Limit distributions of tree parameters

Stephan Wagner

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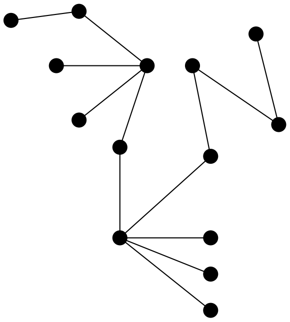
FPSAC, 4 July 2019

Why study trees?



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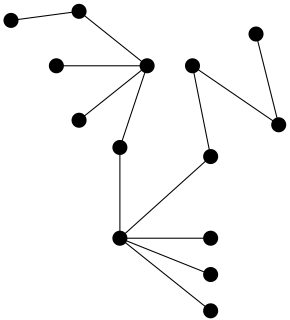

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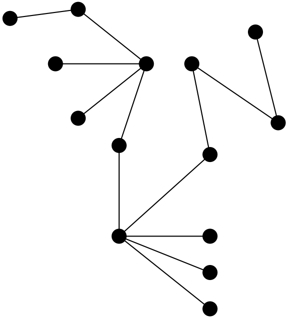

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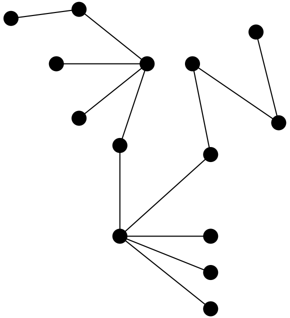

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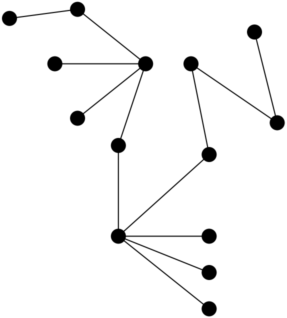

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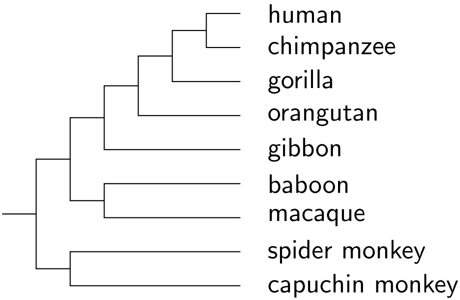

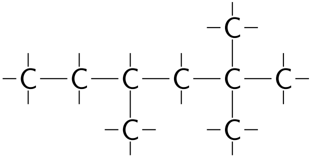

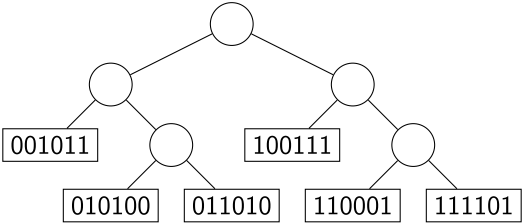

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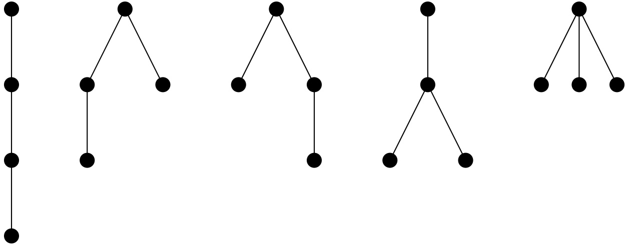
- have various restrictions (labels, vertex degrees, ...).

Depending on these, many different classes of trees have been studied in the literature.

Families of trees

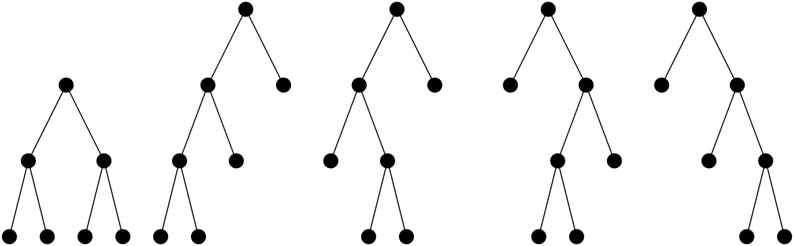
(Planted) plane trees: rooted trees embedded in the plane



The number of plane trees with n vertices is the *Catalan number* $\frac{1}{n}\binom{2n-2}{n-1}$.

