National Research Center for Environment and Health

Institute of Biomathematics & Biometry

Wedgelets Partitions and Image Processing

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Contents

Classical Compression Standards

Original Image

JPEG (6.8 KB)
DCT

JPEG2000 (6.5 KB)
FWT + Contexts

Zoom

Original Image

JPEG (6.8 KB)
DCT

JPEG2000 (6.5 KB)
FWT + Contexts

Mathematical Background

Justification Approximation Theory

Image: real-valued function, continuous domain
Ansatz natural images have some regularity

$$f \in X \text{ (quasi?)-Banach space }, X \subset\subset L^2(\Omega)$$

Approximation $\widehat{f}_n = \sum_i \alpha_i \varphi_i, \varphi \in \mathcal{A}$, n-approximation
We look for $\mathcal{A}$, such that

$$\|f - \widehat{f}_n\|_2^2 = O\left(\frac{1}{n^\alpha}\right), \text{ for some } \alpha > 0, \text{ and } f \in X$$

Critics

- Asymptotic results
- Continuous vs Discrete

Old and new Ansätze

Orthogonal Transforms

- FOURIER: non optimal (bad for local singularities)
- WAVELETS : optimal Non Linear Approximation rates for Besov spaces and Bounded Variation

$+$ in 2D Isotropic vs Anisotropic Methods

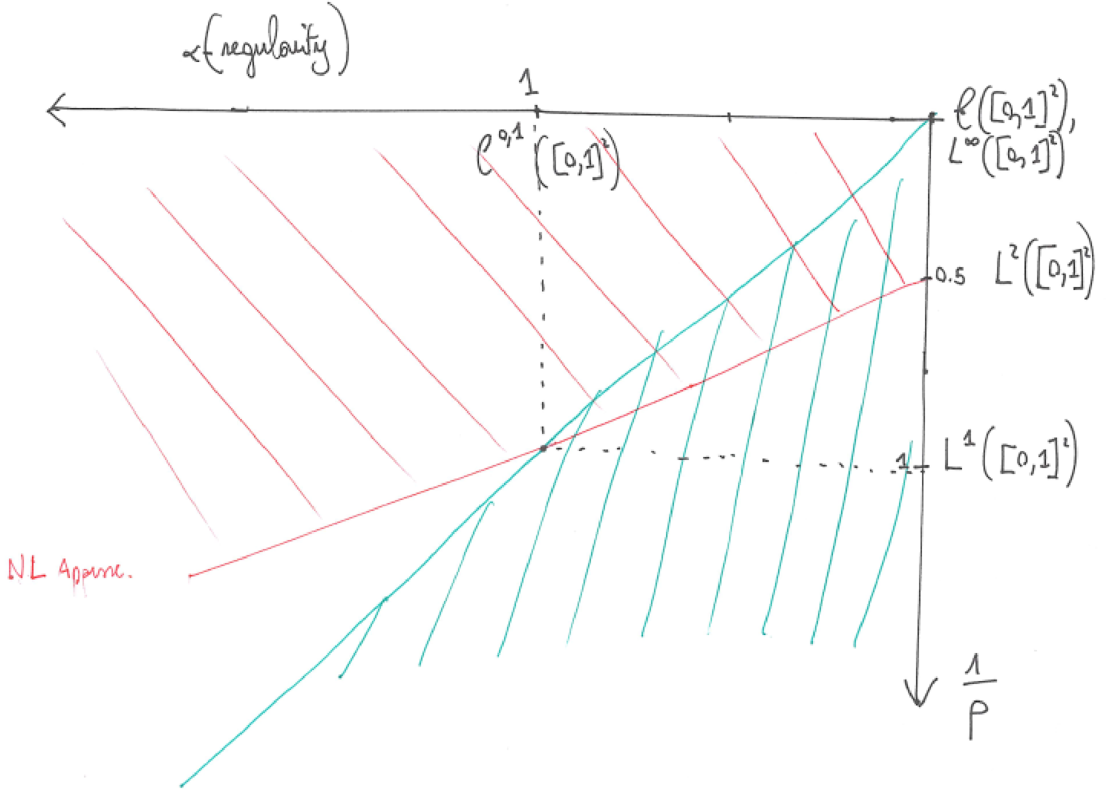
$\Longrightarrow$ Structure of the set of singularities

