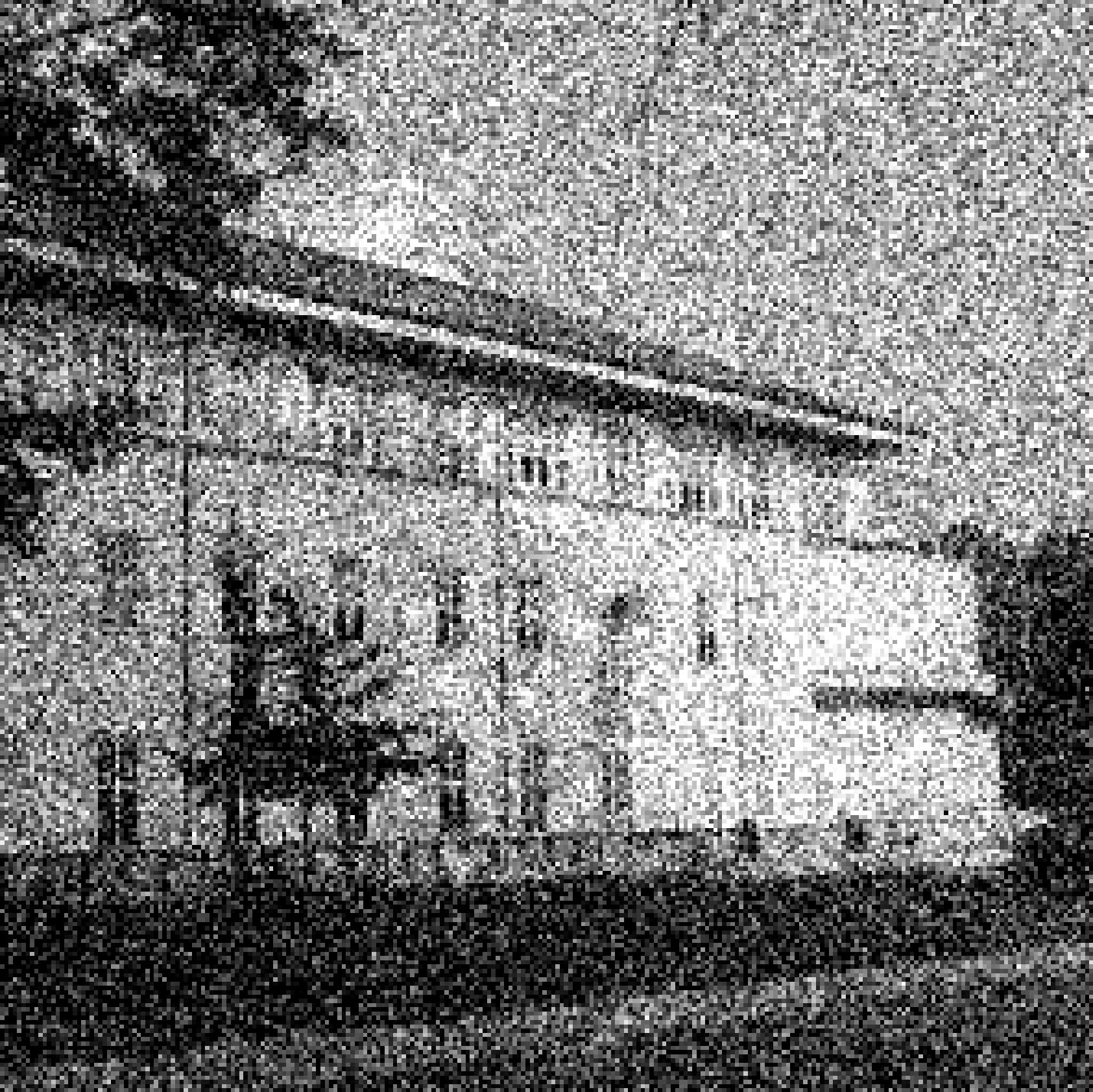
- ◆ Links to other wavelet types, including those designed to represent anisotropy
- ◆ Application of similar ideas