# Chip-Firing and Algebraic Combinatorics

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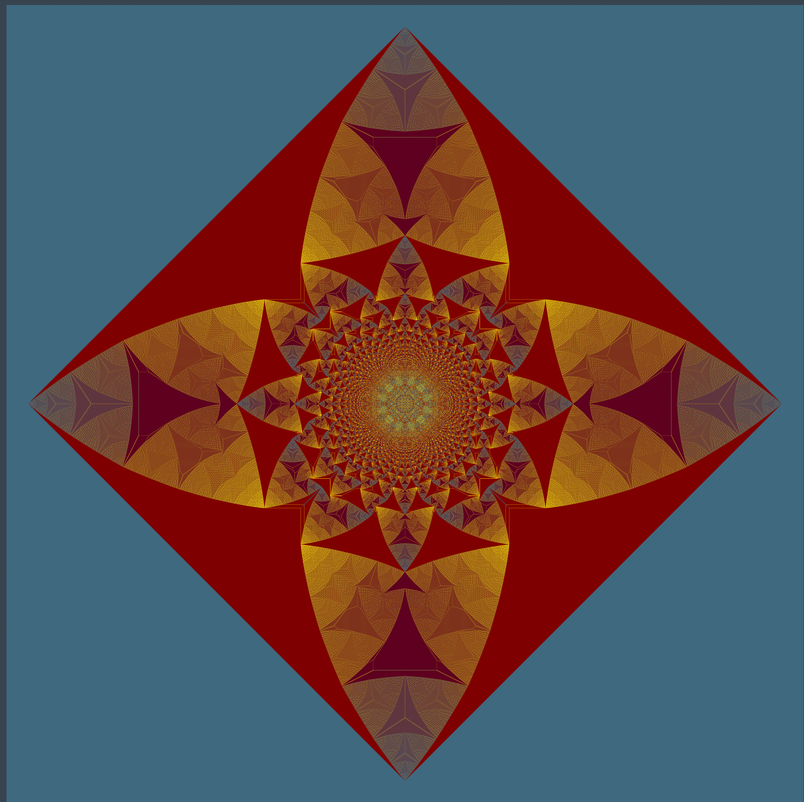
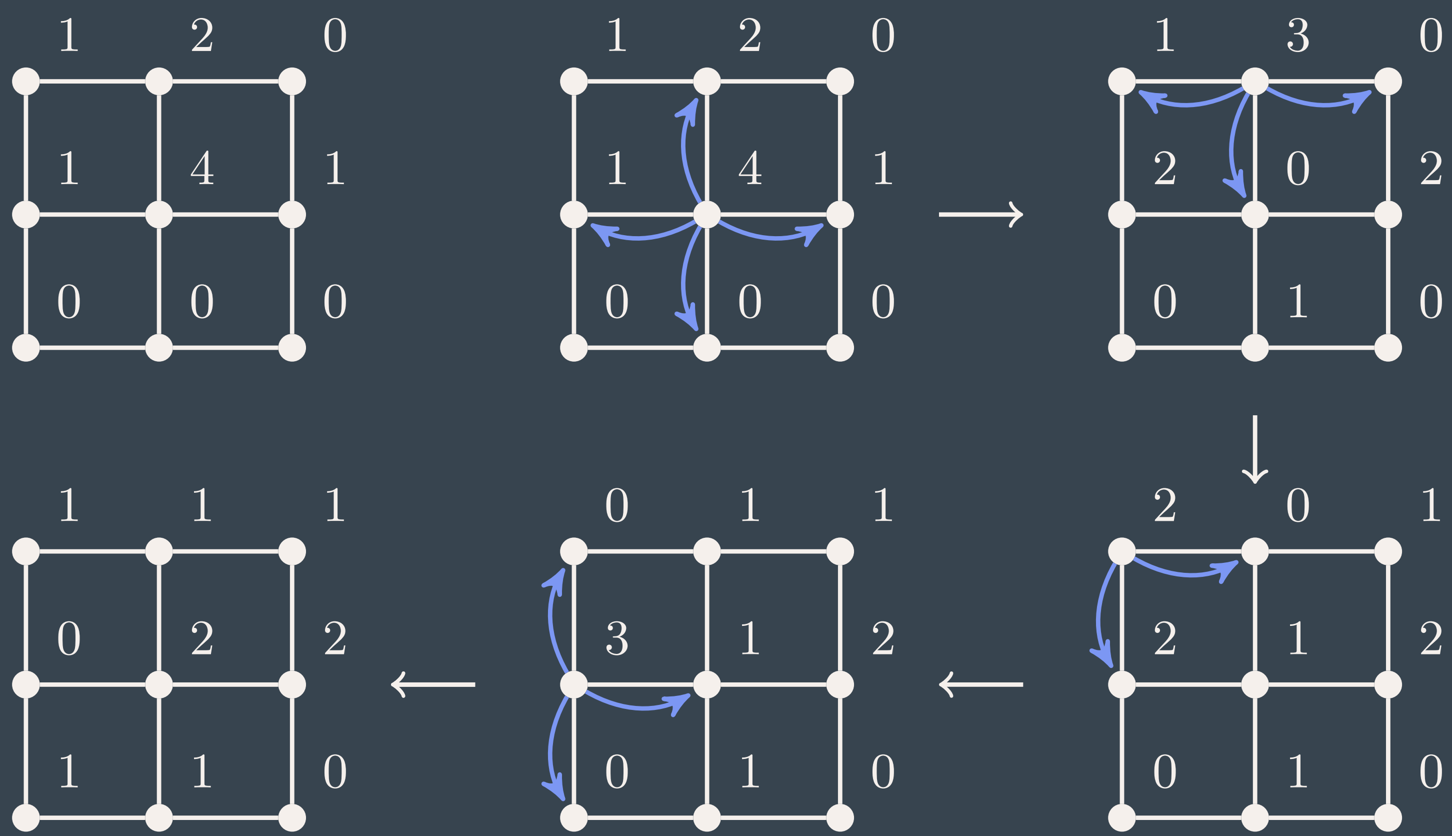
Brown University

