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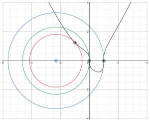
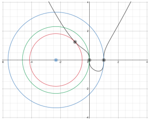


Tangency and Discriminants

FPSAC 2019, Lubljana

Sandra Di Rocco,

Goal



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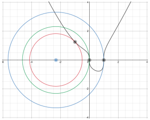

- Discriminants: tangency and duality

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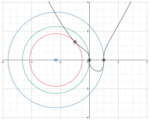

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- Discriminants: tangential intersections

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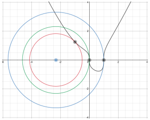

- Discriminants: tangency and duality
- Discriminants: tangential intersections
- Generalized Schäfli decomposition

Natural concept

The discriminant is a concept occurring naturally in connection with the way we grasp 3D objects.

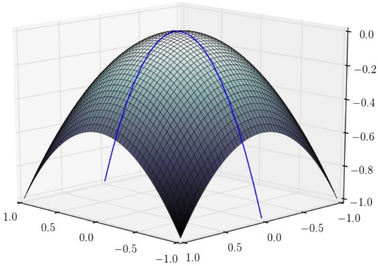

Figure: Boundary locally defined by $f(x, y, z) = 0$. The **Discriminant with respect to** x is the "plane curve" defined by the equation obtained by eliminating x from $\{f = 0, \frac{\partial f}{\partial x} = 0\}$

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- the **discriminant** of $c_2x^2 + c_1x + c_0$ is $c_1^2 - 4c_2c_0$.
- for higher degrees the **discriminant** D_n is a $(2n-1) \times (2n-1)$ determinant :

$$D_n = (1/c_n)det \begin{bmatrix} c_n & c_{n-1} & \cdots & c_0 & 0 & \cdots & 0 \\ 0 & c_n & c_{n-1} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ nc_n & (n-1)c_{n-1} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \cdots & \cdots & 2c_2 & c_1 \end{bmatrix}$$

Algebra vs Geometry

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Algebra: $D_n = Res(p(x), p'(x))$

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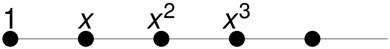
Geometry: 

$\mathbb{P}^1 \hookrightarrow \mathbb{P}^d$

$J_1 = \mathcal{O}_{\mathbb{P}^1}(d-1) \oplus \mathcal{O}_{\mathbb{P}^1}(d-1)$ and $\deg(c_1(J_1)) = 2d-2$

The definition of discriminant

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$$\mathcal{A} = \{m_0, m_1, \ldots, m_n\}$$

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A polynomial p in d variables is **supported on** $\mathcal{A}$ if

$$p(x_1, \ldots, x_d) = \sum_{m_i \in \mathcal{A}} c_i x^{m_i}$$

where $x^m = x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}$ if $m = (k_1, \ldots, k_d) \in \mathcal{A} \in \mathbb{Z}^d$.

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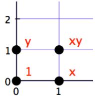

Figure: Quadrics $c_0 + c_1 x + c_2 y + c_3 xy$

The definition of discriminant

Definition

Let $\mathcal{A} = \{m_0, m_1, \ldots, m_n\} \subset \mathbb{Z}^d$. The **discriminant** of $\mathcal{A}$ is (if it exists!) a polynomial $D_{\mathcal{A}}(c_0, \ldots, c_n)$ in $n+1$ variables vanishing whenever the corresponding polynomial $p(x) = \sum_{m_i \in \mathcal{A}} c_i x^{m_i}$ has some multiple root in $(\mathbb{C}^*)^d$.

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$$D_{\mathcal{A}}(c_0, \dots, c_n) = 0 \Leftrightarrow \begin{array}{c} \text{there is } x \in (\mathbb{C}^*)^d \text{ s.t.} \\ p(x) = \dots = \frac{\partial p}{\partial x_j}(x) = \dots = 0 \end{array}$$

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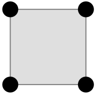
Existence does not mean an efficient algorithm and hence a formula!