Families of trees

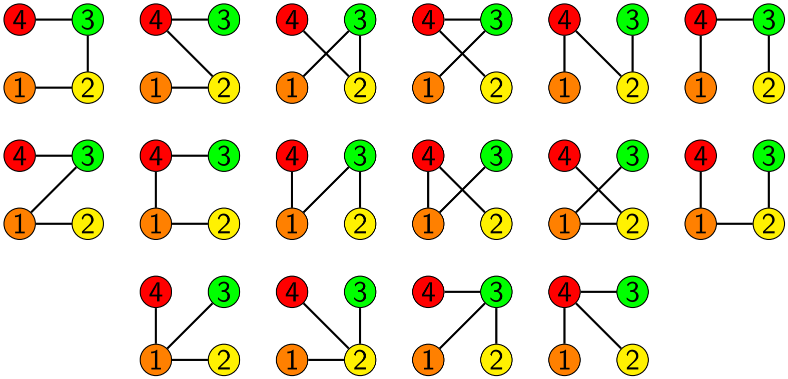
Binary trees: rooted trees where every vertex is either a leaf or has exactly two children (left and right).



The number of binary trees with n internal vertices is the *Catalan number* $\frac{1}{n+1}\binom{2n}{n}$.

Families of trees

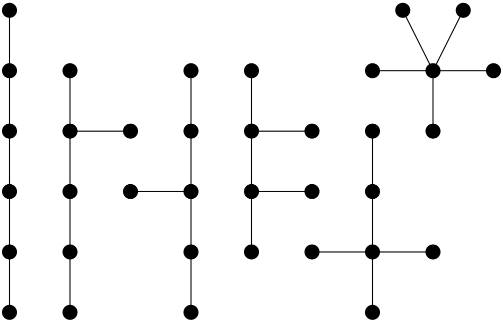
Labelled trees: each vertex has a unique label from 1 up to n (can be rooted or unrooted).



The number of labelled (unrooted) trees with n vertices is n^{n-2}.

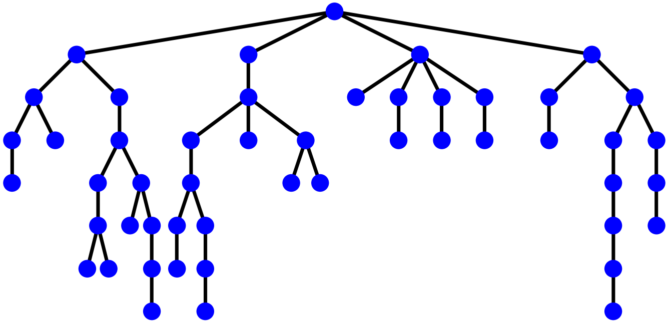
Families of trees

Unlabelled (unrooted) trees:



There is no simple formula for the number of unlabelled trees of a given size. The counting sequence starts $1, 1, 1, 2, 3, 6, 11, 23, 47, \ldots$, and there is an asymptotic formula for the number of trees with n vertices: $0.53495 \cdot n^{-5/2} \cdot 2.95577^n$.

Random trees



A random tree with 50 vertices. What is the underlying model?

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- Branching processes (e.g. Galton-Watson trees),
- Increasing tree models (e.g. recursive trees),
- Models based on random strings or permutations (e.g. tries, binary search trees).

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In particular, this is the case for *simply generated families of trees*.

Simply generated families

On the set of all rooted ordered (plane) trees, we impose a weight function by first specifying a sequence $1 = w_0, w_1, w_2, \ldots$ and then setting

$$w(T) = \prod_{i \geq 0} w_i^{N_i(T)},$$

where $N_i(T)$ is the number of vertices of outdegree i in T. Then we pick a tree of given order n at random, with probabilities proportional to the weights. For instance,

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- $w_i = \frac{1}{i!}$ generates random rooted labelled trees.

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Simply generated trees and Galton-Watson trees are essentially equivalent. For example, a geometric distribution for branching will result in a random plane tree, a Poisson distribution in a random rooted labelled tree.

