

Computer Algebra for Lattice Path Combinatorics

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inria
informatics mathematics

FPSAC 2019
Ljubljana, Slovenia
July 1st, 2019

Enumerative Combinatorics: science of counting

Area of mathematics primarily concerned with counting discrete objects.

▷ Main outcome: theorems

Computer Algebra: effective mathematics

Area of computer science primarily concerned with the algorithmic manipulation of algebraic objects.

▷ Main outcome: algorithms

Computer Algebra for **Enumerative Combinatorics**

Today: **Algorithms** for proving **Theorems** on **Lattice Paths Combinatorics**.

An (innocent looking) combinatorial question

Let $\mathcal{S} = \{\uparrow, \leftarrow, \searrow\}$. An \mathcal{S} -walk is a path in \mathbb{Z}^2 using only steps from \mathcal{S} . Show that, for any integer n , the following quantities are equal:

- (i) number a_n of n -steps \mathcal{S} -walks confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and finish at the origin $(0,0)$ (*excursions*);
- (ii) number b_n of n -steps \mathcal{S} -walks confined to the quarter plane \mathbb{N}^2 that start at the origin $(0,0)$ and finish on the diagonal of \mathbb{N}^2 (*diagonal walks*).

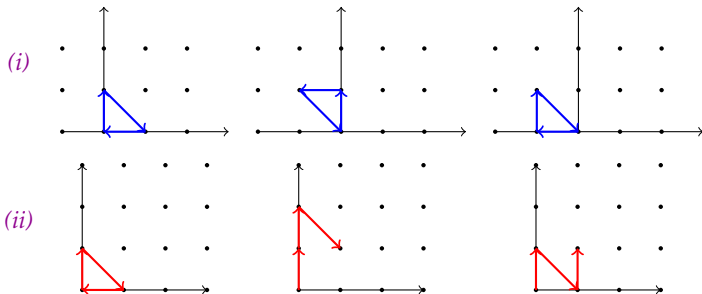
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(ii) number b_n of n -steps \mathcal{S} -walks confined to the quarter plane \mathbb{N}^2 that start at the origin $(0,0)$ and finish on the diagonal of \mathbb{N}^2 (*diagonal walks*).

For instance, for $n = 3$, this common value is $a_3 = b_3 = 3$:



Teaser 1: This “exercise” is non-trivial

Teaser 2: ...but it can be solved using **Computer Algebra**

Teaser 3: ...by two robust and efficient algorithmic techniques,
Guess-and-Prove and **Creative Telescoping**

Why care about counting walks?

Many objects can be encoded by walks:

- probability theory (voting, games of chance, branching processes, ...)
- discrete mathematics (permutations, trees, words, urns, ...)
- statistical physics (Ising model, ...)
- operations research (queueing theory, ...)

**7TH INTERNATIONAL CONFERENCE ON
LATTICE PATH COMBINATORICS AND APPLICATIONS**



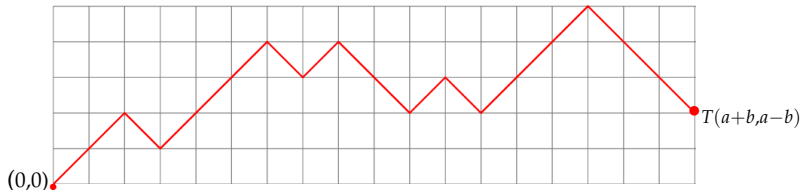
Siena, Italy July 4-7, 2010

HOME	TOPICS to be covered include (but are not limited to) :	
Photo	Lattice path enumeration	Random walks
Program	Plane Partitions	Non parametric statistical inference
Proceedings	Young tableaux	Discrete distributions and urn models
Submission	q-calculus	Queueing theory
Important dates	Orthogonal polynomials	Analysis of algorithms
Participants		Graph Theory and Applications
General Information		Self-dual codes and unimodular lattices
		Bijections between paths and other combinatoric structures

Counting walks is an old topic: the ballot problem [Bertrand, 1887]

Suppose that candidates A and B are running in an election. If a votes are cast for A and b votes are cast for B , where $a > b$, then the probability that A stays ahead of B throughout the counting of the ballots is $(a - b)/(a + b)$.

Lattice path reformulation: find the number of paths in \mathbb{Z}^2 with a upsteps ↗ and b downsteps ↘ that start at the origin and never touch the x -axis

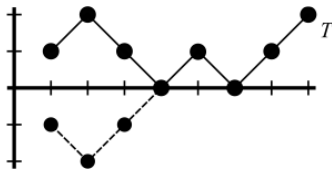


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Lattice path reformulation: find the number of paths in \mathbb{Z}^2 with $a - 1$ upsteps \nearrow and b downsteps \searrow that start at $(1, 1)$ and never touch the x -axis

Reflection principle [Aebly, 1923]: paths in \mathbb{Z}^2 from $(1, 1)$ to $T(a + b, a - b)$ that do touch the x -axis are in bijection with paths in \mathbb{Z}^2 from $(1, -1)$ to T



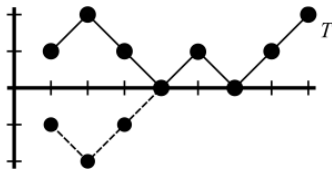
$$\text{Answer: } \underbrace{\binom{a+b-1}{a-1}}_{\text{(paths in } \mathbb{Z}^2 \text{ from } (1, 1) \text{ to } T)} - \underbrace{\binom{a+b-1}{b-1}}_{\text{(paths in } \mathbb{Z}^2 \text{ from } (1, -1) \text{ to } T)}$$

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Answer: $\underbrace{(\text{paths in } \mathbb{Z}^2 \text{ from } (1, 1) \text{ to } T) - (\text{paths in } \mathbb{Z}^2 \text{ from } (1, -1) \text{ to } T)}$

$$\binom{a+b-1}{a-1} - \binom{a+b-1}{b-1} = \frac{a-b}{a+b} \binom{a+b}{a}$$

Lot of recent activity; many recent contributors:

Arquès, Bacher, Banderier, Bernardi, Bostan, Bousquet-Mélou, Budd, Chyzak, Cori, Courtiel, Denisov, Dreyfus, Du, Duchon, Dulucq, Duraj, Fayolle, Fisher, Flajolet, Fusy, Garbit, Gessel, Gouyou-Beauchamps, Guttmann, Guy, Hardouin, van Hoeij, Hou, Iasnogorodski, Johnson, Kauers, Kenyon, Koutschan, Krattenthaler, Kreweras, Kurkova, Malyshev, Melczer, Miller, Mishna, Niederhausen, Pech, Petkovšek, Prellberg, Raschel, Rechnitzer, Roques, Sagan, Salvy, Sheffield, Singer, Viennot, Wachtel, Wang, Wilf, D. Wilson, M. Wilson, Yatchak, Yeats, Zeilberger, ...

etc.

...but it is still a very hot topic

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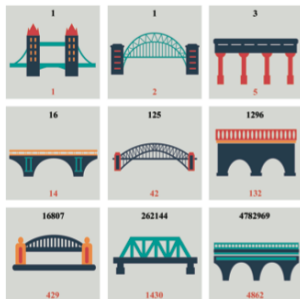
~~Specific question
Ad hoc solution~~



Systematic approach

DISCRETE MATHEMATICS AND ITS APPLICATIONS

HANDBOOK OF ENUMERATIVE COMBINATORICS



Edited by
Miklós Bóna

 **CRC Press**
Taylor & Francis Group
A CHAPMAN & HALL BOOK

Chapter 10

Lattice Path Enumeration

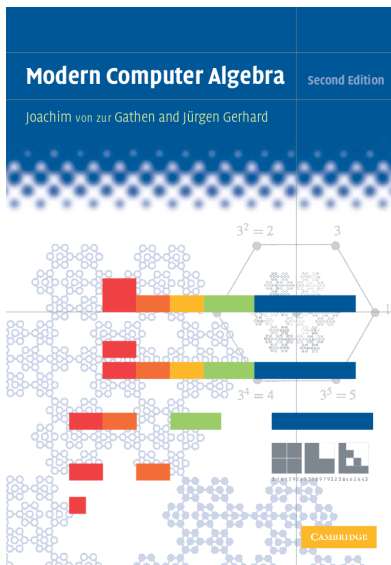
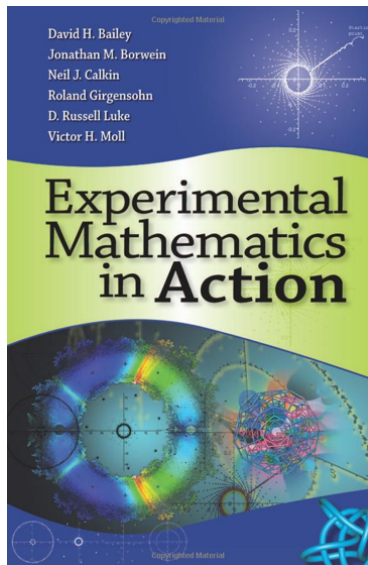
Christian Krattenthaler