Example 1

For the configuration $\mathcal{A} = \{(0,0),(1,0),(0,1),(1,1)\} \subset \mathbb{Z}^2$

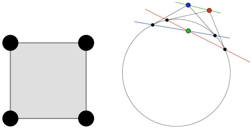


The discriminant is given by an homogeneous polynomial $\Delta_{\mathcal{A}}(c_0, c_1, c_2, c_3)$ vanishing whenever the corresponding quadric has a singular point in $(\mathbb{C}^*)^2$. I

$$D_{\mathcal{A}}(c_0, c_1, c_2, c_3) = \det(M) = c_0 c_3 - c_1 c_2.$$

Geometry

Geometry



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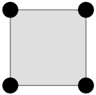

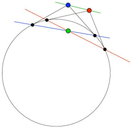

Let $Q \subset \mathbb{C}^2$ and $p \in \mathbb{C}^2$

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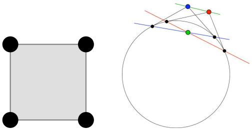

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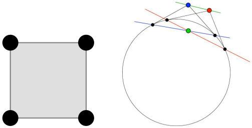

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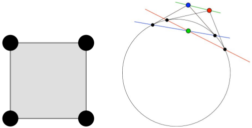

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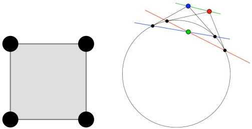
- general tangent lines to Q do not contain the point p
- exceptional locus: $\{x \in Q \mid p \in \mathbb{T}_{Q,x}\}$ has degree 2.
- It gives the degree of the discriminant.

Geometry



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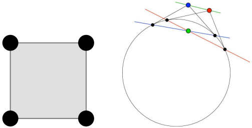

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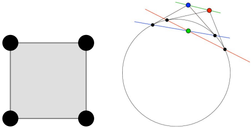

- The polar classes P_i are codimension i cycles on $X \hookrightarrow \mathbb{P}^N$
- P_1 on Q is a zero-cycle of degree 2.

Projective duality

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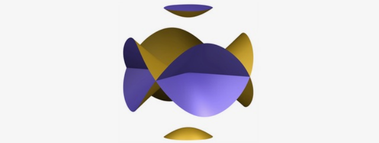
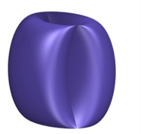


$x^4 - 2x^2y^2 + y^4 - 2x^2w^2 - 2y^2w^2 - 16w^2 + w^4 = 0$



$16x^2 - y^4 + 2y^2z^2 - 8x^2y^2 - z^4 - 8x^2z^2 - 16x^4 = 0$

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- It is defined by an irreducible polynomial D_X, called the discriminant.

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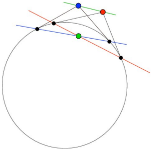

Figure: C^* is another conic, $\deg(P_1(X)) = 2$

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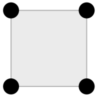

$(x, y) \to (1, x, y, xy)$

$3! \cdot Area - 2!(perimeter) + 4 =$

$6 - 8 + 4 = 2$

Can a discriminant govern multiple roots of systems of polynomials?

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As before: Let $\mathcal{A}_1, \ldots, \mathcal{A}_d$ be (finite) in $\mathbb{Z}^d$ and let $f_1, \ldots, f_d$ be Laurent polynomials with these support sets and coefficients in an alg. cl. field K, e.g. $\mathbb{C}$:

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If the coefficients $c_{i,a}$ are generic then, by *Bernstein's Theorem*, the number of common solutions in the algebraic torus $(\mathbb{C}^*)^d$ equals the *mixed volume* $MV(Q_1, Q_2, \ldots, Q_d)$ of the Newton polytopes $Q_i = conv(A_i)$ in $\mathbb{R}^d$.