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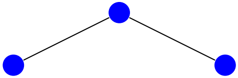

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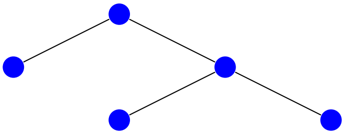

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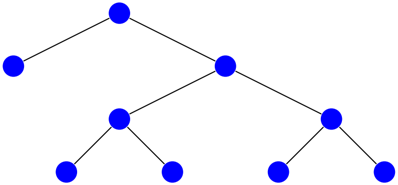

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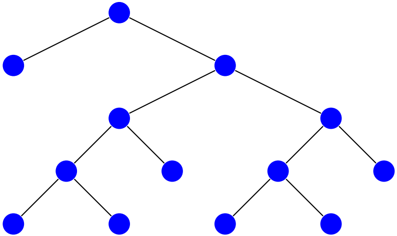

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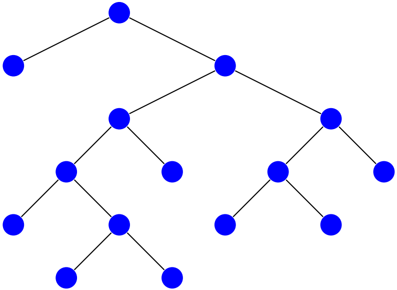

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Simply generated and Galton-Watson trees

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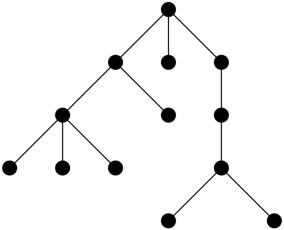

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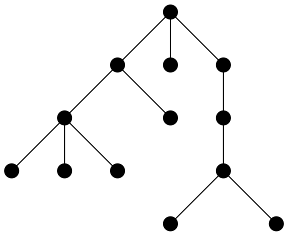

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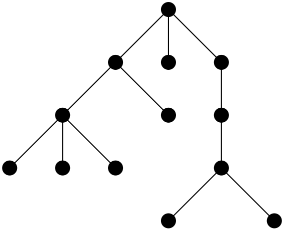

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The model can be modified by not choosing a parent uniformly at random, but depending on the current outdegrees (to generate, for example, binary increasing trees).

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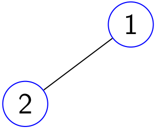

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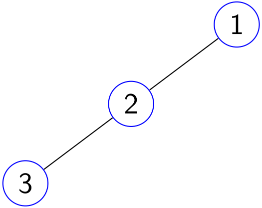

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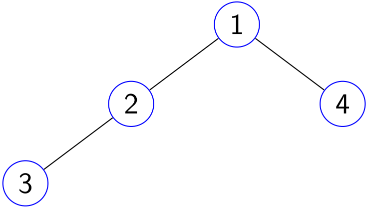

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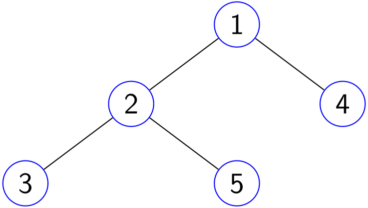

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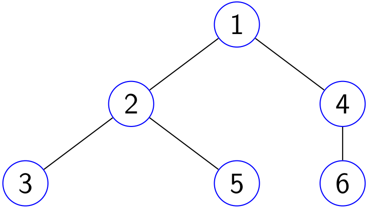

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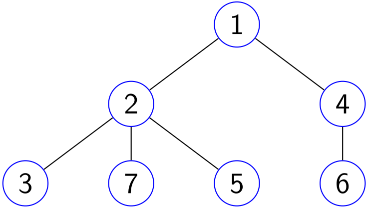

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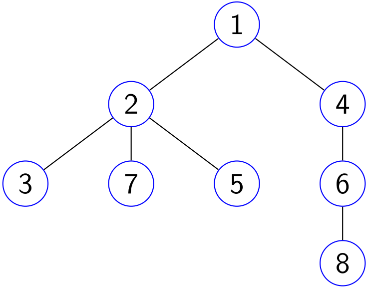

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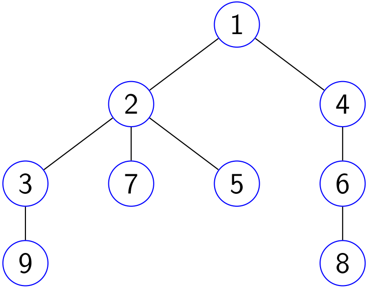

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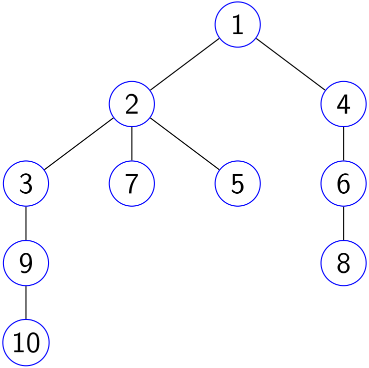