Geometrical Methods

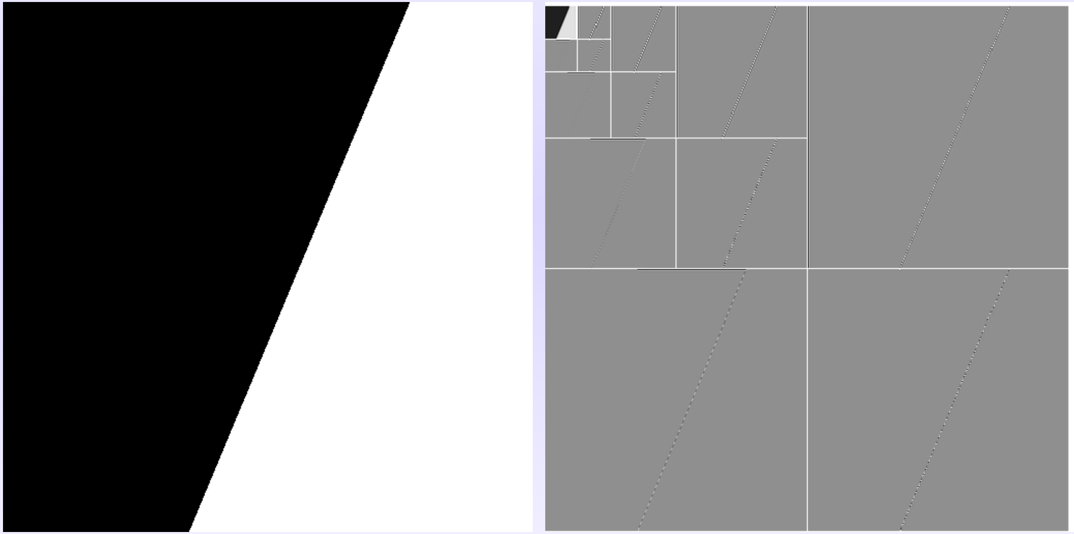
- CURVELETS (Candès 99-04) + SHEARLETS (Labate et al. 2005) not adaptive but quasi-optimal (flexible geometrical features)
- BANDELETS (Mallat-LePennec 99)
- TRIANGULATIONS: good theoretical Approximation rates (Mallat 04, Demaret-Iske 06)
- WEDGELETS (Donoho 99)

DIAGRAMM OF FUNCTIONAL SPACES

(IN 2D)



Wavelets and Contours



Wedge (left) and its Wavelet coefficients (right)

Geometrical Segmentations

$S \subset \mathbb{Z}^2$	set of pixels
$f \in \mathbb{R}^S$	image
$\mathfrak{P}$	family of partitions $\mathcal{P} \subset 2^S$ of S
$f_{\mathcal{P}} \in \mathbb{R}^S$	best constant approximation with $f_{\mathcal{P}}\|_r$ constant, $r \in \mathcal{P}$
$\mathfrak{S}$	segmentations $(\mathcal{P}, f_{\mathcal{P}})$
$\gamma \geq 0$	penalisation parameter

- Goal: Efficient Minimisation of the penalised Functional

$$H_{f,\gamma} : \mathfrak{S} \to \mathbb{R}, \quad (\mathcal{P}, f_{\mathcal{P}}) \mapsto \gamma \cdot |\mathcal{P}| + \|f - f_{\mathcal{P}}\|_2^2 \quad (\gamma \geq 0).$$

- Result

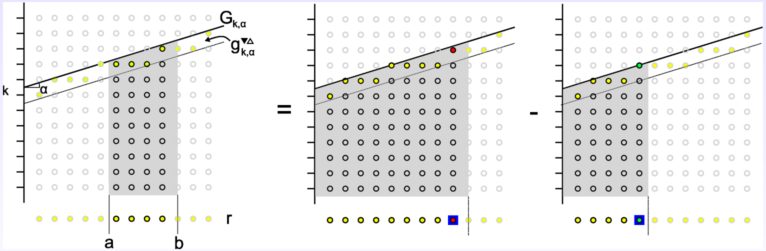
$$(\hat{\mathcal{P}}, \hat{f}_{\hat{\mathcal{P}}}) \in \operatorname*{argmin}_{(\mathcal{P}, f_{\mathcal{P}})} H_{z,\gamma}$$