Example

Let $n = 2$ and $\mathcal{A}_1 = \mathcal{A}_2 = \{(0,0),(1,0),(0,1),(1,1)\}$ be the unit square, $f_1 = a_{00} + a_{10}x_1 + a_{01}x_2 + a_{11}x_1x_2$, $f_2 = b_{00} + b_{10}x_1 + b_{01}x_2 + b_{11}x_1x_2$.

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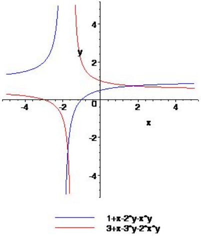

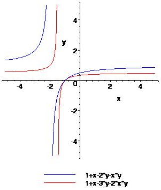

tangential intersections

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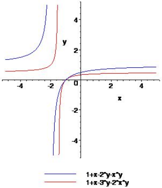
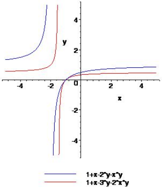


Given p_{A_1}, p_{A_2}, we say that x is a tangential solution of the system $p_{A_1}(u) = p_{A_2}(u) = 0$ if x is a regular point of the hypersurfaces $p_{A_i} = 0$ and their normal lines are dependent.

tangential intersections



Given p_{A_1}, p_{A_2}, we say that x is a tangential solution of the system $p_{A_1}(u) = p_{A_2}(u) = 0$ if x is a regular point of the hypersurfaces $p_{A_i} = 0$ and their normal lines are dependent.

Definition

Given a system of type $(A_0, \ldots, A_r)$. We call an isolated solution $u \in (\mathbb{C}^*)^n$ a *non-degenerate multiple root* if the $r+1$ gradient vectors $\nabla_x p_{A_i}(u), i = 0, \ldots, r$ are linearly dependent.

The mixed discriminant

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Given $\mathcal{A}_0, \ldots, \mathcal{A}_r \subset \mathbb{Z}^d$

Definition

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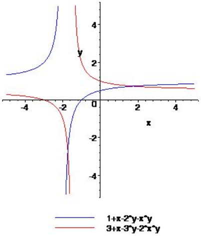
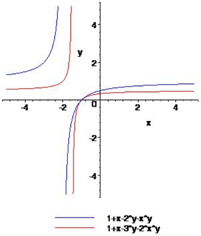
When $\mathcal{A}_0 = \cdots = \mathcal{A}_r = \mathcal{A}$ we denote it by $M(r, \mathcal{A})$.

Example

Let $n = 2$ and $\mathcal{A}_1 = \mathcal{A}_2 = \{(0,0),(1,0),(0,1),(1,1)\}$ be the unit square, $f_1 = a_{00} + a_{10}x_1 + a_{01}x_2 + a_{11}x_1x_2$, $f_2 = b_{00} + b_{10}x_1 + b_{01}x_2 + b_{11}x_1x_2$.





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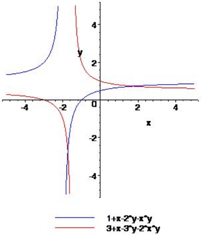

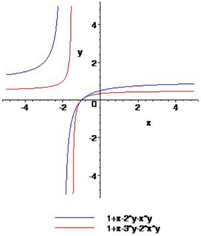

$\Delta_{\mathcal{A}_1,\mathcal{A}_2}$ is the **hyperdeterminant** of format $2\times2\times2$:

$$a_{00}^2b_{11}^2 - 2a_{00}a_{01}b_{10}b_{11} - 2a_{00}a_{10}b_{01}b_{11} - 2a_{00}a_{11}b_{00}b_{11} + 4a_{00}a_{11}b_{01}b_{10} + a_{01}^2b_{10}^2 + 4a_{01}a_{10}b_{00}b_{11} - 2a_{01}a_{10}b_{01}b_{10} - 2a_{01}a_{11}b_{00}b_{10} + a_{10}^2b_{01}^2 - 2a_{10}a_{11}b_{00}b_{01} + a_{11}^2b_{00}^2$$