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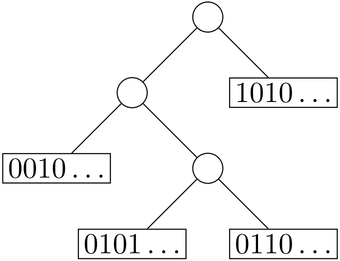

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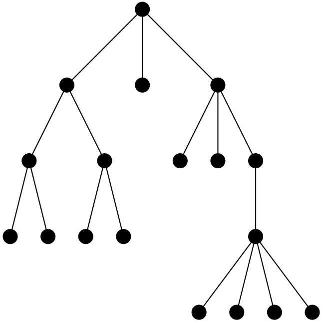
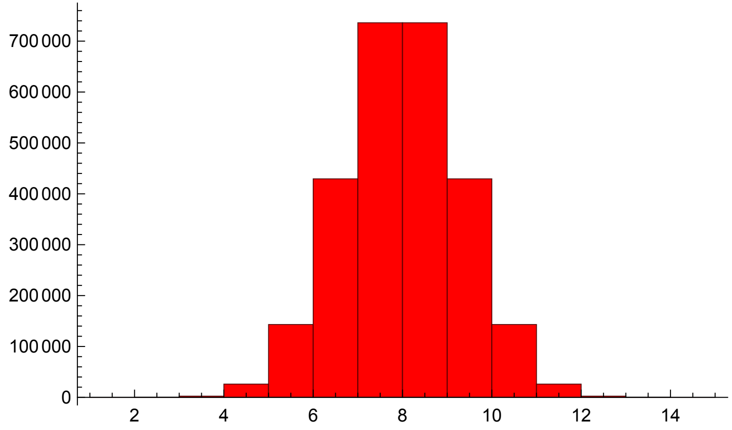
These questions become particularly relevant when n is large.

Some examples of parameters



The tree above has 11 leaves, 2 "cherries", height 4, path length 44, 384 automorphisms and 3945 subtrees.

Distribution of parameters: some examples



Distribution of the number of leaves in plane trees with 15 vertices. Plane trees with n vertices and k leaves are counted by the Narayana numbers $N_{n,k} = \frac{1}{n-1}\binom{n-1}{k}\binom{n-1}{k-1}$.

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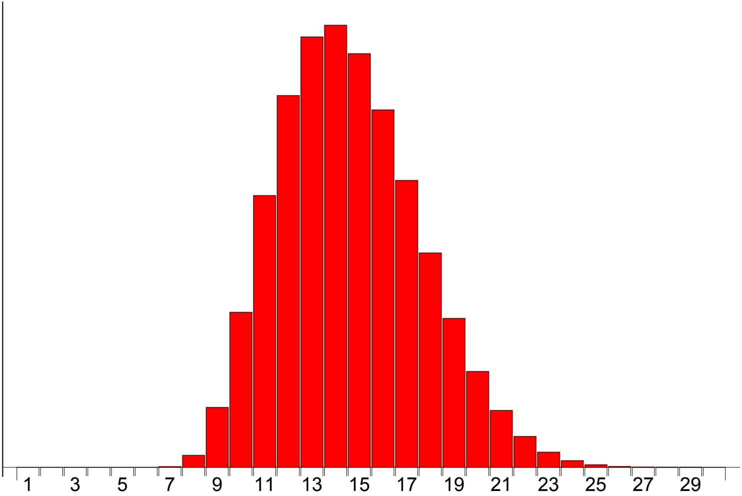
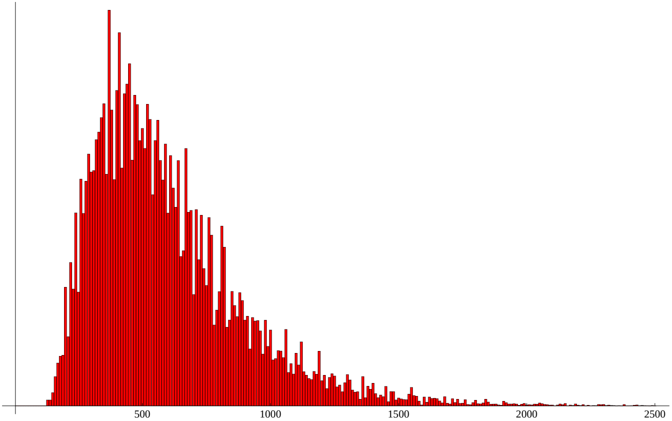


Distribution of the height in binary trees with 30 internal vertices.