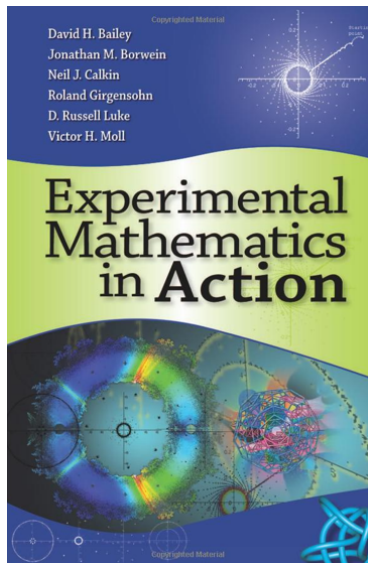
Universität Wien

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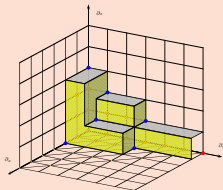
Our approach: Experimental Mathematics using Computer Algebra





Algorithmes Efficaces en Calcul Formel

Alin BOSTAN
Frédéric CHYZAK
Marc GIUSTI
Romain LEBRETON
Grégoire LECERF
Bruno SALVY
Éric SCHOST



▷ Nearest-neighbor walks in the quarter plane:

\mathcal{S} -walks in \mathbb{N}^2 : starting at $(0,0)$ and using steps in a *fixed* subset \mathcal{S} of

$$\{\swarrow, \leftarrow, \nearrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}$$

▷ Counting sequence $q_{\mathcal{S}}(n)$: number of \mathcal{S} -walks of length n

▷ Generating function:

$$Q_{\mathcal{S}}(t) = \sum_{n=0}^{\infty} q_{\mathcal{S}}(n)t^n \in \mathbb{Z}[[t]]$$

▷ Nearest-neighbor walks in the quarter plane:

\mathcal{S} -walks in \mathbb{N}^2 : starting at $(0,0)$ and using steps in a *fixed* subset \mathcal{S} of

$$\{\swarrow, \leftarrow, \nearrow, \uparrow, \searrow, \rightarrow, \downarrow\}$$

▷ Counting sequence $q_{\mathcal{S}}(i, j; n)$: number of walks of length n ending at (i, j)

▷ Complete generating function (with “catalytic” variables x, y):

$$Q_{\mathcal{S}}(x, y; t) = \sum_{i, j, n=0}^{\infty} q_{\mathcal{S}}(i, j; n) x^i y^j t^n \in \mathbb{Z}[[x, y, t]]$$

Entire books dedicated to small step walks in the quarter plane!

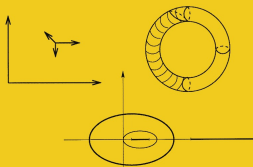
Applications of Mathematics
Stochastic Modelling and Applied Probability

40

Guy Fayolle
Roudolf Iasnogorodski
Vadim Malyshev

Random Walks in the Quarter-Plane

Algebraic Methods,
Boundary Value Problems
and Applications



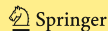
Probability Theory and Stochastic Modelling 40

Guy Fayolle
Roudolf Iasnogorodski
Vadim Malyshev

Random Walks in the Quarter Plane

Algebraic Methods, Boundary Value
Problems, Applications to Queueing
Systems and Analytic Combinatorics

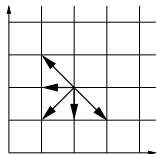
Second Edition



Among the 2^8 step sets $\mathcal{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, some are:

Small-step models of interest

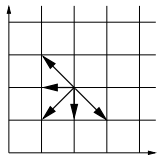
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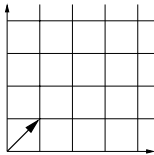
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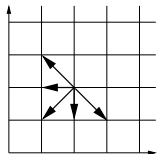
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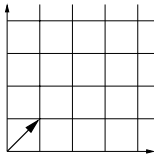
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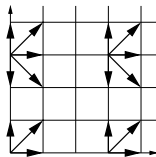
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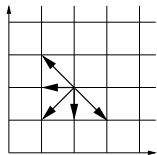
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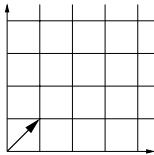
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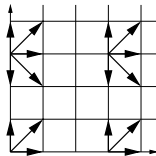
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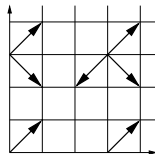
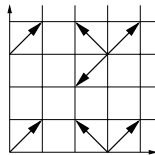
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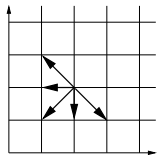
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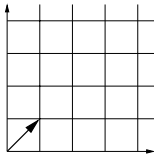
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Small-step models of interest

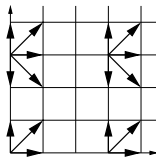
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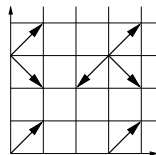
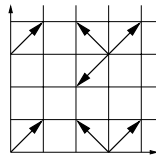
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One is left with [79 interesting distinct models](#).

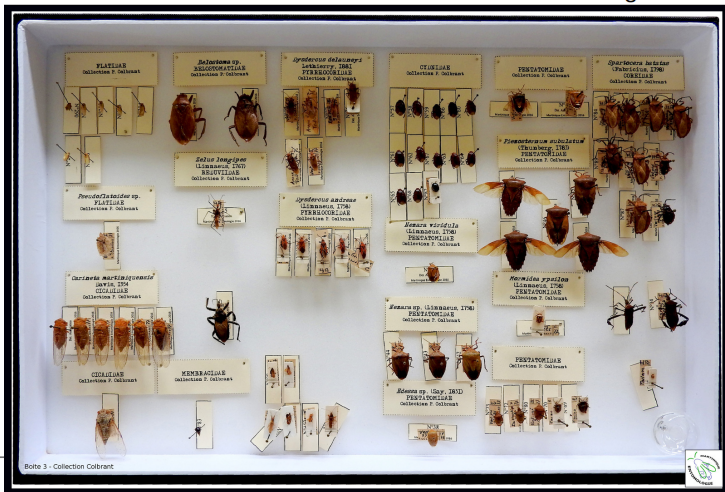
The 79 small steps models of interest



Task: classify their generating functions!