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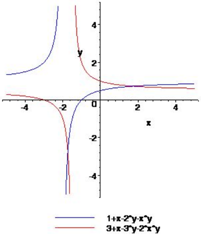

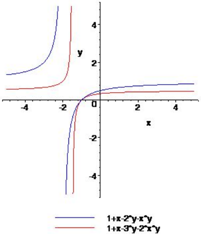

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bidegree $= (2,2)$

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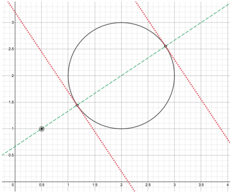

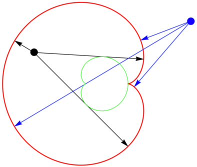

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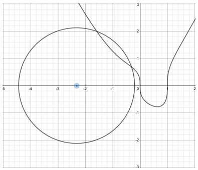
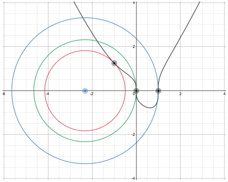
Consider now a plane curve. Equivalently one looks at the circles admitting tangent solutions with the curve.





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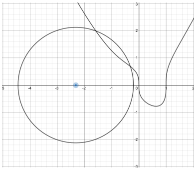

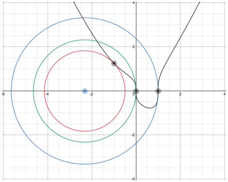

C is a conic: 3x3 matrix $M(c_{ij})$ and the circle by the the 3x3 symmetric matrix $M(u, r)$.
The *Mixed Discriminant* is given by the 2x3x3 hyperderminant: $H(c_{ij}, u, r)$.

This proves:

Theorem (Cayley)

Let C be an irreducible conic, then

- *EDD(Circle) = 2*
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The key tool is the use of *Schläfli decomposition*

$$MD(A_1, A_2) = Hyperdet([M_1, M_2]) = Disc_t(det(M_1 + tM_2)).$$

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Key tools:

- Classified by the Hyperdeterminant, i.e. **discriminant of Segre embeddings**
- The hyperdeterminant can be computed by **iteration**

Two Main Questions:

- **Question 1** Can the mixed discriminant be computed via iteration?
- **Question 2** What about singular intersection of higher dimensional quadrics?

Towards an answer to question 1

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Definition

Let $\mathcal{A} \subset \mathbb{Z}^d$, such that $D_{\mathcal{A}} \neq 1, \deg(D_{\mathcal{A}}) = \delta$, and let $(\lambda_0, \ldots, \lambda_r) \in \mathbb{C}^{r+1}$. Define the **iterated discriminant** as:

$$ID_{r,\mathcal{A}} = D_{\delta\Delta_r}(D_{\mathcal{A}}(\lambda_0 f_0 + \ldots + \lambda_r f_r))$$

Abuse of notation: $f_i = (c_0^i, \ldots, c_N^i)$
$\deg(I_{r,\mathcal{A}}) = \delta(\delta - 1)(r + 1)$

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$\mathcal{A} \subset \mathbb{Z}^d$, $D_{\mathcal{A}} \neq 1$ and $0 \leqslant r \leqslant d$. Then, the mixed discriminant $MD_{r,\mathcal{A}} \neq 1$ divides the iterated discriminant $ID_{r,\mathcal{A}}$. Moreover,

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3. *If $codim_{X_{\mathcal{A}}^*}(sing(X_{\mathcal{A}}^*)) < r$, $ID_{r,\mathcal{A}} = 0$.*

Answer to question 2

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Theorem (Dickenstein-DR-Morrison)

Let Q_1, Q_2 be two d-dimensional quadric hypersurfaces then:

$$Q_1 \cap Q_2 \text{ singular if and only if } I_{1,2\Delta_d} = MD_{1,2\Delta_d}$$

$$(\text{THANK YOU})^n$$

$n >> 1$