Distribution of parameters: some examples



Distribution of the number of subtrees in labelled trees with 15 vertices.

Distributional results

In the following, let $\mathcal{F}$ be either a simply generated family of trees or the family of unlabelled rooted trees (Pólya trees), which is not simply generated, but has similar properties.

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We consider a random element $\mathcal{T}_n$ of $\mathcal{F}$ with n vertices.

For some parameter P, what can we say about the distribution of $P(\mathcal{T}_n)$?

The number of leaves

Theorem (Kolchin 1984, Drmota + Gittenberger 1999, Janson 2016)

For every family $\mathcal{F}$*, there exist constants* $\mu > 0$ *and* $\sigma^2 > 0$ *such that the number of leaves* $L(\mathcal{T}_n)$ *of a random tree* $\mathcal{T}_n$ *in* $\mathcal{F}$ *has mean* $\mu_n \sim \mu n$ *and variance* $\sigma_n^2 \sim \sigma^2 n$*.*

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Moreover, the renormalised random variable

$$X_n = \frac{L(\mathcal{T}_n) - \mu n}{\sqrt{\sigma^2 n}}$$

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$$X_n = \frac{L(\mathcal{T}_n) - \mu n}{\sqrt{\sigma^2 n}}$$

converges weakly to a standard normal distribution $N(0,1)$.

The theorem generalises to the number of vertices with a given degree or the number of fringe subtrees of a given shape.

The height

Theorem (Flajolet, Gao, Odlyzko + Richmond 1993, Drmota + Gittenberger 2010)

For every family $\mathcal{F}$, there exists a constant $\mu > 0$ such that the height $H(\mathcal{T}_n)$ of a random tree $\mathcal{T}_n$ in $\mathcal{F}$ has mean $\mu_n \sim \mu\sqrt{n}$.

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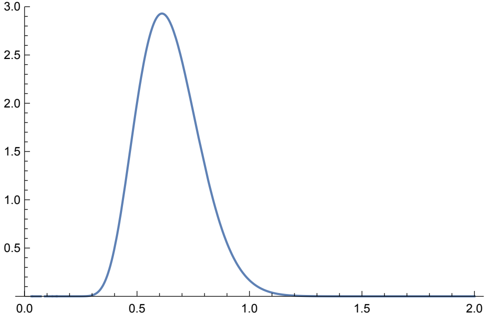
Moreover, the renormalised random variable

$$X_n = \frac{H(\mathcal{T}_n)}{c\sqrt{n}},$$

where $c = \frac{45\zeta(3)\mu}{2\pi^{5/2}}$, converges weakly to a so-called theta distribution, characterised by the density function

$$f(t) = \frac{4\pi^{5/2}}{3\zeta(3)} t^4 \sum_{m \geq 1} (m\pi)^2 (2(m\pi t)^2 - 3) \exp(-(m\pi t)^2).$$

The height



The theta distribution: limiting distribution of the height.

Path length and Wiener index

Theorem (Takács 1993, Janson 2003, SW 2012)

For every family $\mathcal{F}$, there exists a constant $\mu > 0$ such that the path length $D(\mathcal{T}_n)$ and the Wiener index $W(\mathcal{T}_n)$ of a random tree $\mathcal{T}_n$ in $\mathcal{F}$ have means $\mu_n^D \sim \mu n^{3/2}$ and $\mu_n^W \sim \frac{\mu}{2} n^{5/2}$ respectively.

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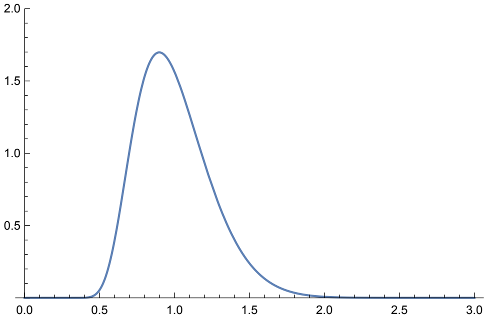
Moreover, the renormalised random variables

$$X_n = \frac{D(\mathcal{T}_n)}{\mu n^{3/2}} \qquad \text{and} \qquad Y_n = \frac{W(\mathcal{T}_n)}{\mu n^{5/2}}$$

converge weakly to random variables given in terms of a normalised Brownian excursion $e(t)$ on $[0,1]$:

$$\sqrt{\frac{8}{\pi}} \int_0^1 e(t)\,dt \quad \text{and} \quad \sqrt{\frac{8}{\pi}} \iint_{0<s<t<1} \Big(e(s) + e(t) - 2 \min_{s \le u \le t} e(u)\Big)\,ds\,dt.$$

The path length



The Airy distribution: limiting distribution of the path length.