optimal tradeoff between penalisation and reconstruction quality

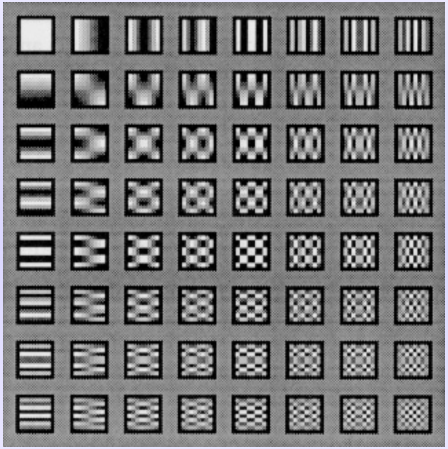
Wedgelet Segmentations

$$H_{f,\gamma} : \mathfrak{S} \to \mathbb{R}, \quad (\mathcal{P}, f_{\mathcal{P}}) \mapsto \gamma \cdot |\mathcal{P}| + \|f - f_{\mathcal{P}}\|_2^2 \quad (\gamma \geq 0).$$

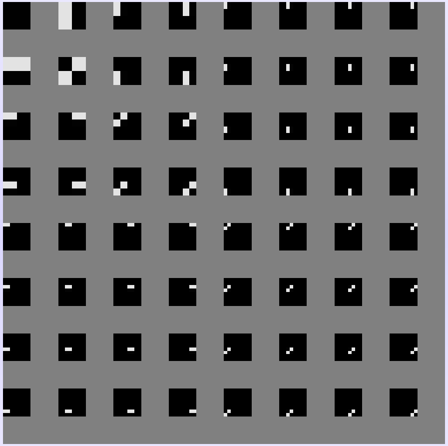
- **Problem** Size of the search space :($|\mathfrak{P}| > 2^{|S|}!$)
  - MCMC: slow and not exact
- Restriction of the search space
  - discrete wedges
  - nested Quadtree structure
- fast moment computation: Green-like formula



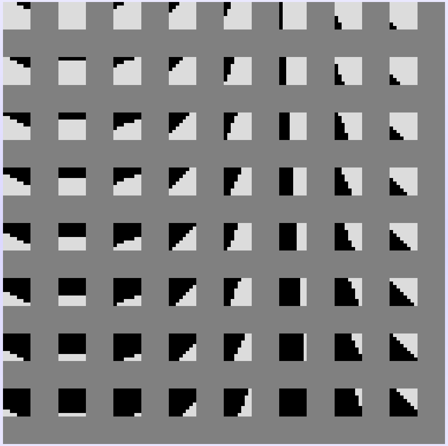
Representation Elements



DCT basis (JPEG)