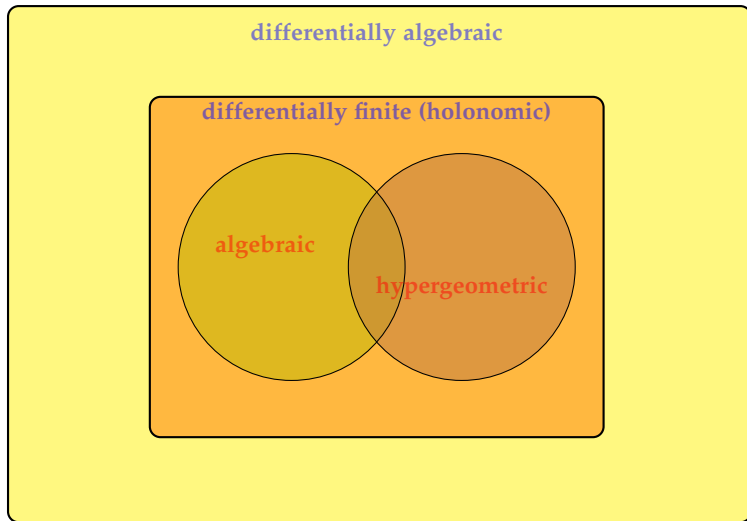


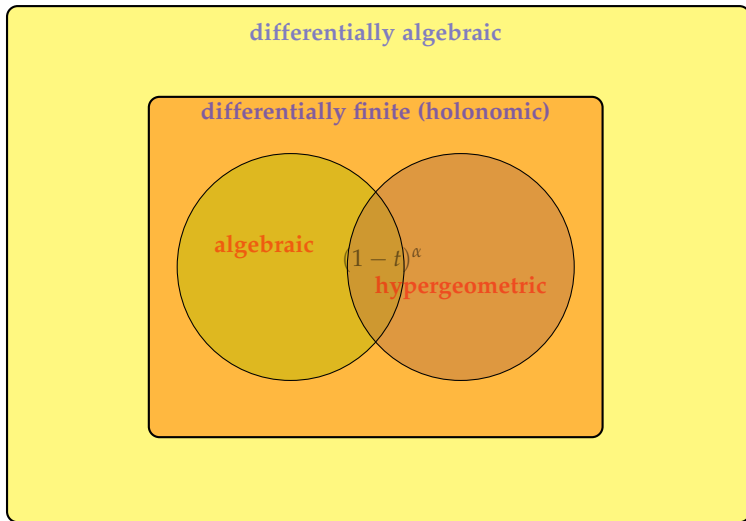
Non-singular

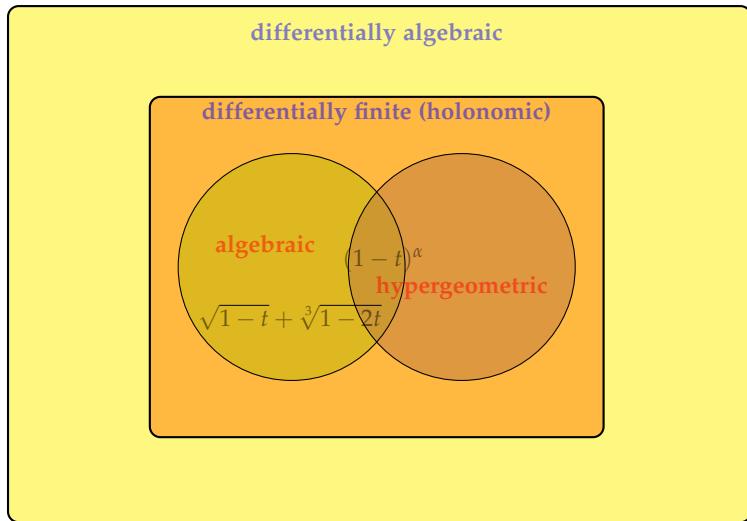


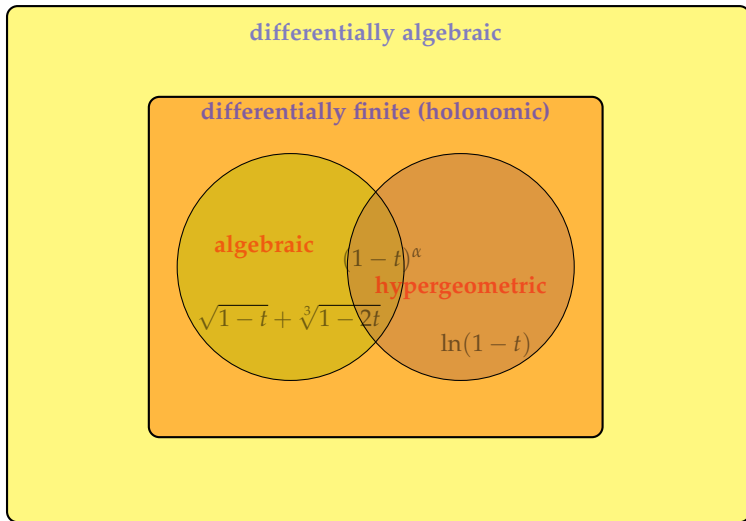
Singular

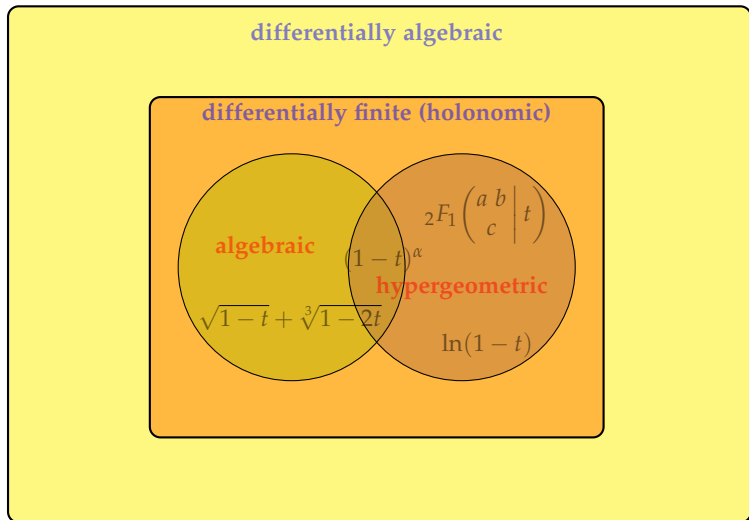




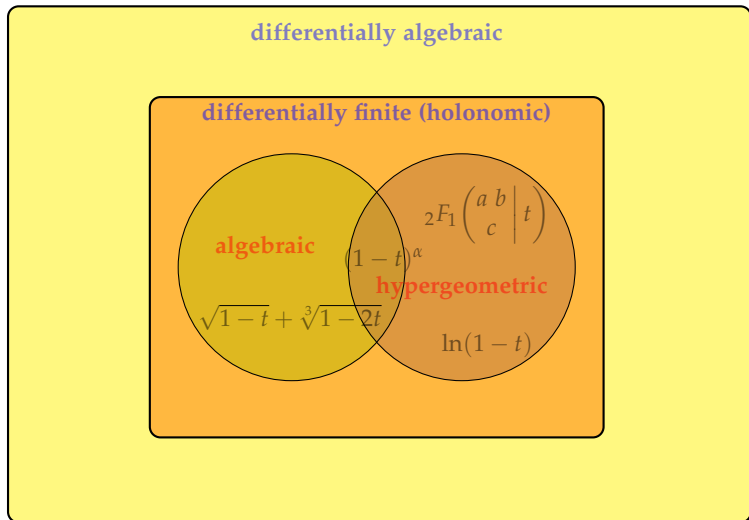






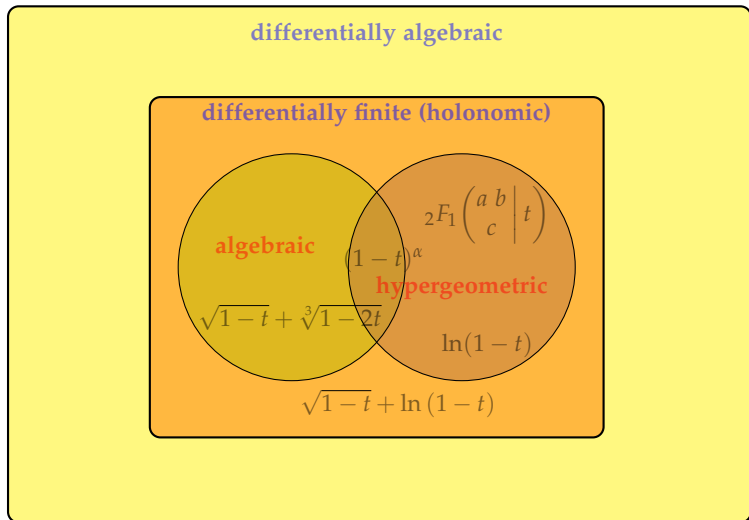


$${}_2F_1\left(\begin{matrix} a & b \\ c \end{matrix} \middle| t\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad \text{where } (a)_n = a(a+1) \cdots (a+n-1).$$



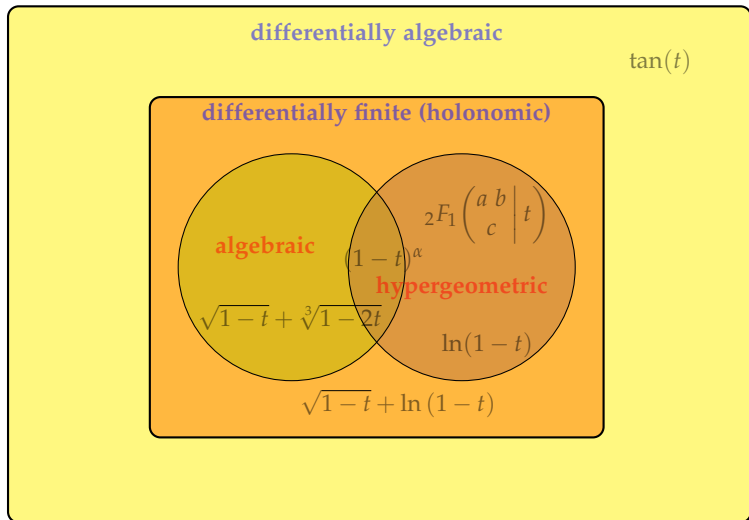
E.g., $(1-t)^\alpha = {}_2F_1\left(\begin{matrix} -\alpha & 1 \\ 1 \end{matrix} \middle| t\right)$, $\ln(1-t) = -t \cdot {}_2F_1\left(\begin{matrix} 1 & 1 \\ 2 \end{matrix} \middle| t\right) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$

Classification criterion: properties of generating functions



$${}_2F_1\left(\begin{matrix} a & b \\ c \end{matrix} \middle| t\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad \text{where } (a)_n = a(a+1) \cdots (a+n-1).$$

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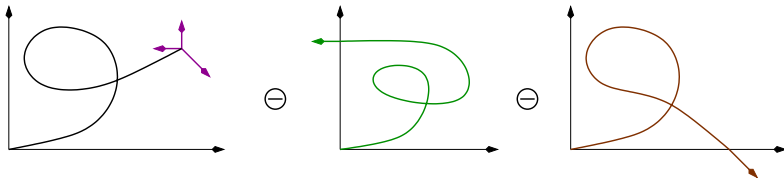
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Algebraic reformulation of main task: solving a functional equation

Generating function: $Q(x, y) \equiv Q(x, y; t) = \sum_{i, j, n=0}^{\infty} q(i, j; n) x^i y^j t^n \in \mathbb{Z}[[x, y, t]]$

Recursive construction yields the *kernel equation*

$$Q(x, y) = 1 + t \left(y + \frac{1}{x} + x \frac{1}{y} \right) Q(x, y) - t \frac{1}{x} Q(0, y) - t x \frac{1}{y} Q(x, 0)$$

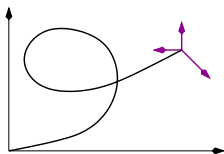


Algebraic reformulation of main task: solving a functional equation

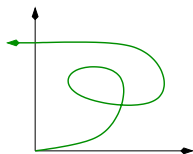
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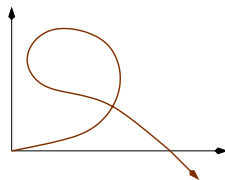
$$\left(1 - t \left(y + \frac{1}{x} + x \frac{1}{y}\right)\right) xyQ(x, y) = xy - tyQ(0, y) - tx^2Q(x, 0)$$



⊖



⊖

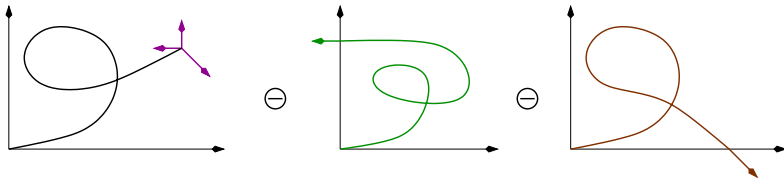


Algebraic reformulation of main task: solving a functional equation

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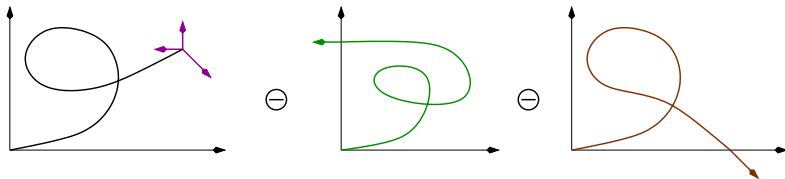
New task: Solve this functional equation!