Additive functionals: a general concept

A tree parameter is called an *additive functional* if it can be computed by adding its values for all the branches and adding a "toll function" that also depends on the tree.

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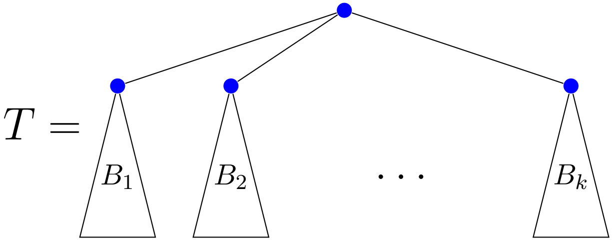

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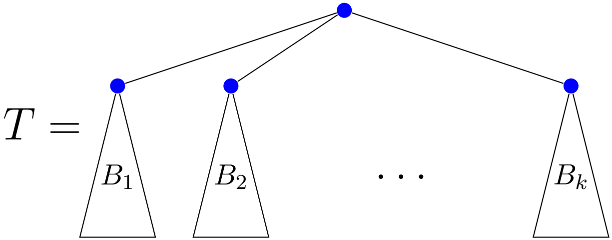

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Remark

The recursion remains true for the tree $T = \bullet$ of order 1 if we assume without loss of generality that $f(\bullet) = F(\bullet)$.

An equivalent definition

The fringe subtree T_v associated with a vertex v of a tree T is the subtree consisting of v and all its descendants.

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One can see by induction that the recursion

$$F(T) = F(B_1) + F(B_2) + \cdots + F(B_k) + f(T)$$

is equivalent to the formula

$$F(T) = \sum_v f(T_v).$$

Some examples

- The number of leaves, corresponding to the toll function

$$f(T) = \begin{cases} 1 & |T| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

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- The number of vertices whose outdegree is a fixed number k:

$$f(T) = \begin{cases} 1 & \text{if the root of } T \text{ has outdegree } k, \\ 0 & \text{otherwise.} \end{cases}$$

Some more examples

- The path length, i.e., the sum of the distances from the root to all vertices, can be obtained from the toll function $f(T) = |T| - 1$:

$$P(T) = \sum_{i=1}^{k} (P(B_i) + |B_i|) = |T| - 1 + \sum_{i=1}^{k} P(B_i).$$

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- The log-product of the subtree sizes, also called the "shape functional", corresponds to $f(T) = \log |T|$. It is related to the number of linear extensions:

$$\mathrm{LE}(T) = \binom{|T| - 1}{|B_1|, |B_2|, \ldots, |B_k|} \prod_{i=1}^{k} \mathrm{LE}(B_i),$$

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$$\log \frac{|T|!}{\mathrm{LE}(T)} = \log |T| + \sum_{i=1}^{n} \log \frac{|B_i|!}{\mathrm{LE}(B_i)}.$$

Even more examples

- The size of the automorphism group: if $c_1, c_2, \ldots, c_r$ are the multiplicities of the different isomorphism classes of branches, we have

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- The multiplicity of some eigenvalue λ:

$$N_\lambda(T) = \sum_{i=1}^{k} N_\lambda(B_i) + \epsilon_\lambda(T),$$

  where $\epsilon_\lambda(T) \in \{-1, 0, 1\}$.

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Hence

$$\log(1 + s_1(T)) = \sum_{i=1}^{k} \log(1 + s_1(B_i)) + \log(1 + s_1(T)^{-1}).$$

This means that $\log(1 + s_1(T))$ is additive with toll function $f(T) = \log(1 + s_1(T)^{-1})$.

General results

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In a nutshell, there are two types of conditions:

- The toll function f is “small” (at least on average) for large trees.
- The toll function f is “local” (only depends on a small neighbourhood of the root), at least approximately.

General results

Similar results are known for other tree models, specifically:

- increasing tree families: recursive trees, d-ary increasing trees, (generalised) plane-oriented recursive trees (Holmgren + Janson 2015, Holmgren + Janson + Šileikis 2017, Ralaivaosaona + SW 2019)
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Proofs involve:

- combinatorial techniques (generating functions, analytic combinatorics, . . .)
- probabilistic techniques (growth processes, urn models, . . .)

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- the average subtree size (N).

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Future work

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