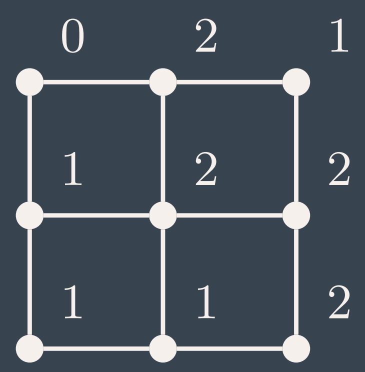


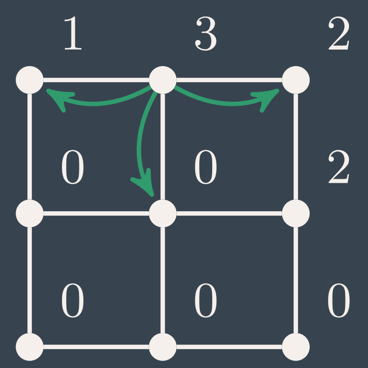
# Chip-Firing – Basic Dynamics

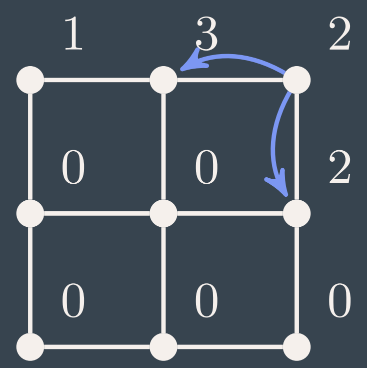


# Chip-Firing – Basic Dynamics

- Does the process stop?
- Order of firings?

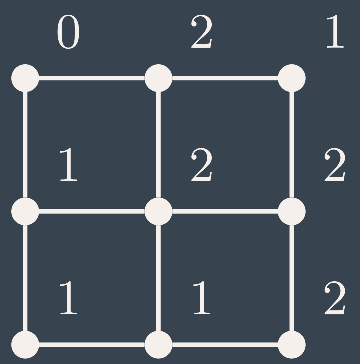


# Chip-Firing – Basic Dynamics

- Does the process stop?




Three Regimes Theorem

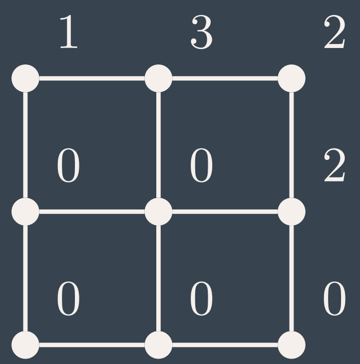
(Björner, Lovász, Shor '91)

$N$ = Number of chips.

- $N$ Large – infinite
- $N$ Small – finite
- $a \leq N \leq b$ –
  can always achieve both.

# Chip-Firing – Basic Dynamics

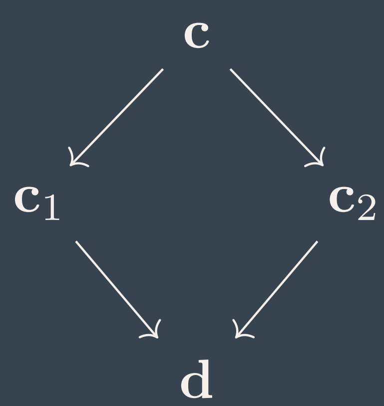
- Order of firings?




- Local confluence

  (Diamond lemma)

  (Church–Rosser Property)

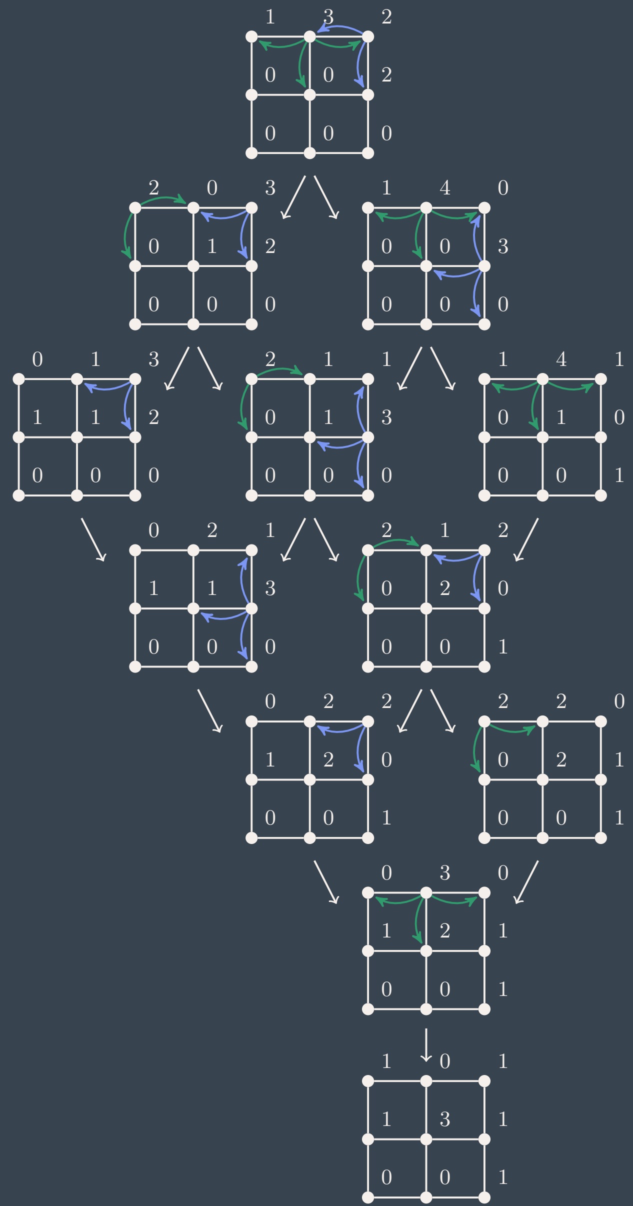



- Local + Finite = Global

  (Newman Lemma)

# Chip-Firing – Basic Dynamics

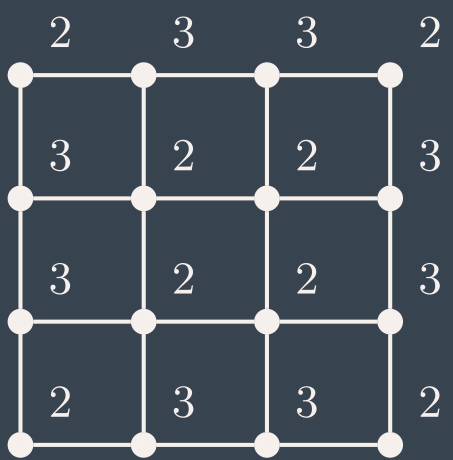
- Order of firings?