Algebraic reformulation of main task: solving a functional equation

Generating function: $Q(x, y) \equiv Q(x, y; t) = \sum_{i, j, n=0}^{\infty} q(i, j; n) x^i y^j t^n \in \mathbb{Z}[[x, y, t]]$

Recursive construction yields the *kernel equation*


$$\left(1 - t \left(y + \frac{1}{x} + x \frac{1}{y}\right)\right) xyQ(x, y) = xy - tyQ(0, y) - tx^2Q(x, 0)$$



New task: For the other models – solve 78 similar equations!

“Special” models of walks in the quarter plane

Dyck: 


Motzkin: 


Pólya: 

Kreweras: 

Gessel: 

Gouyou-Beauchamps: 

King walks: 

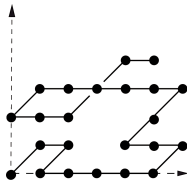
Tandem walks: 



- $g(n)$ = number of n -steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$ -walks in \mathbb{N}^2
1, 2, 7, 21, 78, 260, 988, 3458, 13300, 47880, ...

Question: What is the nature of the generating function

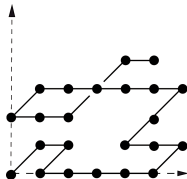
$$G(t) = \sum_{n=0}^{\infty} g(n) t^n ?$$



- $g(i, j; n)$ = number of n -steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$ -walks in \mathbb{N}^2 from $(0, 0)$ to (i, j)

Question: What is the nature of the generating function

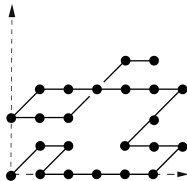
$$G(x, y; t) = \sum_{i, j, n=0}^{\infty} g(i, j; n) x^i y^j t^n ?$$



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$$G(x, y; t) = \sum_{i, j, n=0}^{\infty} g(i, j; n) x^i y^j t^n ?$$



Theorem [B., Kauers, 2010]

$G(x, y; t)$ is an algebraic function[†].

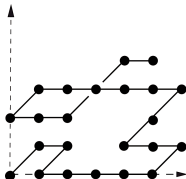
▷ computer-driven discovery / proof via *algorithmic Guess-and-Prove*

[†] Minimal polynomial $P(G(x, y; t); x, y, t) = 0$ has $> 10^{11}$ terms; ≈ 30 Gb (6 DVDs!)

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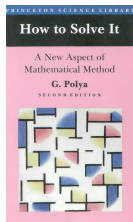
$$G(t) = \sum_{n=0}^{\infty} g(n) t^n ?$$



Corollary [B., Kauers, 2010] (former conjecture of Gessel's)

$$(3n + 1) g(2n) = (12n + 2) g(2n - 1) \text{ and } (n + 1) g(2n + 1) = (4n + 2) g(2n)$$

▷ computer-driven discovery/proof via *algorithmic Guess-and-Prove*



Guessing and Proving

George Pólya

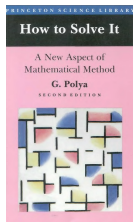


What is “scientific method”? Philosophers and non-philosophers have discussed this question and have not yet finished discussing it. Yet as a first introduction it can be described in three syllables:

Guess and test.

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

First guess, then prove.



Guessing and Proving

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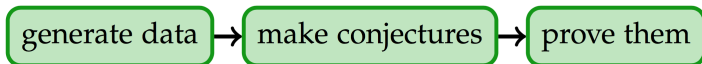


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Guess-and-Prove: a toy example

Question: Find $B_{i,j} :=$ the number of $\{\rightarrow, \uparrow\}$ -walks in \mathbb{N}^2 from $(0,0)$ to (i,j)

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- ① There are 2 ways to get to (i,j) , either from $(i-1,j)$, or from $(i,j-1)$:

$$B_{i,j} = B_{i-1,j} + B_{i,j-1}$$

- ② There is only one way to get to a point on an axis: $B_{i,0} = B_{0,j} = 1$

▷ These two rules completely determine all the numbers $B_{i,j}$

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⋮

(I) Generate data:

1	7	28	84	210	462	924	
1	6	21	56	126	252	462	
1	5	15	35	70	126	210	
1	4	10	20	35	56	84	
1	3	6	10	15	21	28	
1	2	3	4	5	6	7	
1	1	1	1	1	1	1	...

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(II) Guess:

$$\begin{aligned} &\longrightarrow \dots \\ &\longrightarrow \frac{(i+1)(i+2)}{2} \\ &\longrightarrow i+1 \\ &\longrightarrow 1 \end{aligned}$$

Guess-and-Prove: a toy example

Question: Find $B_{i,j} :=$ the number of $\{\rightarrow, \uparrow\}$ -walks in \mathbb{N}^2 from $(0,0)$ to (i,j)

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$$B_{i,j} \stackrel{?}{=} \frac{(i+j)!}{i!j!}$$

Guess-and-Prove: a toy example

Question: Find $B_{i,j} :=$ the number of $\{\rightarrow, \uparrow\}$ -walks in \mathbb{N}^2 from $(0,0)$ to (i,j)

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1	1	1	1	1	1	1	...

(III) Prove: If

$C_{i,j} \stackrel{\text{def}}{=} \frac{(i+j)!}{i!j!}$, then

$$\frac{C_{i-1,j}}{C_{i,j}} + \frac{C_{i,j-1}}{C_{i,j}} = \frac{i}{i+j} + \frac{j}{i+j} = 1$$

and $C_{i,0} = C_{0,j} = 1$.

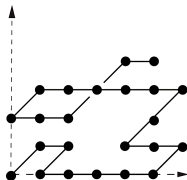
Thus $B_{i,j} = C_{i,j}$

Guess-and-Prove for Gessel walks

- $g(i, j; n)$ = number of n -steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$ -walks in \mathbb{N}^2 from $(0, 0)$ to (i, j)

Question: What is the nature of the generating function

$$G(x, y; t) = \sum_{i, j, n=0}^{\infty} g(i, j; n) x^i y^j t^n ?$$



Answer: [B., Kauers, 2010] $G(x, y; t)$ is an algebraic function[†].

Approach:

- ① **Generate data:** compute G to precision t^{1200} (≈ 1.5 billion coeffs!)
- ② **Guess:** conjecture polynomial equations for $G(x, 0; t)$ and $G(0, y; t)$ (degree 24 each, coeffs. of degree (46, 56), with 80-bits digits coeffs.)
- ③ **Prove:** multivariate resultants of (very big) polynomials (30 pages each)

[†] Minimal polynomial $P(G(x, y; t); x, y, t) = 0$ has $> 10^{11}$ terms; ≈ 30 Gb (6 DVDs!)

A typical Guess-and-Prove algorithmic proof

Theorem ["Gessel excursions are algebraic"]

$$g(t) := G(0,0; \sqrt{t}) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n \text{ is algebraic.}$$

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Proof: First **guess** a polynomial $P(t, T)$ in $\mathbb{Q}[t, T]$, then **prove** that P admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

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- 1 Find P such that $P(t, g(t)) = 0 \pmod{t^{100}}$ by **(structured) linear algebra**.

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- 3 $r(t) = \sum_{n=0}^{\infty} r_n t^n$ **being algebraic, it is D-finite**, and so (r_n) is **P-recursive**:

$$(n+2)(3n+5)r_{n+1} - 4(6n+5)(2n+1)r_n = 0, \quad r_0 = 1$$

$$\Rightarrow \text{solution } r_n = \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} 16^n = g_n, \text{ thus } g(t) = r(t) \text{ is algebraic.}$$