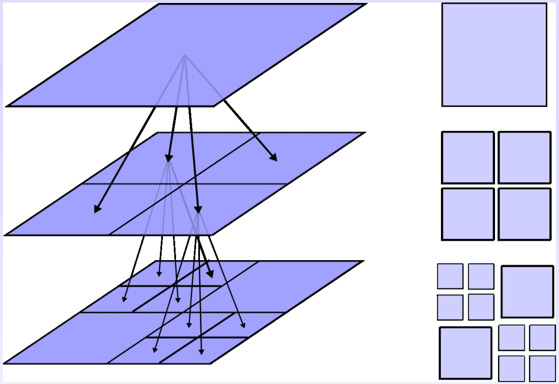


(Haar) Wavelet basis

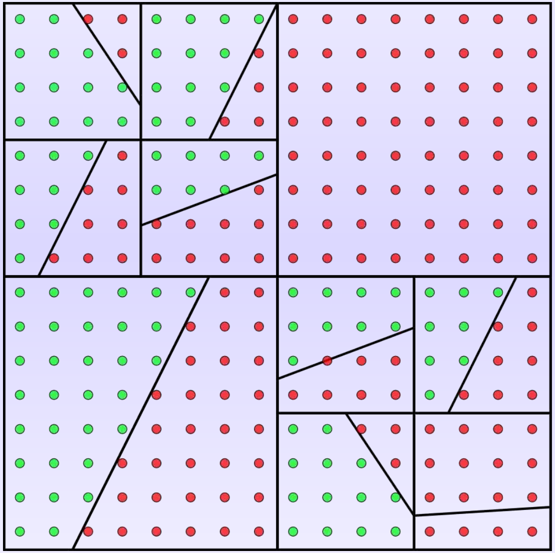


Wedgelet partitions

Data Structure



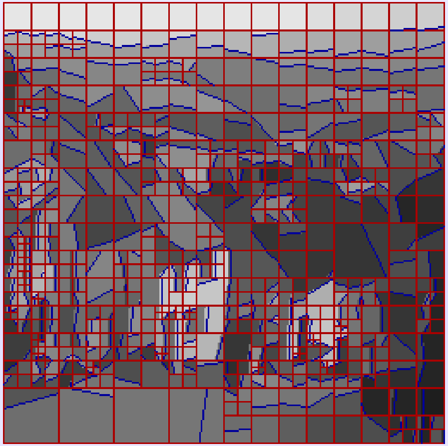
Quadtree Partition



dyadic Wedge Partition

Example

f



$(\mathcal{W}, f_{\mathcal{W}})$

$f_{\mathcal{W}}$

Compression: Algorithm

Idee Wedgelet representation contains too much redundancies

$\Longrightarrow$ Correlation Model between neighbours

ALGORITHM

- Tree Coding
- Model Coding
- IF (Model = constant over square)

  (quantised) mean value encoded
- IF (Modell = constant over each Wedge)

  Angle Encoding and relative position

  Coding of the (quantised) mean values

Compression: Features

- mixed Models (e.g. square constant, wedge constant, wedge linear ...)
- corresponding penalisation : estimation of the coding costs

$$H_\gamma : (f, (\mathcal{P}, f_\mathcal{P})) \longmapsto \gamma(\sum_i |C(W_i)| + \sum_j |C(Q_j)|) + \|f - f_\mathcal{P}\|_2^2,\ \gamma \geq 0,$$

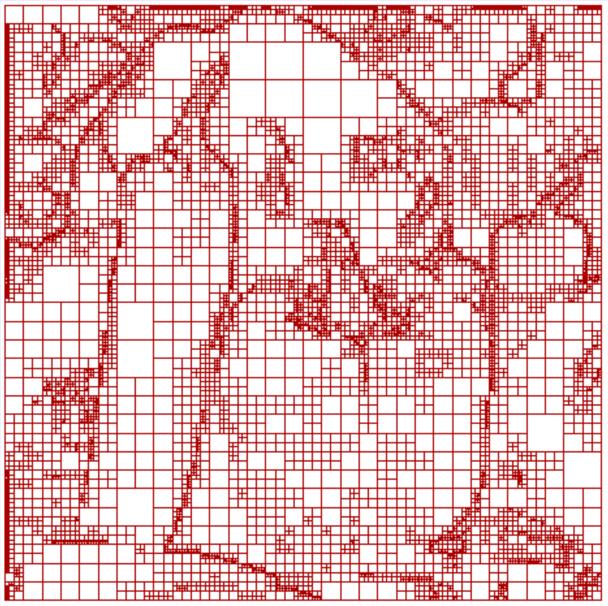
W_i : wedge, Q_j : square, C estimator for the coding costs

- Coding
  - combinatorial encoding
  - angle coding : resolution-adaptive
- **Prediction Method**

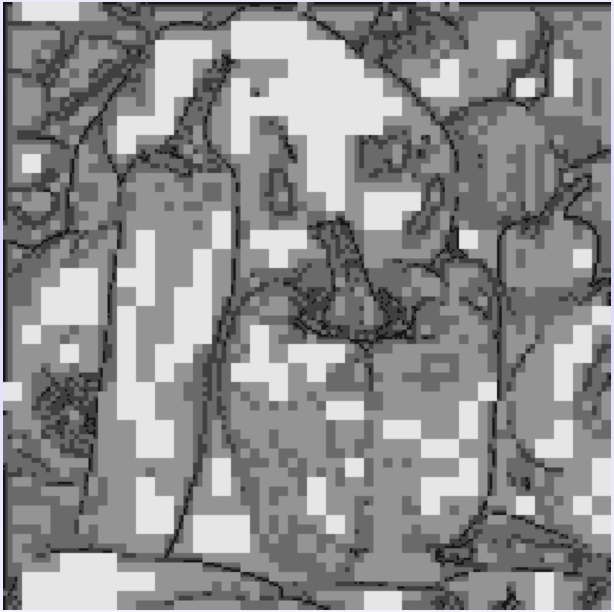
Prediction

- **Observation** Representation still strongly redundant
  - "not natural", arbitrary quadtree structure
- **Main idea**
  - Multiresolution differential coding
    only "Brotherhood" correlations
  - Extraction of spatial correlation between quadtree "cousins"
- Current piece coded from the causal (already coded) information

How to Code the Leaves ?