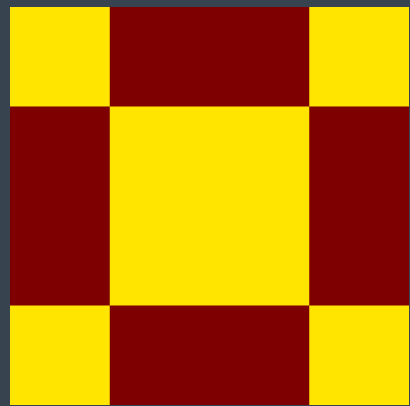
From a fixed initial configuration:
If the process is finite then it terminates at a unique final configuration.

Let's look at some larger examples. How can we visualize them?

| Color | Number of chips |
|---|---|
|  | 0 |
|  | 1 |
|  | 2 |
|  | 3 |



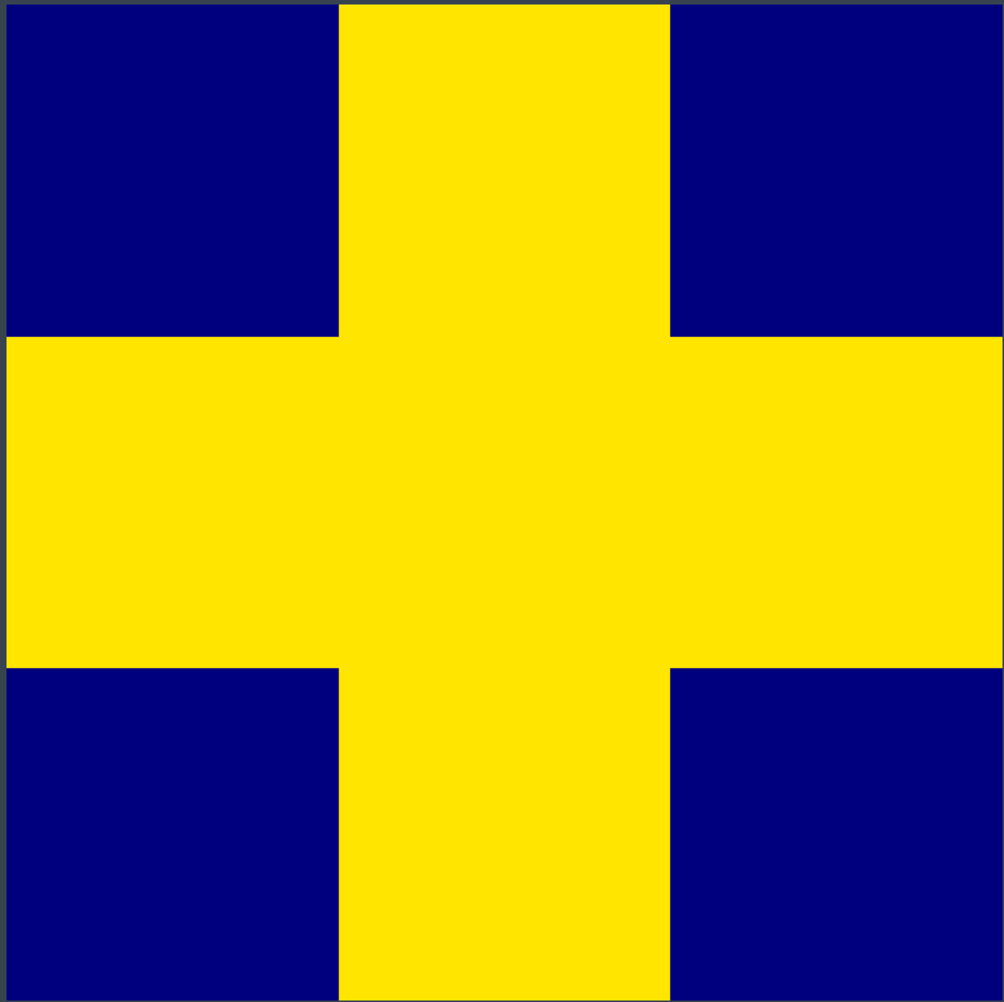

| | | | |
|---|---|---|---|
| 2 | 3 | 3 | 2 |
| 3 | 2 | 2 | 3 |
| 3 | 2 | 2 | 3 |
| 2 | 3 | 3 | 2 |



# Pattern Formation

Suppose we drop $N$ chips at the origin of the two-dimensional grid.