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













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```
> P:=gfun:-listtoalgeq([seq(pochhammer(5/6,n)*pochhammer(1/2,n)/
  pochhammer(5/3,n)/pochhammer(2,n)*16^n, n=0..100)], g(t)):
> gfun:-diffeqtoec(gfun:-algeqtodiffeq(P[1], g(t)), g(t), r(n));
```

Algorithmic classification of models with D-Finite $Q_{\mathcal{S}}(t) := Q_{\mathcal{S}}(1, 1; t)$

	OEIS	\mathcal{S}	Pol size	LDE size	Rec size		OEIS	\mathcal{S}	Pol size	LDE size	Rec size
1	A005566		—	(3, 4)	(2, 2)	13	A151275		—	(5, 24)	(9, 18)
2	A018224		—	(3, 5)	(2, 3)	14	A151314		—	(5, 24)	(9, 18)
3	A151312		—	(3, 8)	(4, 5)	15	A151255		—	(4, 16)	(6, 8)
4	A151331		—	(3, 6)	(3, 4)	16	A151287		—	(5, 19)	(7, 11)
5	A151266		—	(5, 16)	(7, 10)	17	A001006		(2, 2)	(2, 3)	(2, 1)
6	A151307		—	(5, 20)	(8, 15)	18	A129400		(2, 2)	(2, 3)	(2, 1)
7	A151291		—	(5, 15)	(6, 10)	19	A005558		—	(3, 5)	(2, 3)
8	A151326		—	(5, 18)	(7, 14)						
9	A151302		—	(5, 24)	(9, 18)	20	A151265		(6, 8)	(4, 9)	(6, 4)
10	A151329		—	(5, 24)	(9, 18)	21	A151278		(6, 8)	(4, 12)	(7, 4)
11	A151261		—	(4, 15)	(5, 8)	22	A151323		(4, 4)	(2, 3)	(2, 1)
12	A151297		—	(5, 18)	(7, 11)	23	A060900		(8, 9)	(3, 5)	(2, 3)

Equation sizes = (order, degree)

▷ Computerized discovery: enumeration + guessing [B., Kauers, 2009]
























Algorithmic classification of models with D-Finite $Q_{\mathcal{S}}(t) := Q_{\mathcal{S}}(1, 1; t)$

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






















- ▷ Computerized discovery: enumeration + guessing [B., Kauers, 2009]
- ▷ 1–22: DF confirmed by human proofs in [Bousquet-Mélou, Mishna, 2010]
- ▷ 23: DF confirmed by a human proof in [B., Kurkova, Raschel, 2017]
- ▷ All: explicit eqs. proved via CA [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Algorithmic classification of models with D-Finite $Q_{\mathcal{S}}(t) := Q_{\mathcal{S}}(1, 1; t)$

	OEIS	\mathcal{S}	algebraic?	asymptotics		OEIS	\mathcal{S}	algebraic?	asymptotics
1	A005566		N	$\frac{4}{\pi} \frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi} \frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi} \frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi} \frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi} \frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi} \frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi} \frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2} \sqrt{\frac{3}{\pi}} \frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2} \sqrt{\frac{3}{\pi}} \frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2} \sqrt{\frac{5}{2\pi}} \frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2} \sqrt{\frac{3}{\pi}} \frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}} \frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi} \frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}} \frac{6^n}{n^{1/2}}$	$A = 1 + \sqrt{2}, B = 1 + \sqrt{3}, C = 1 + \sqrt{6}, \lambda = 7 + 3\sqrt{6}, \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$				
9	A151302		N	$\frac{1}{3} \sqrt{\frac{5}{2\pi}} \frac{5^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)} \frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3} \sqrt{\frac{7}{3\pi}} \frac{7^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)} \frac{3^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)} \frac{6^n}{n^{3/4}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi} \frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)} \frac{4^n}{n^{2/3}}$

► Computerized discovery: convergence acceleration + LLL [B., Kauers, '09]

Algorithmic classification of models with D-Finite $Q_{\mathcal{S}}(t) := Q_{\mathcal{S}}(1, 1; t)$

	OEIS	\mathcal{S}	algebraic?	asymptotics		OEIS	\mathcal{S}	algebraic?	asymptotics
1	A005566		N	$\frac{4}{\pi} \frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi} \frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi} \frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi} \frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi} \frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi} \frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi} \frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2} \sqrt{\frac{3}{\pi}} \frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2} \sqrt{\frac{3}{\pi}} \frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2} \sqrt{\frac{5}{2\pi}} \frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2} \sqrt{\frac{3}{\pi}} \frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}} \frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi} \frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}} \frac{6^n}{n^{1/2}}$	$A = 1 + \sqrt{2}, B = 1 + \sqrt{3}, C = 1 + \sqrt{6}, \lambda = 7 + 3\sqrt{6}, \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$				
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- ▷ Computerized discovery: convergence acceleration + LLL [B., Kauers, '09]
- ▷ Asympt. confirmed by human proofs via ACSV in [Melczer, Wilson, 2016]
- ▷ Transcendence proofs via CA [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Let \mathcal{S} be one of the models 1–19. Then

- $Q_{\mathcal{S}}(x, y; t)$ is expressible using iterated integrals of ${}_2F_1$ expressions.
- $Q_{\mathcal{S}}(x, y; t)$ is transcendental.

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Let \mathcal{S} be one of the models 1–19. Then

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Example (King walks in the quarter plane, A151331)

$$Q_{\begin{smallmatrix} \swarrow \uparrow \\ \searrow \downarrow \end{smallmatrix}}(t) = \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\frac{3}{2} \mid \frac{3}{2} \mid \frac{16x(1+x)}{(1+4x)^2}\right) dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \dots$$

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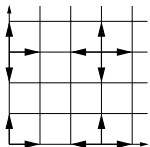
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- ▷ Computer-driven discovery and proof; no human proof yet.
- ▷ Proof uses: (1) kernel method + (2) **creative telescoping**.

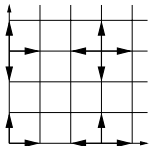
(1) Kernel method [Bousquet-Mélou, Mishna, 2010]



The kernel $K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left(x + \frac{1}{x} + y + \frac{1}{y} \right)$ is left **invariant** under the change of (x, y) into the elements of