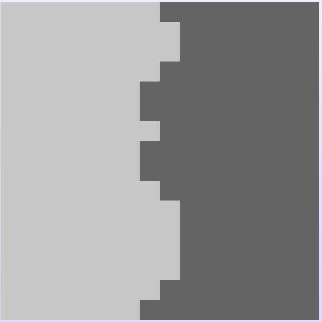


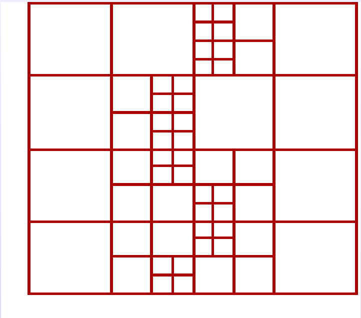
Tree



Levels of the leaves

Predictive Coding: an Illustration





0	0	0	0	1	0	0	0
0	0	0	0	1	0	0	0
0	0	0	1	0	0	0	0
0	0	0	1	0	0	0	0
0	0	0	1	0	0	0	0
0	0	0	0	1	0	0	0
0	0	0	0	1	0	0	0
0	0	0	1	0	0	0	0

	Context	
	= 0	> 0
0	41	15
1	1	7

- Binary Tree : 45 bits
- Bottom to Top Non-Predictive: $\log_2(64) + \log_2(\binom{64}{8}) = 39$ bits
- Bottom to Top Predictive: $\log_2(64) + \log_2(\binom{42}{1}) + \log_2(\binom{22}{7}) = 31$ bits

First results (1)

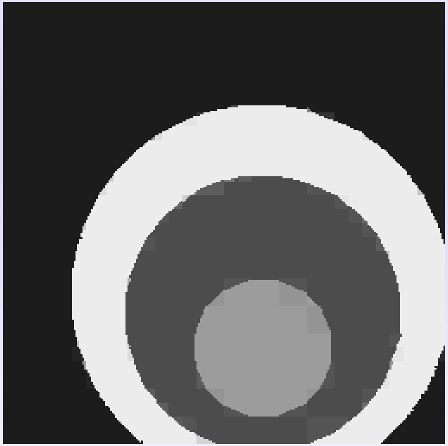
Comparison between "pure Wedge" and "Wedge+Constant" Models with higher penalisation for Wedges versus Squares

$$C(W_i) = 3.5 \times C(Q_i)$$

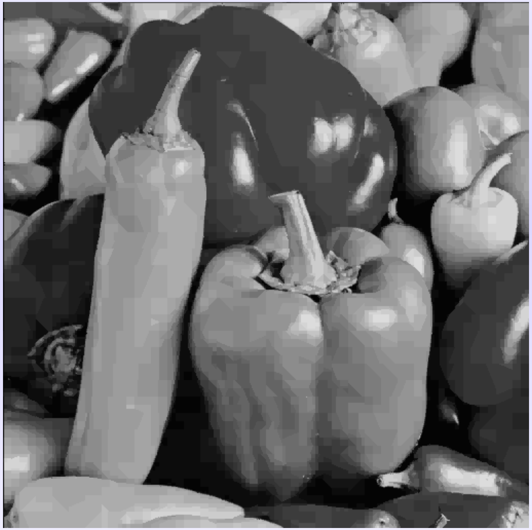
(a) Original Image (b) Squares: 87784 b, 30,54 dB(c) only Wedges: 84632 b, 30,42 dB (d) Wedges + Squares: 76184 b, 30,60 dB

Model		Tree	Models	Const. values	Angles	Line number	Wedge values	Total
pure squares	bits	16960		70336				87296
	symb.	20801		15601				36402
pure wedges	bits	5776			9808	20376	48296	84256
	symb.	7021			5266	5266	10532	28085
wedges + squares	bits	10088	6440	34824	2928	6784	14904	75968
	symb.	12337	9253	7582	1671	1671	3342	35856

First Results (2)



Circles, WC, 533 B, PSNR: 27.50 dB



Peppers, WC, 10.5 KB, PSNR: 31.50 dB
Compression Rate 1:25

Work in Progress

- systematic investigation of the penalisation functional
  - *rate-distortion* Optimisation
  - Depends on the resolution
  - Contexts change penalty
- Contextual Encoding
- Compression with richer regression models (e.g. linear)
  - aim: avoid bloc artefacts
- Correct theoretical framework for discrete Data
  - non asymptotical results