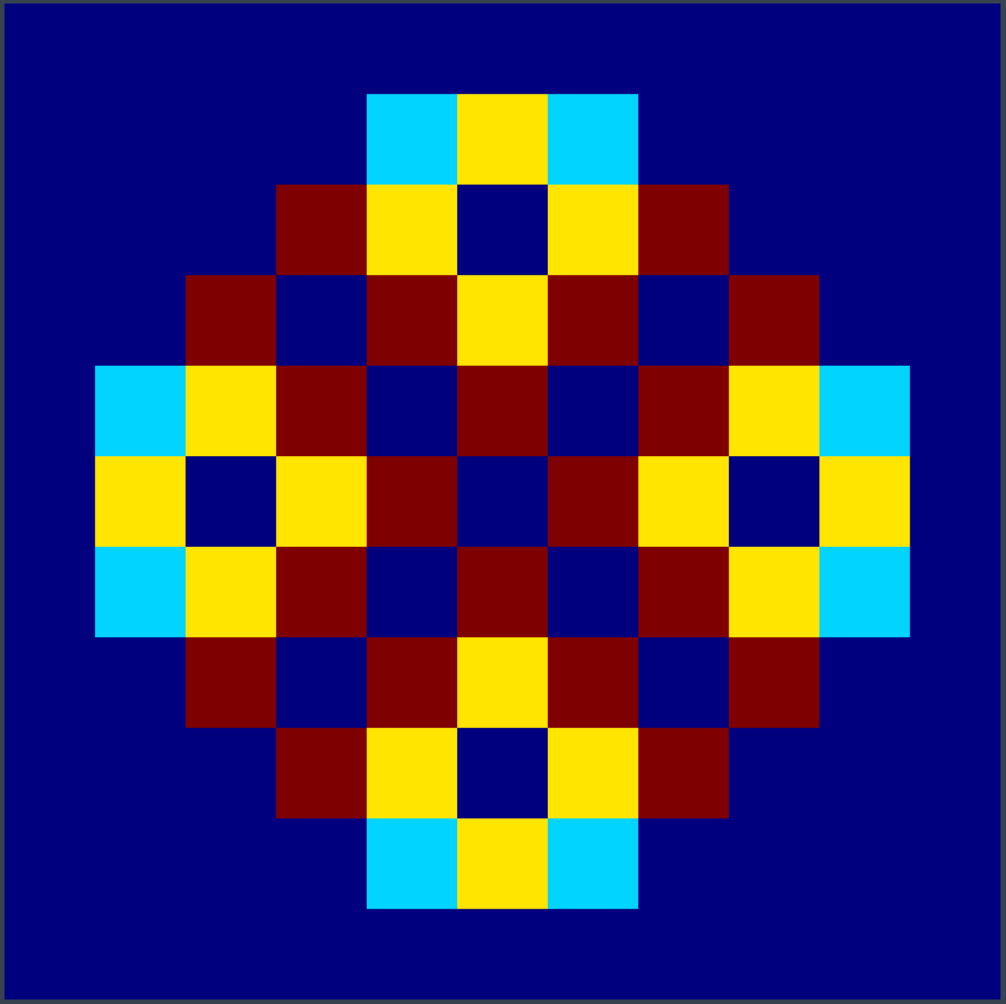
$N = 10$



# Pattern Formation

Suppose we drop $N$ chips at the origin of the two-dimensional grid.

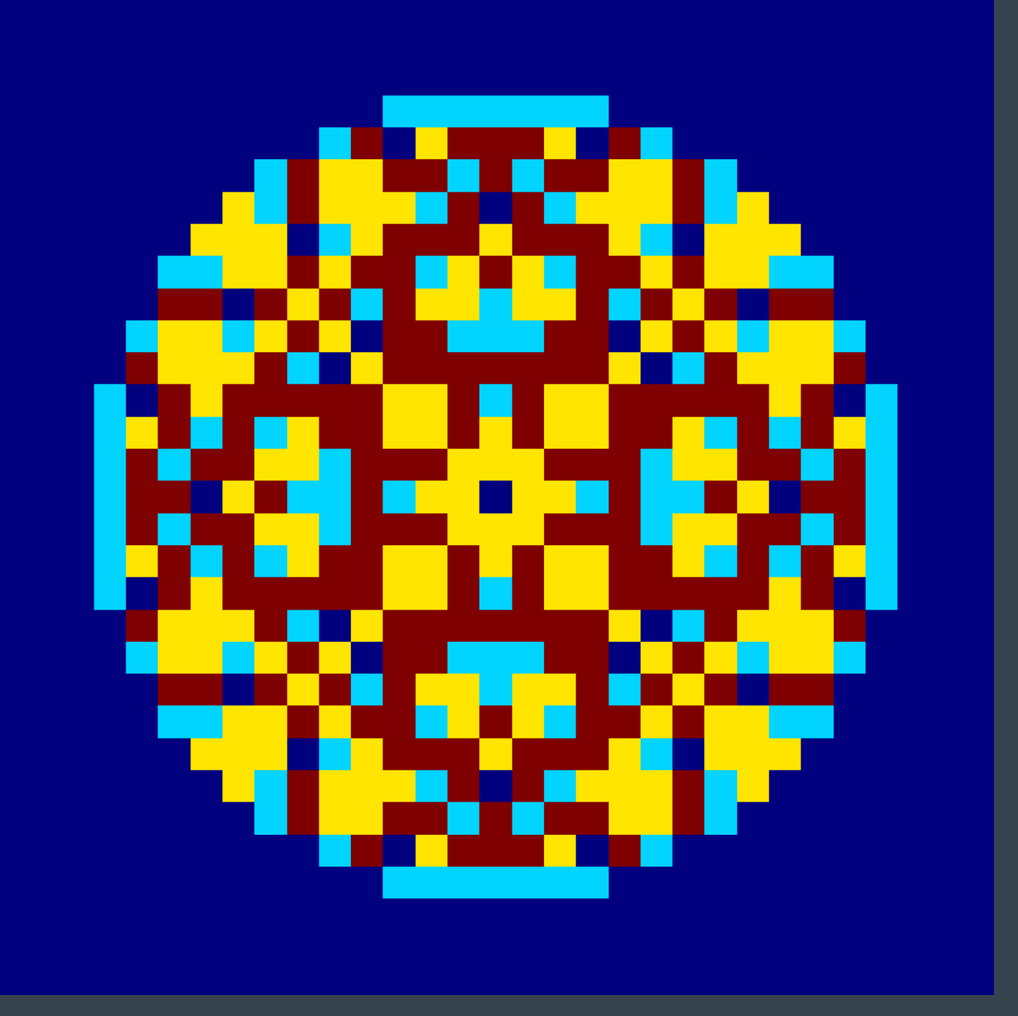
$N = 100$



# Pattern Formation

Suppose we drop $N$ chips at the origin of the two-dimensional grid.