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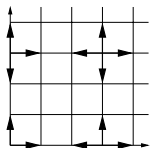
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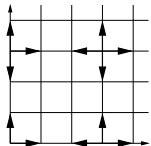
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$$\begin{aligned} K(x, y; t)xyQ(x, y; t) &= xy - txQ(x, 0; t) - tyQ(0, y; t) \\ -K(x, y; t)\frac{1}{x}yQ\left(\frac{1}{x}, y; t\right) &= -\frac{1}{x}y + t\frac{1}{x}Q\left(\frac{1}{x}, 0; t\right) + tyQ(0, y; t) \end{aligned}$$

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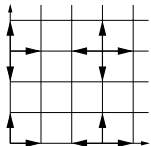
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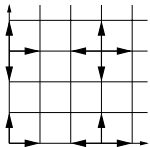
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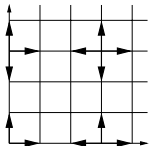
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Summing up yields the orbit equation:

$$\sum_{\theta \in \mathcal{G}} (-1)^{\theta} \theta(xyQ(x, y; t)) = \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{K(x, y; t)}$$

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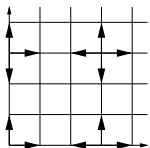
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Taking positive parts yields:

$$[x^>y^>] \sum_{\theta \in \mathcal{G}} (-1)^{\theta} \theta(xy Q(x, y; t)) = [x^>y^>] \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{K(x, y; t)}$$

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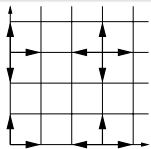
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Summing up and taking positive parts yields:

$$xyQ(x, y; t) = [x > y] \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{K(x, y; t)}$$

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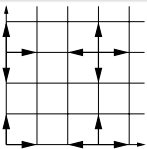
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$$\text{GF} = \text{PosPart} \left(\frac{\text{OS}}{\text{kernel}} \right) = \iint \text{RatFrac}$$

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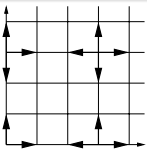
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$$\text{GF} = \text{PosPart} \left(\frac{\text{OS}}{\text{ker}} \right) \text{ is D-finite [Lipshitz, 1988]}$$

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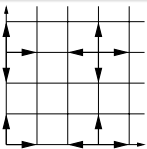
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$$\begin{aligned} K(x, y; t)xyQ(x, y; t) &= xy - txQ(x, 0; t) - tyQ(0, y; t) \\ -K(x, y; t)\frac{1}{x}yQ\left(\frac{1}{x}, y; t\right) &= -\frac{1}{x}y + t\frac{1}{x}Q\left(\frac{1}{x}, 0; t\right) + tyQ(0, y; t) \\ K(x, y; t)\frac{1}{x}\frac{1}{y}Q\left(\frac{1}{x}, \frac{1}{y}; t\right) &= \frac{1}{x}\frac{1}{y} - t\frac{1}{x}Q\left(\frac{1}{x}, 0; t\right) - t\frac{1}{y}Q\left(0, \frac{1}{y}; t\right) \\ -K(x, y; t)x\frac{1}{y}Q\left(x, \frac{1}{y}; t\right) &= -x\frac{1}{y} + txQ(x, 0; t) + t\frac{1}{y}Q\left(0, \frac{1}{y}; t\right) \end{aligned}$$

$$\text{GF} = \text{PosPart} \left(\frac{\text{OS}}{\text{ker}} \right) \text{ is D-finite [Lipshitz, 1988]}$$

▷ Argument works if $\text{OS} \neq 0$: algebraic version of the reflection principle

(1) Kernel method [Bousquet-Mélou, Mishna, 2010]



The kernel $K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left(x + \frac{1}{x} + y + \frac{1}{y} \right)$ is left invariant under the change of (x, y) into the elements of

$$\mathcal{G}_{\mathcal{S}} := \left\{ (x, y), \left(\frac{1}{x}, y \right), \left(\frac{1}{x}, \frac{1}{y} \right), \left(x, \frac{1}{y} \right) \right\}$$

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GF = PosPart $\left(\frac{\text{OS}}{\text{ker}} \right)$ is D-finite [Lipshitz, 1988]

▷ Creative Telescoping finds a differential equation for GF = $\int \int$ RatFrac

(2) Creative Telescoping

“An algorithmic toolbox for multiple sums and integrals with parameters”

Example [Apéry 1978]: $A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfies the recurrence

$$(n+1)^3 A_{n+1} + n^3 A_{n-1} = (2n+1)(17n^2 + 17n + 5)A_n.$$

▷ Key fact used to prove that $\zeta(3) := \sum_{n \geq 1} \frac{1}{n^3} \approx 1.202056903 \dots$ is irrational.

1. Journées Arithmétiques de Marseille-Luminy, June 1978

The board of programme changes informed us that R. Apéry (Caen) would speak Thursday, 14.00 “Sur l’irrationalité de $\zeta(3)$.” Though there had been earlier rumours of his claiming a proof, scepticism was general. The lecture tended to strengthen this view to rank disbelief. Those who listened casually, or who were afflicted with being non-Francophone, appeared to hear only a sequence of unlikely assertions.

7. ICM '78, Helsinki, August 1978

Neither Cohen nor I had been able to prove $\textcircled{5}$ or $\textcircled{5}$ in the intervening 2 months. After a few days of fruitless effort the specific problem was mentioned to Don Zagier (Bonn), and with irritating speed he showed that indeed the sequence $\{b'_n\}$ satisfies the recurrence (4). This more or less broke the dam and $\textcircled{5}$ and $\textcircled{5}$ were quickly conquered. Henri Cohen addressed a very well-attended meeting at 17.00 on Friday, August 18 in the language of the majority, proving $\textcircled{5}$ and explaining how this implied the

[Van der Poorten, 1979: “A proof that Euler missed”]

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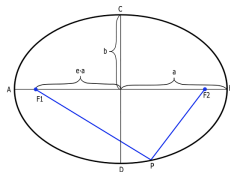
[Zeilberger, 1990: “The method of creative telescoping”]

(2) Creative Telescoping

“An algorithmic toolbox for multiple sums and integrals with parameters”

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity e , semi-major axis 1

$$p(e) = 4 \int_0^1 \sqrt{\frac{1 - e^2 u^2}{1 - u^2}} du = 4 \oint \frac{du dv}{1 - \frac{1 - e^2 u^2}{(1 - u^2) v^2}}$$



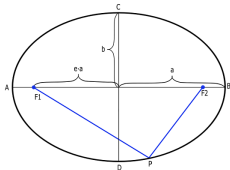
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$$\begin{aligned} & \left((e - e^3) \partial_e^2 + (1 - e^2) \partial_e + e \right) \cdot \left(\frac{1}{1 - \frac{1 - e^2 u^2}{(1 - u^2)v^2}} \right) = \\ & \partial_u \left(- \frac{e(-1 - u + u^2 + u^3)v^2(-3 + 2u + v^2 + u^2(-2 + 3e^2 - v^2))}{(-1 + v^2 + u^2(e^2 - v^2))^2} \right) \\ & + \partial_v \left(\frac{2e(-1 + e^2)u(1 + u^3)v^3}{(-1 + v^2 + u^2(e^2 - v^2))^2} \right) \end{aligned}$$

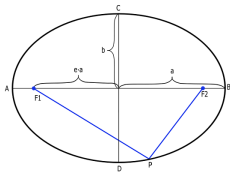
▷ Conclusion: $(e - e^3) \cdot p''(e) + (1 - e^2) \cdot p'(e) + e \cdot p(e) = 0$.