to other PDE image filters

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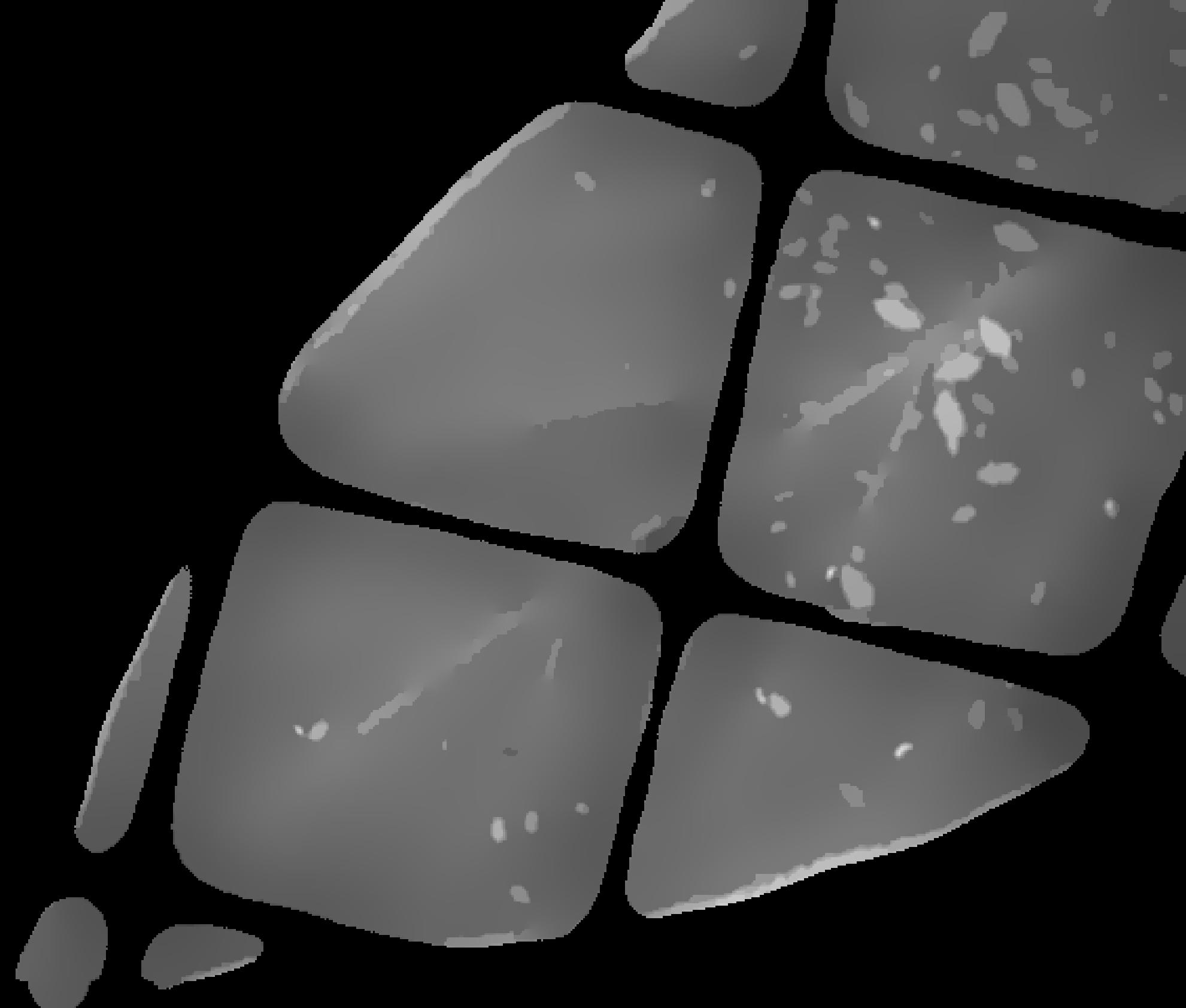
Thank you for your attention!