$N = 1,000$



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Suppose we drop $N$ chips at the origin of the two-dimensional grid.

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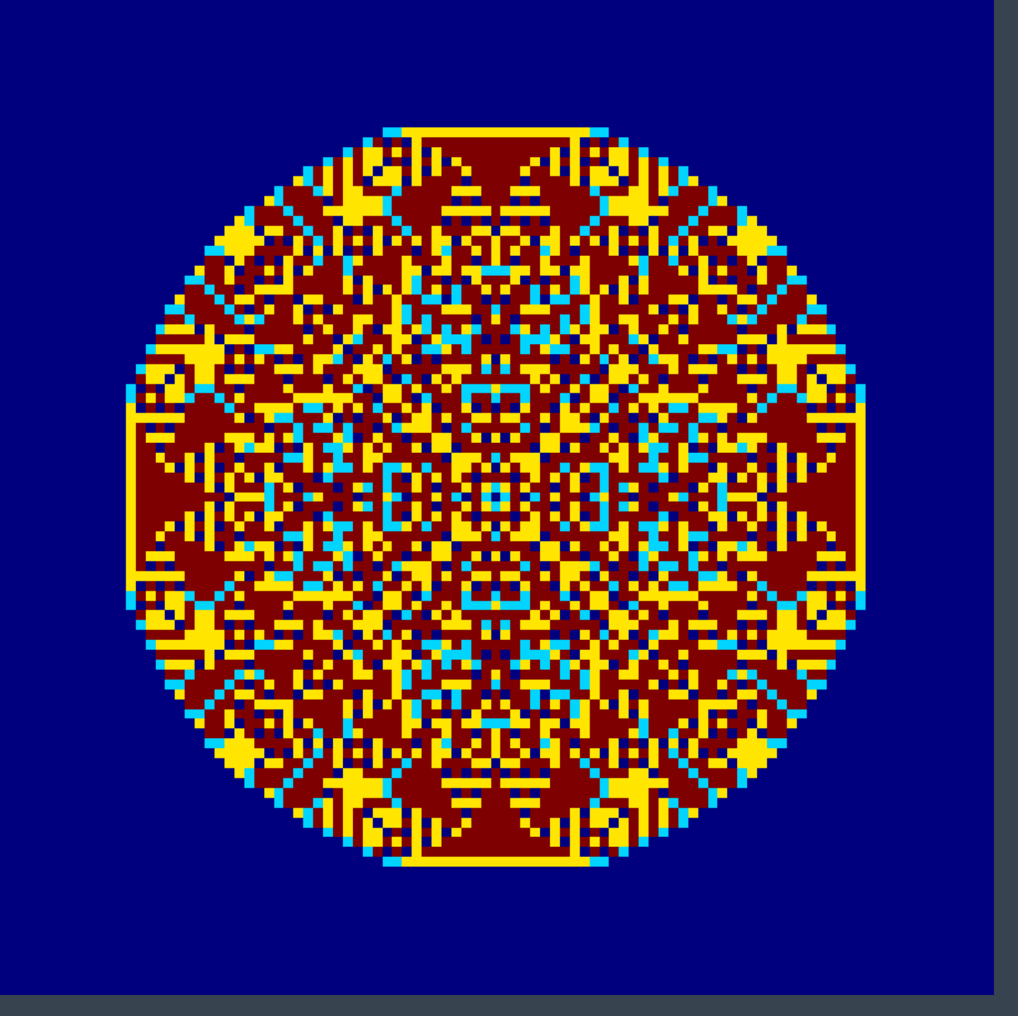

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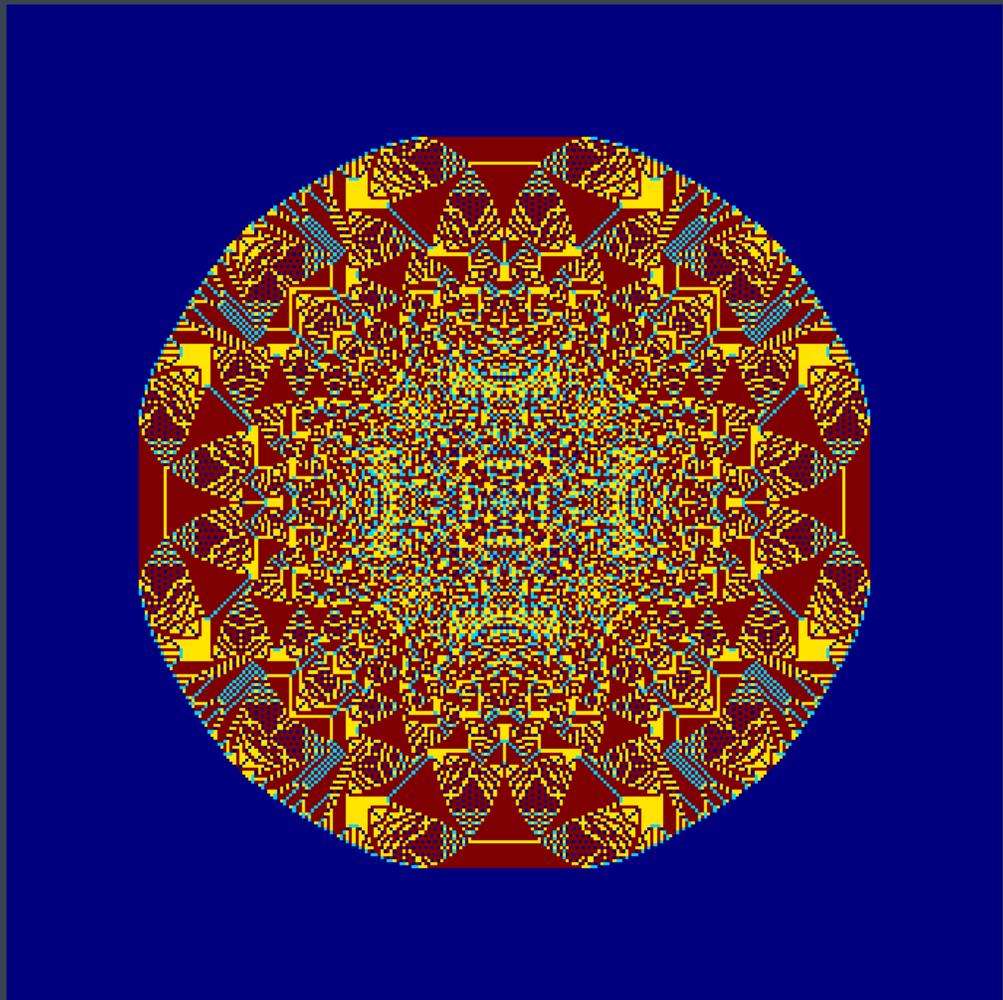