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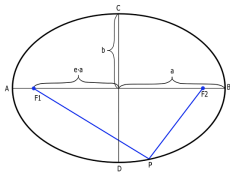
▷ Conclusion: $p(e) = \frac{\pi}{2} \cdot {}_2F_1 \left(-\frac{1}{2}, \frac{1}{2} \mid e^2 \right) = 2\pi - \frac{\pi}{2}e^2 - \frac{3\pi}{32}e^4 - \dots$

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▷ Drawback: Size(certificate) \gg Size(telescopier).

(2) 4G Creative Telescoping

Algorithm for the integration of rational functions [B., Lairez, Salvy, 2013]

- **Input:** $R(e, \mathbf{x})$ a rational function in e and $\mathbf{x} = x_1, \dots, x_n$.
- **Output:** A linear ODE $T(e, \partial_e)y = 0$ satisfied by $y(e) = \int R(e, \mathbf{x})d\mathbf{x}$.
- **Complexity:** $\mathcal{O}(D^{8n+2})$, where $D = \deg R$.
- **Output size:** T has order $\leq D^n$ in ∂_e and degree $\leq D^{3n+2}$ in e .

- ▷ Avoids the (costly) computation of **certificates**, of size $\Omega(D^{n^2}/2)$.
- ▷ Previous algorithms: complexity (at least) doubly exponential in n .
- ▷ Very efficient in practice.

Theorem (Apéry's power series is transcendental)

$$f(t) = \sum_n A_n t^n, \quad \text{where } A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2, \quad \text{is transcendental.}$$

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compute least-order L_f^{\min} in $\mathbb{Q}(t)\langle\partial_t\rangle$ such that $L_f^{\min}(f) = 0$

A toy *transcendence proof*: blending **Guess-and-Prove** and **CT**

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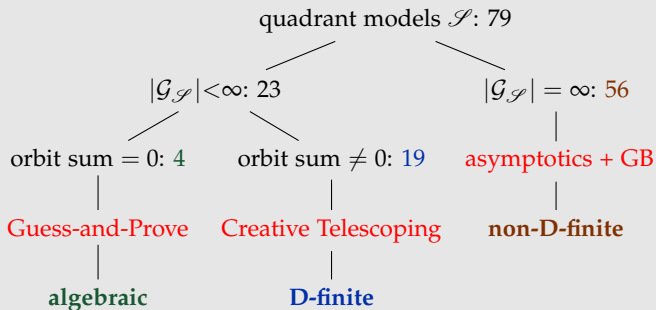
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⑤ **Conclusion:** f is transcendental[†]

[†] f algebraic would imply a full **basis of algebraic solutions** for L_f^{\min} [Tannery, 1875].

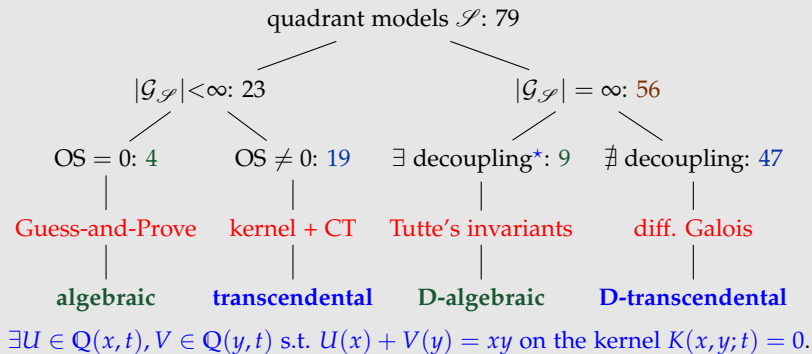
Summary: classification of walks with small steps in \mathbb{N}^2

$Q_{\mathcal{S}}$ is D-finite \iff a certain group $\mathcal{G}_{\mathcal{S}}$ is finite (!)



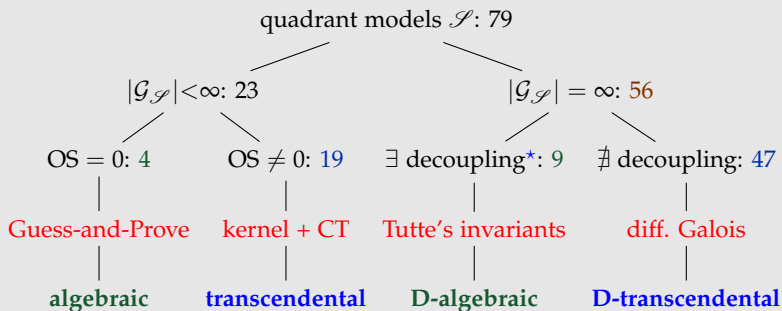
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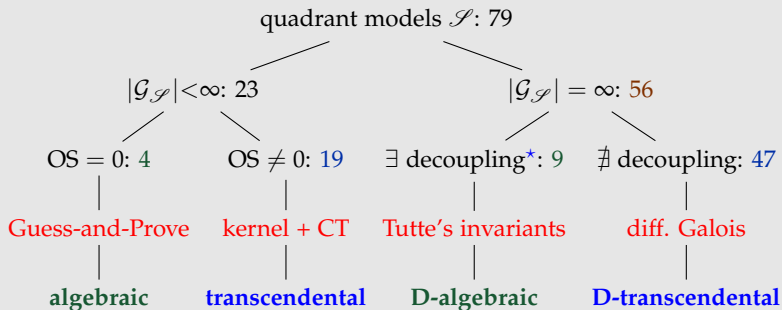
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▷ Proofs use **various tools**: algebra, complex analysis, probability theory, differential Galois theory, computer algebra, etc.



Enumerative Combinatorics and Computer Algebra enrich one another



Classification of $Q(x, y; t)$ **fully completed** for 2D small step walks



Robust algorithmic methods, based on efficient algorithms:

- **Guess-and-Prove**
- **Creative Telescoping**



Brute-force and/or use of naive algorithms = **hopeless**.

E.g. size of algebraic equations for $G(x, y; t) \approx 30\text{Gb}$.



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Lack of “purely human” proofs for some results.

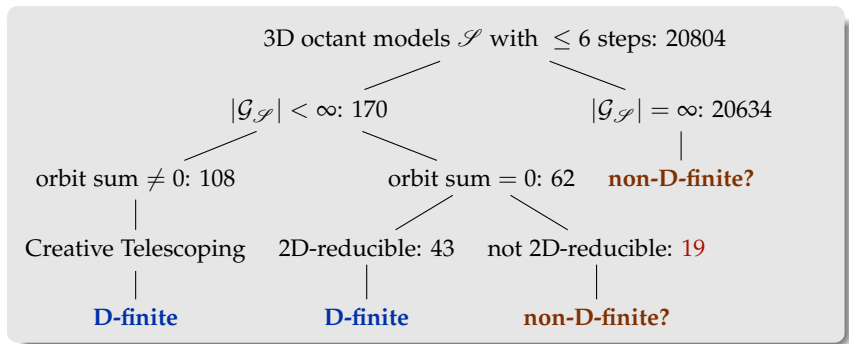


Many beautiful open questions for 2D models with **repeated** or **large** steps, and in **dimension** > 2 .

Bonus

Beyond dimension 2: walks with small steps in \mathbb{N}^3

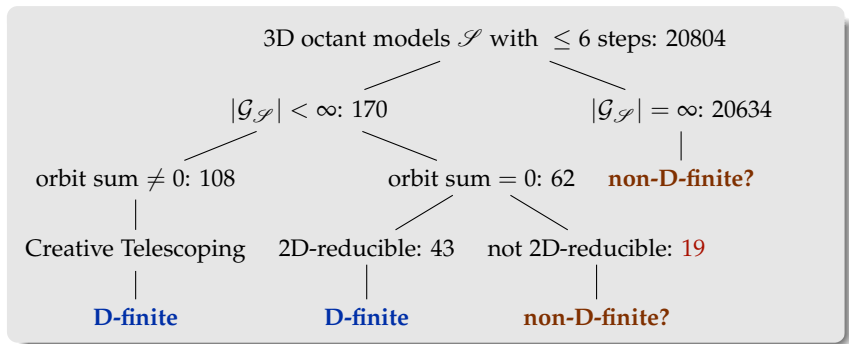
▷ $2^{3^3-1} \approx 67$ million models, of which ≈ 11 million inherently 3D



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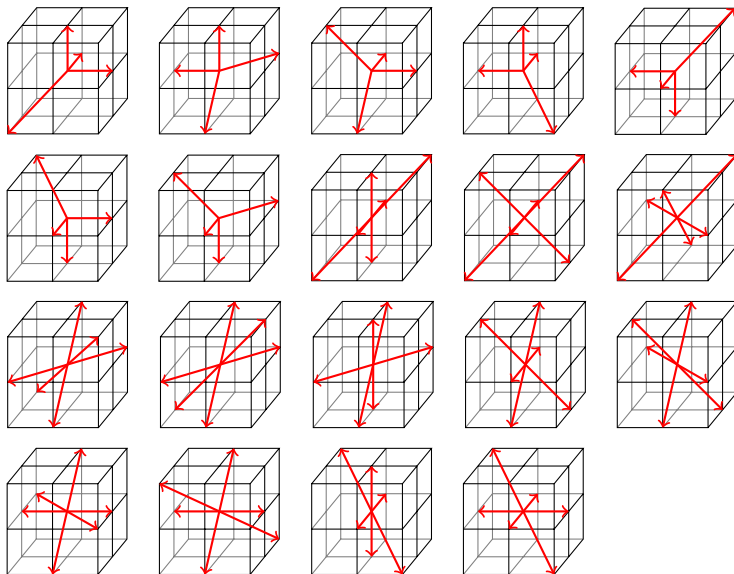


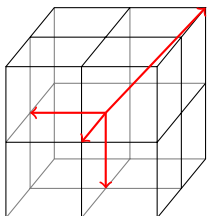
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Question: differential finiteness \iff finiteness of the group?

Answer: probably no

19 mysterious 3D-models: finite $\mathcal{G}_{\mathcal{S}}$ and possibly non-D-finite $Q_{\mathcal{S}}$



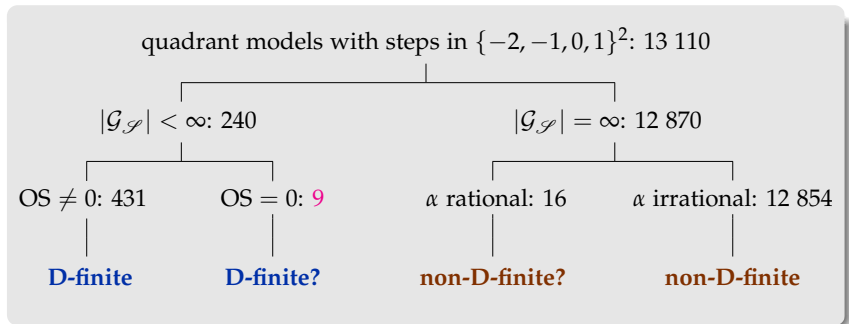


Two different computations suggest:

$$k_{4n} = C \cdot 256^n / n^\alpha, \text{ for } \alpha = 3.3257570041744\dots,$$

so excursions are very probably non-D-finite

Beyond small steps: Walks in \mathbb{N}^2 with large steps




[B., Bousquet-Mélou, Melczer, 2018]

Question: differential finiteness \iff finiteness of the group?

Answer: ?


Two challenging models with large steps

Conjecture 1 [B., Bousquet-Mélou, Melczer, 2018]

For the model  the excursions generating function $Q(0,0;t^{1/2})$ equals

$$\frac{1}{3t} - \frac{1}{6t} \cdot \left(\frac{1-12t}{(1+36t)^{1/3}} \cdot {}_2F_1 \left(\begin{matrix} \frac{1}{6} & \frac{2}{3} \\ 1 \end{matrix} \middle| \frac{108t(1+4t)^2}{(1+36t)^2} \right) + \sqrt{1-12t} \cdot {}_2F_1 \left(\begin{matrix} -\frac{1}{6} & \frac{2}{3} \\ 1 \end{matrix} \middle| \frac{108t(1+4t)^2}{(1-12t)^2} \right) \right).$$

Conjecture 2 [B., Bousquet-Mélou, Melczer, 2018]

For the model  the excursions generating function $Q(0,0;t)$ equals

$$\frac{(1-24U+120U^2-144U^3)(1-4U)}{(1-3U)(1-2U)^{3/2}(1-6U)^{9/2}},$$

where $U = t^4 + 53t^8 + 4363t^{12} + \dots$ is the unique series in $\mathbb{Q}[[t]]$ satisfying

$$U(1-2U)^3(1-3U)^3(1-6U)^9 = t^4(1-4U)^4.$$

- Automatic classification of restricted lattice walks, with M. Kauers. *Proceedings FPSAC*, 2009.
- The complete generating function for Gessel walks is algebraic, with M. Kauers. *Proceedings of the American Mathematical Society*, 2010.
- Explicit formula for the generating series of diagonal 3D Rook paths, with F. Chyzak, M. van Hoeij and L. Pech. *Séminaire Lotharingien de Combinatoire*, 2011.
- Non-D-finite excursions in the quarter plane, with K. Raschel and B. Salvy. *Journal of Combinatorial Theory A*, 2014.
- On 3-dimensional lattice walks confined to the positive octant, with M. Bousquet-Mélou, M. Kauers and S. Melczer. *Annals of Comb.*, 2016.
- A human proof of Gessel's lattice path conjecture, with I. Kurkova, K. Raschel, *Transactions of the American Mathematical Society*, 2017.
- Hypergeometric expressions for generating functions of walks with small steps in the quarter plane, with F. Chyzak, M. van Hoeij, M. Kauers and L. Pech, *European Journal of Combinatorics*, 2017.
- Counting walks with large steps in an orthant, with M. Bousquet-Mélou and S. Melczer, preprint, 2018.
- *Computer Algebra for Lattice Path Combinatorics*, preprint, 2019.

Solution of the “exercise”

- The kernel equation reads (with $K(x, y) = 1 - t(y + \bar{x} + x\bar{y})$):

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$$y_0 = \frac{x - t - \sqrt{(t - x)^2 - 4t^2x^3}}{2tx} = xt + t^2 + (x^2 + \bar{x})t^3 + (3x + \bar{x}^2)t^4 + \dots$$

be the (unique) root in $\mathbb{Q}[x, \bar{x}][[t]]$ of $K(x, y_0) = 0$.

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- Then

$$0 = K(x, y_0)yH(x, y_0) = y_0 - txH(x, 0),$$

thus