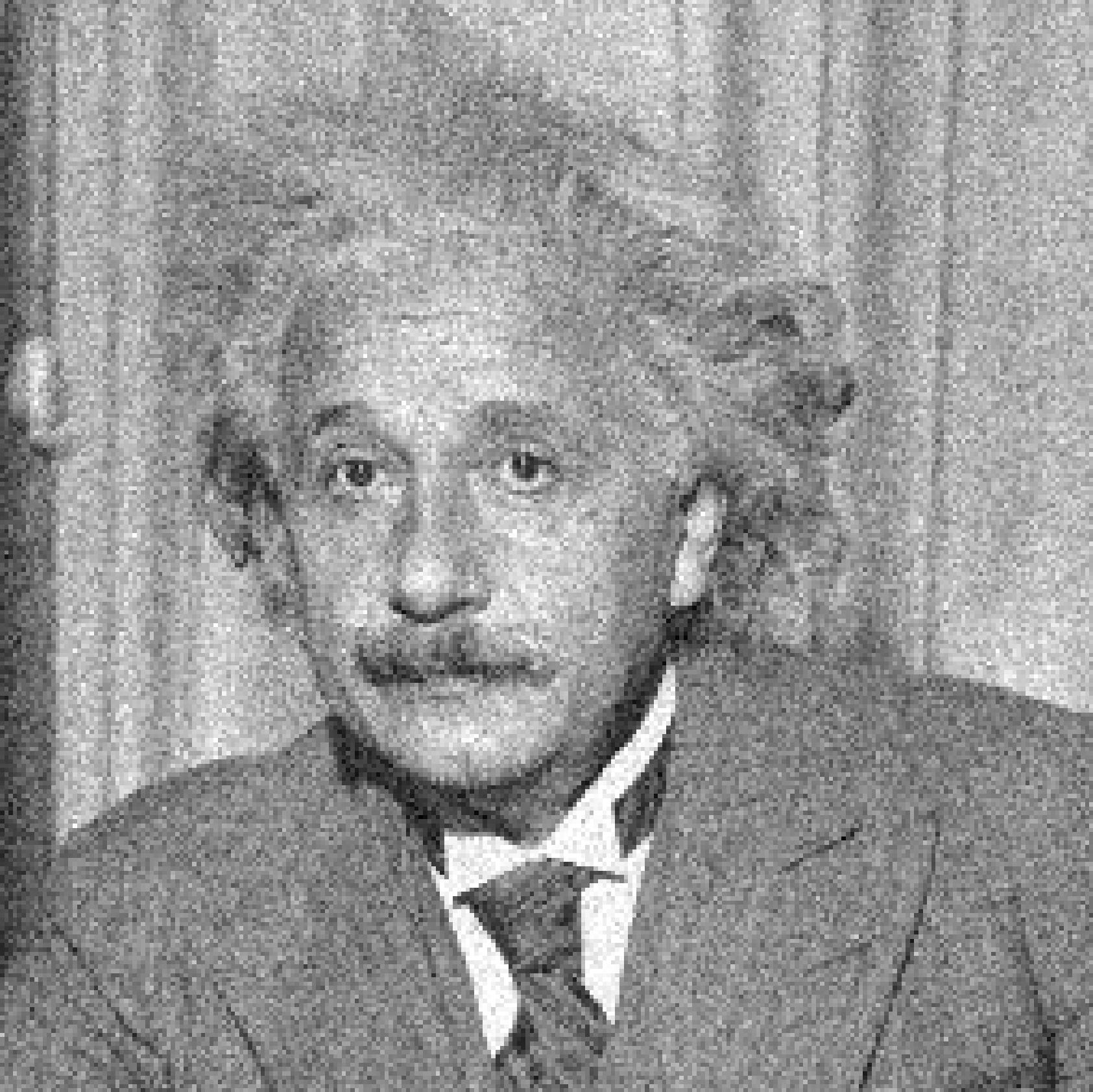


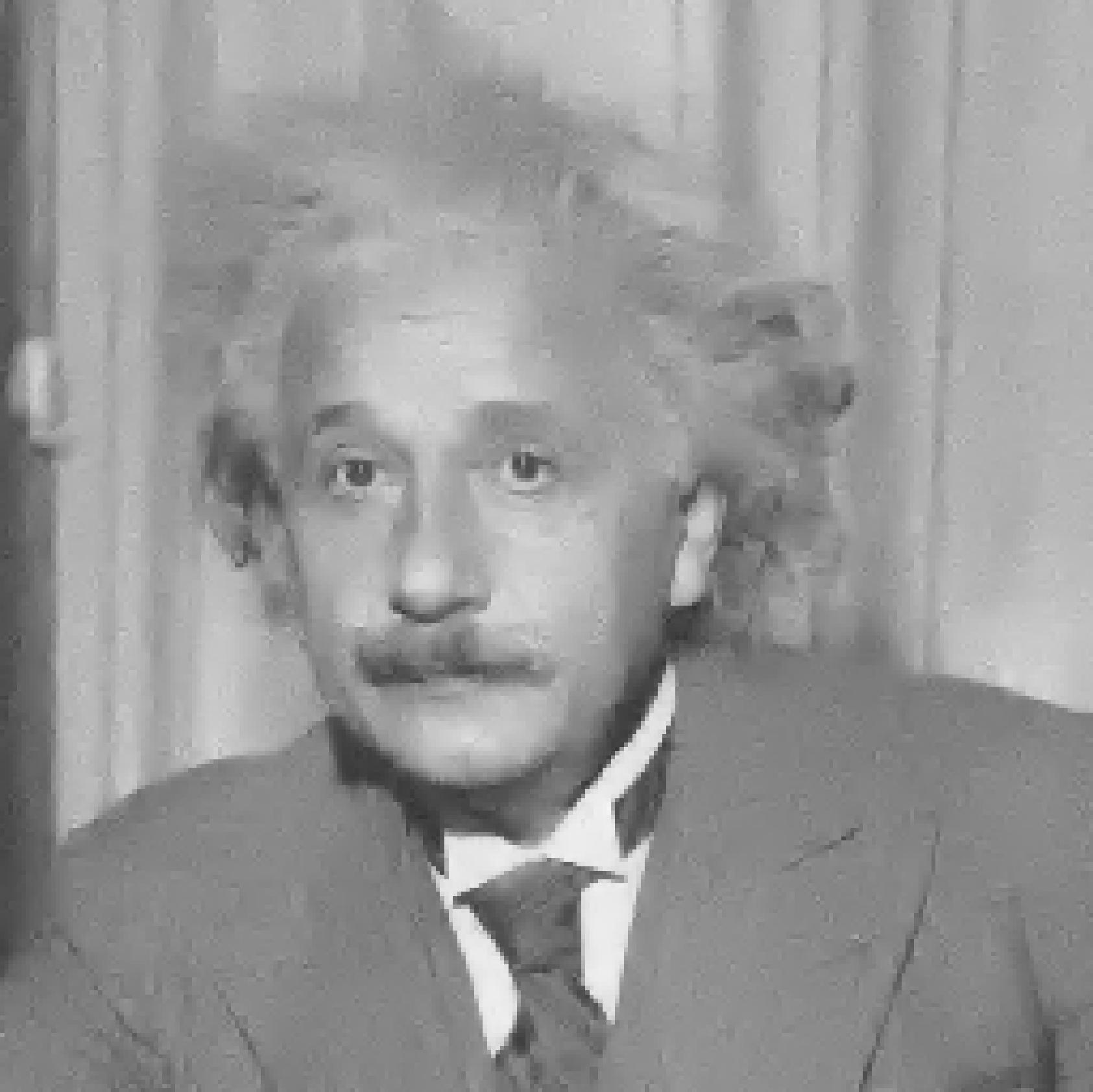