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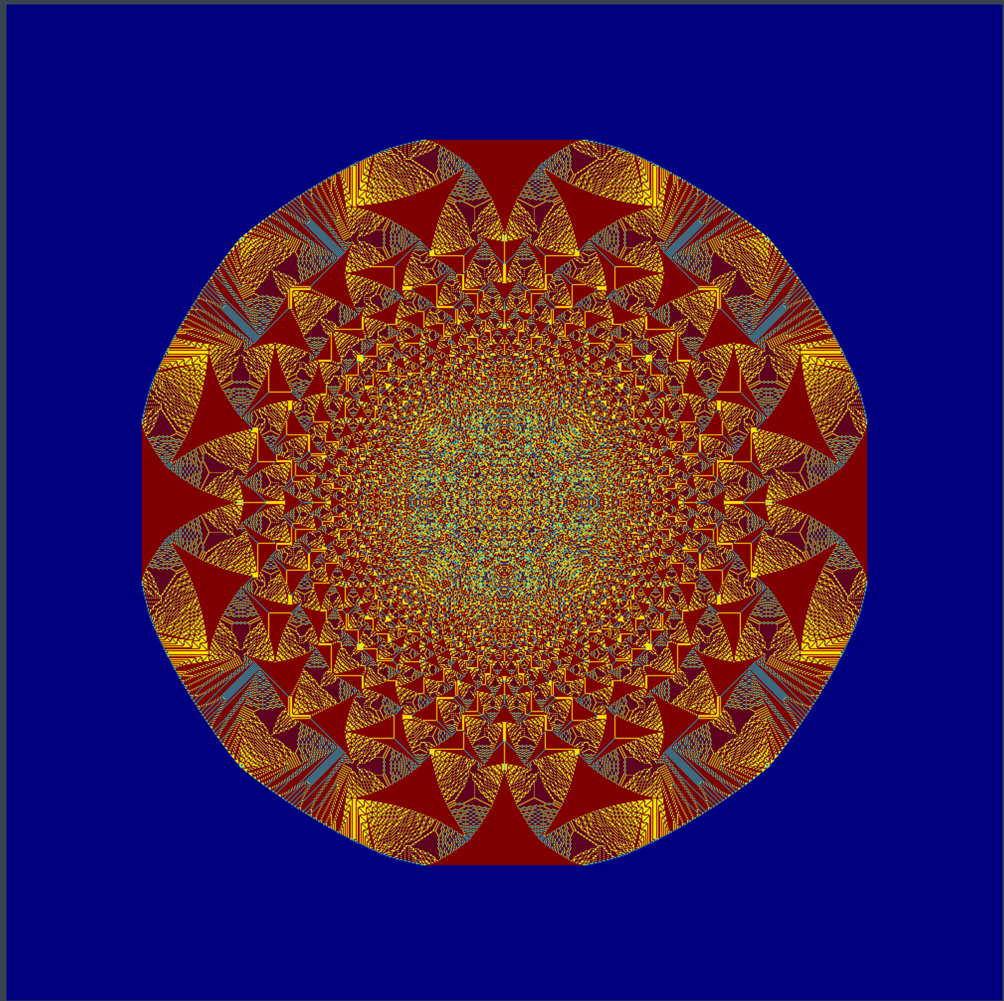