$$H(x, 0) = \frac{y_0}{tx} \quad \text{and} \quad A(t) = \left[x^0 \right] \frac{y_0}{tx}.$$

- The kernel equation reads (with $K(x, y) = 1 - t(y + \bar{x} + x\bar{y})$):

$$K(x, y)yH(x, y) = y - txH(x, 0)$$

- Let

$$y_0 = \frac{x - t - \sqrt{(t-x)^2 - 4t^2x^3}}{2tx} = xt + t^2 + (x^2 + \bar{x})t^3 + (3x + \bar{x}^2)t^4 + \dots$$

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- Creative telescoping** then proves:

$$(27t^4 - t)A''(t) + (108t^3 - 4)A'(t) + 54t^2A(t) = 0.$$

> Zeilberger(1/x * sqrt((t-x)^2 - 4*t^2*x^3)/(2*t^2*x^2), t, x, Dt);

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The group of the model $\{\uparrow, \leftarrow, \searrow\}$

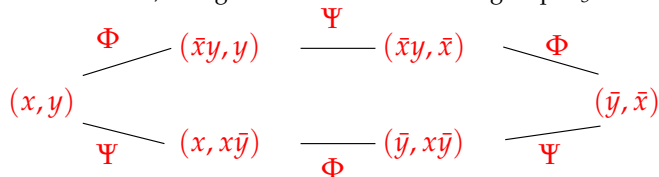
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Φ and Ψ are involutions, and generate a finite dihedral group D_3 of order 6:



- **Orbit equation:**

$$\begin{aligned} xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x}) \\ - \bar{x}\bar{y}Q(\bar{y}, \bar{x}) + x\bar{y}^2Q(\bar{y}, x\bar{y}) - x^2\bar{y}Q(x, x\bar{y}) = \\ \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})} \end{aligned}$$

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- **Corollary** [Bousquet-Mélou & Mishna, 2010]:

$$xyQ(x, y) = [x^{>0}y^{>0}] \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

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- **Corollary [B.-Chyzak-van Hoeij-Kauers-Pech, 2015]:**

$$B(t) = [z^0]Q(z, \bar{z}) = [u^{-1}v^{-1}z^{-1}] \frac{\bar{u}\bar{v} - u\bar{v}^2 + u^2\bar{v} - uv + \bar{u}v^2 - \bar{u}^2v}{z(1 - zu)(1 - v\bar{z})(1 - t(\bar{v} + u + \bar{u}v))}$$

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- **Creative Telescoping** gives a differential equation for $B(t)$:

$$(27t^4 - t)B''(t) + (108t^3 - 4)B'(t) + 54t^2B(t) = 0.$$

We have proved that $A(t)$ and $B(t)$ are both solutions of

$$(27t^4 - t)y''(t) + (108t^3 - 4)y'(t) + 54t^2y(t) = 0.$$

Solving this equation proves:

$$A(t) = B(t) = {}_2F_1\left(\begin{matrix} 1/3 & 2/3 \\ 2 \end{matrix} \middle| 27t^3\right) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3} \frac{t^{3n}}{n+1}.$$

Thus the two sequences are equal to

$$a_{3n} = b_{3n} = \frac{(3n)!}{n!^2 \cdot (n+1)!}, \quad \text{and} \quad a_m = b_m = 0 \quad \text{if } 3 \text{ does not divide } m.$$

Example with infinite group: the scarecrows

[B., Raschel, Salvy, 2014]: $Q_{\mathcal{S}}(0,0;t)$ is not D-finite for the models



▷ For the 1st and the 3rd, the excursions sequence $[t^n] Q_{\mathcal{S}}(0,0;t)$

$1, 0, 0, 2, 4, 8, 28, 108, 372, \dots$

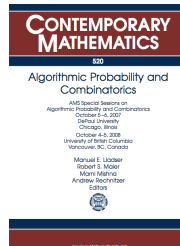
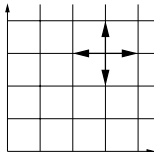
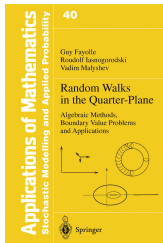
is $\sim K \cdot 5^n \cdot n^{-\alpha}$, with $\alpha = 1 + \pi / \arccos(1/4) = 3.383396\dots$

[Denisov, Wachtel, 2015]

▷ The **irrationality** of α prevents $Q_{\mathcal{S}}(0,0;t)$ from being D-finite.

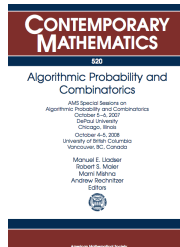
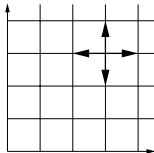
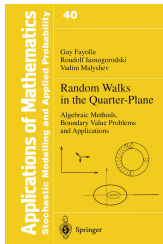
[Katz, 1970; Chudnovsky, 1985; André, 1989]

The group of a model: the simple walk case



The characteristic polynomial $\chi_{\mathcal{L}} := x + \frac{1}{x} + y + \frac{1}{y}$

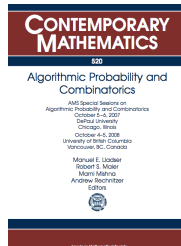
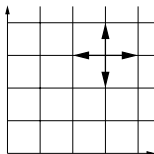
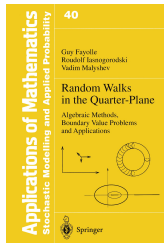
The group of a model: the simple walk case



The characteristic polynomial $\chi_{\mathcal{L}} := x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$$\psi(x, y) = \left(x, \frac{1}{y}\right), \quad \phi(x, y) = \left(\frac{1}{x}, y\right),$$

The group of a model: the simple walk case



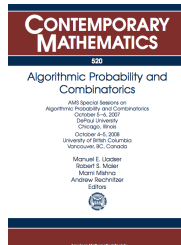
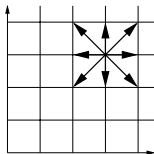
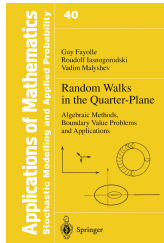
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and thus under any element of the group

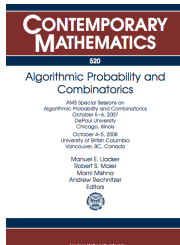
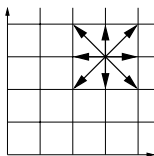
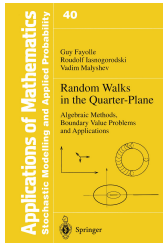
$$\langle \psi, \phi \rangle = \left\{ (x, y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}.$$

The group of a model



The generating polynomial $\chi_{\mathcal{S}} := \sum_{(i,j) \in \mathcal{S}} x^i y^j = \sum_{i=-1}^1 B_i(y) x^i = \sum_{j=-1}^1 A_j(x) y^j$

The group of a model



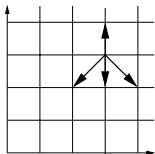
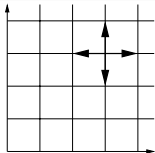
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$$\psi(x, y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y} \right), \quad \phi(x, y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y \right),$$

and thus under any element of the (dihedral) group

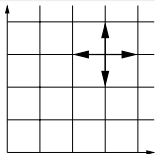
$$\mathcal{G}_{\mathcal{S}} := \langle \psi, \phi \rangle.$$

Examples of groups

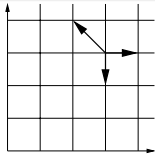


Order 4,

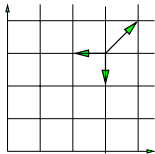
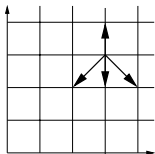
Examples of groups



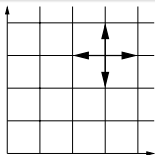
Order 4,



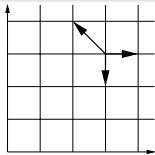
order 6,



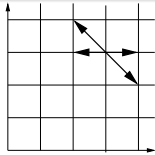
Examples of groups



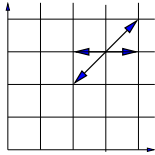
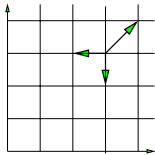
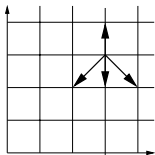
Order 4,



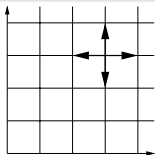
order 6,



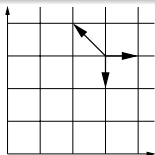
order 8,



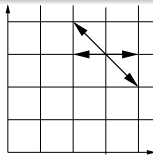
Examples of groups



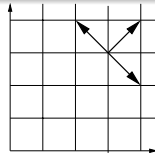
Order 4,



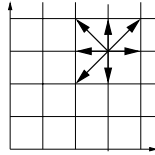
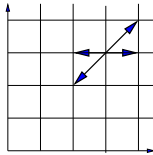
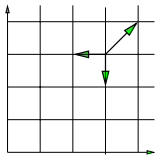
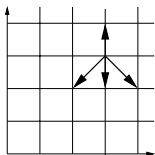
order 6,



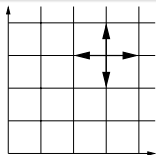
order 8,



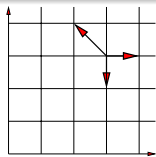
order ∞ .



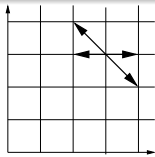
Examples of groups



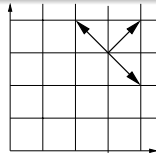
Order 4,



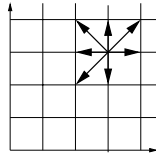
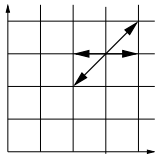
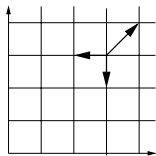
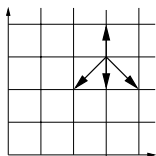
order 6,



order 8,



order ∞ .



$$\begin{array}{ccccc}
 & \Phi & \left(\frac{y}{x}, y\right) & \Psi & \left(\frac{y}{x}, \frac{1}{x}\right) & \Phi & \\
 \left(x, y\right) & \diagdown & & \text{---} & & \diagdown & \left(\frac{1}{y}, \frac{1}{x}\right) \\
 & \Psi & \left(x, \frac{x}{y}\right) & \Phi & \left(\frac{1}{y}, \frac{x}{y}\right) & \Psi &
 \end{array}$$