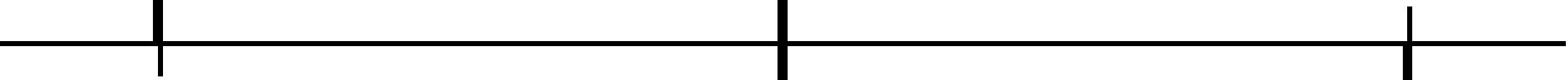
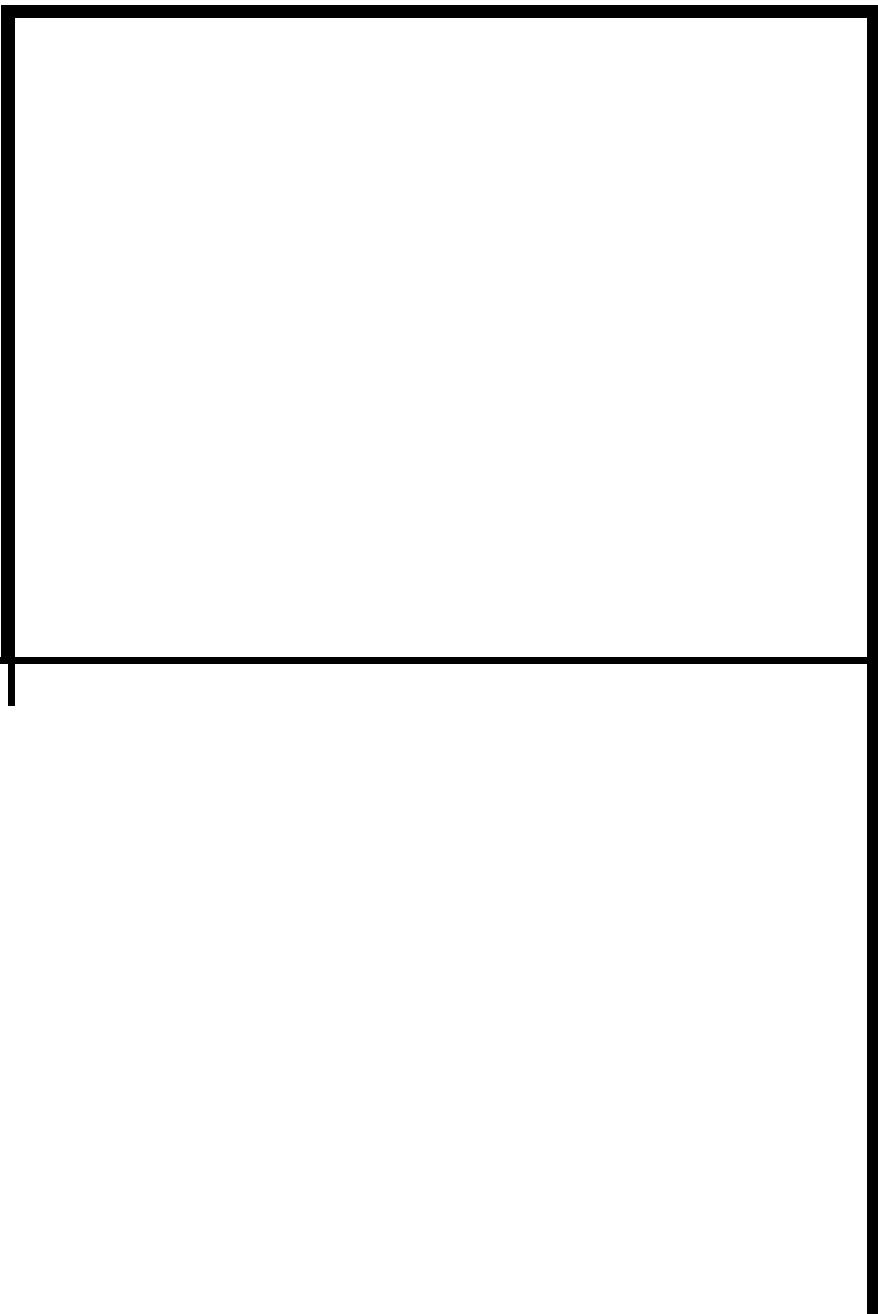








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