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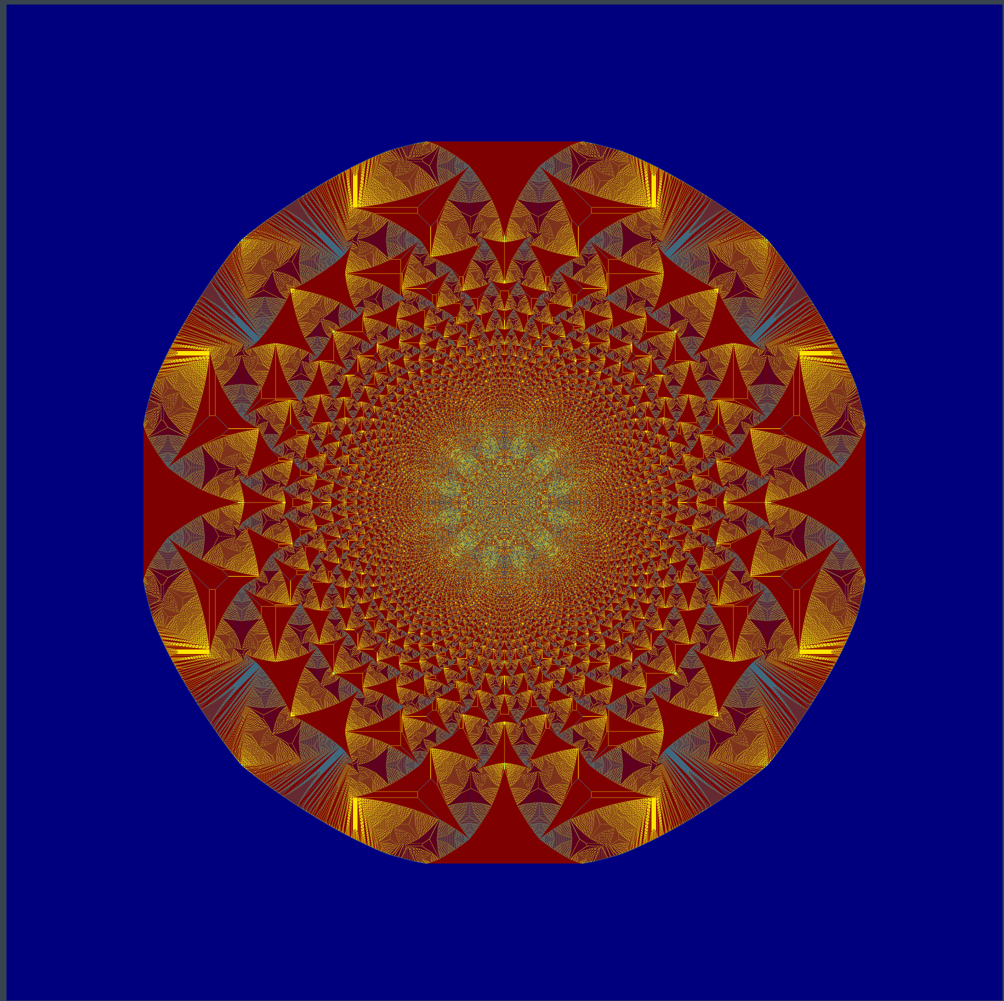
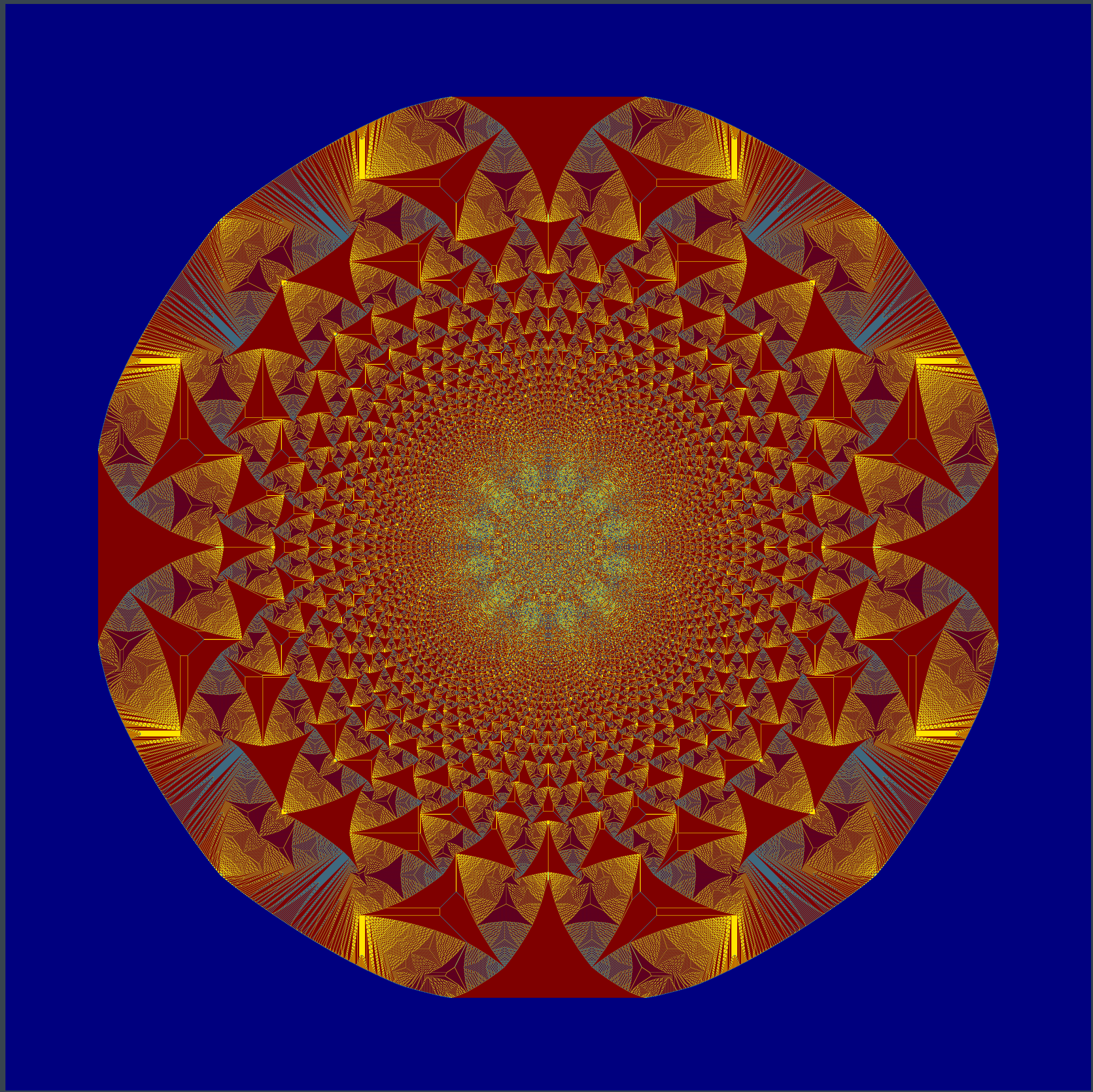


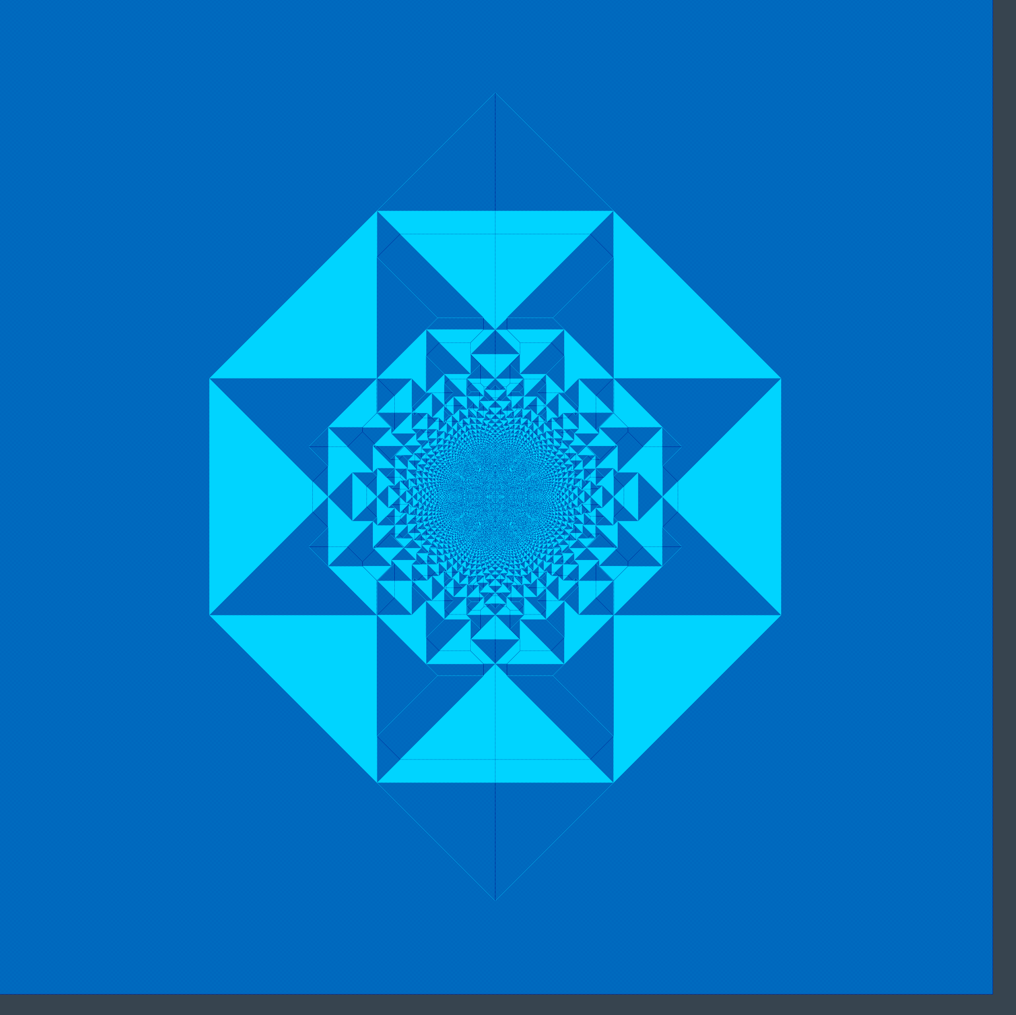


(Bak, Tang, Wiesenfeld '88, Dhar '06, Creutz '04, Pstojic '03, Caracciolo, Paoletti, Sportiello '08, Paoletti '14, Levine, Pegden, Smart '13, '16, '17)

# Pattern Formation

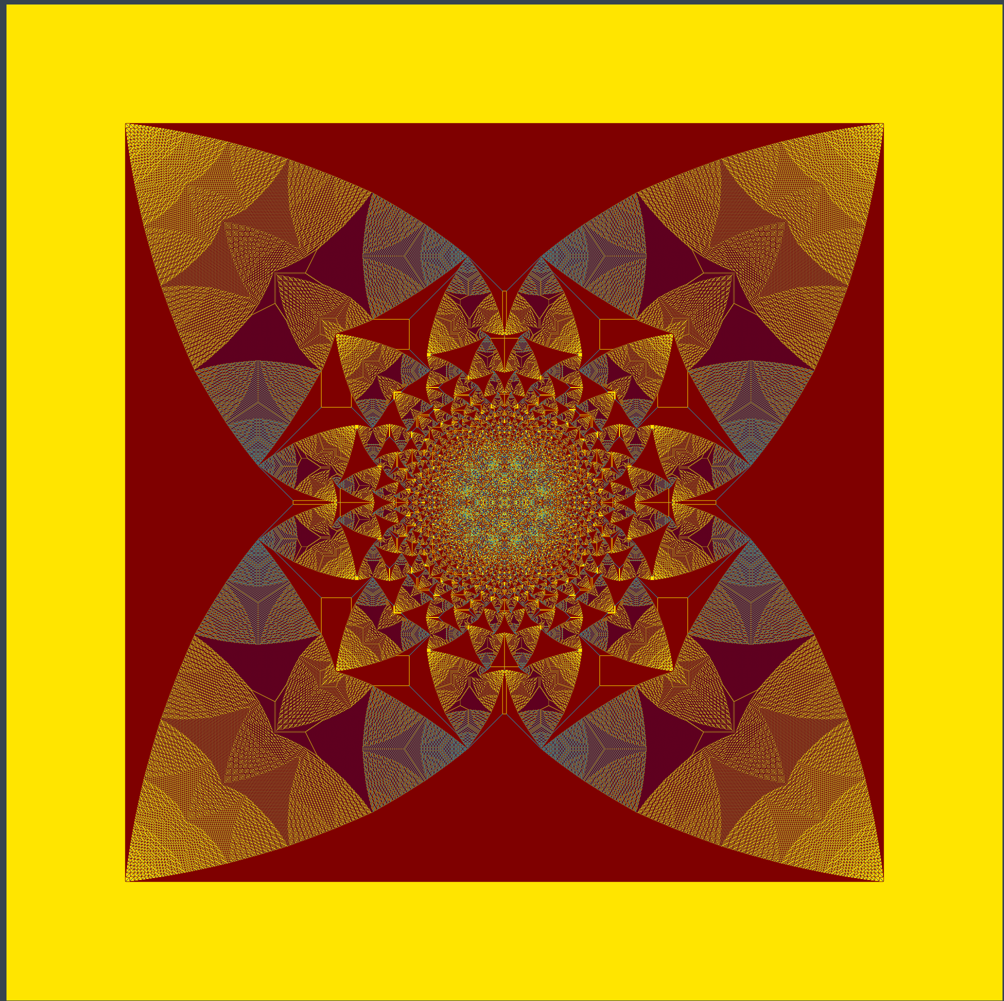
Suppose we drop $N$ chips at the origin of the F-Lattice.

With checkerboard background 0 / 1.



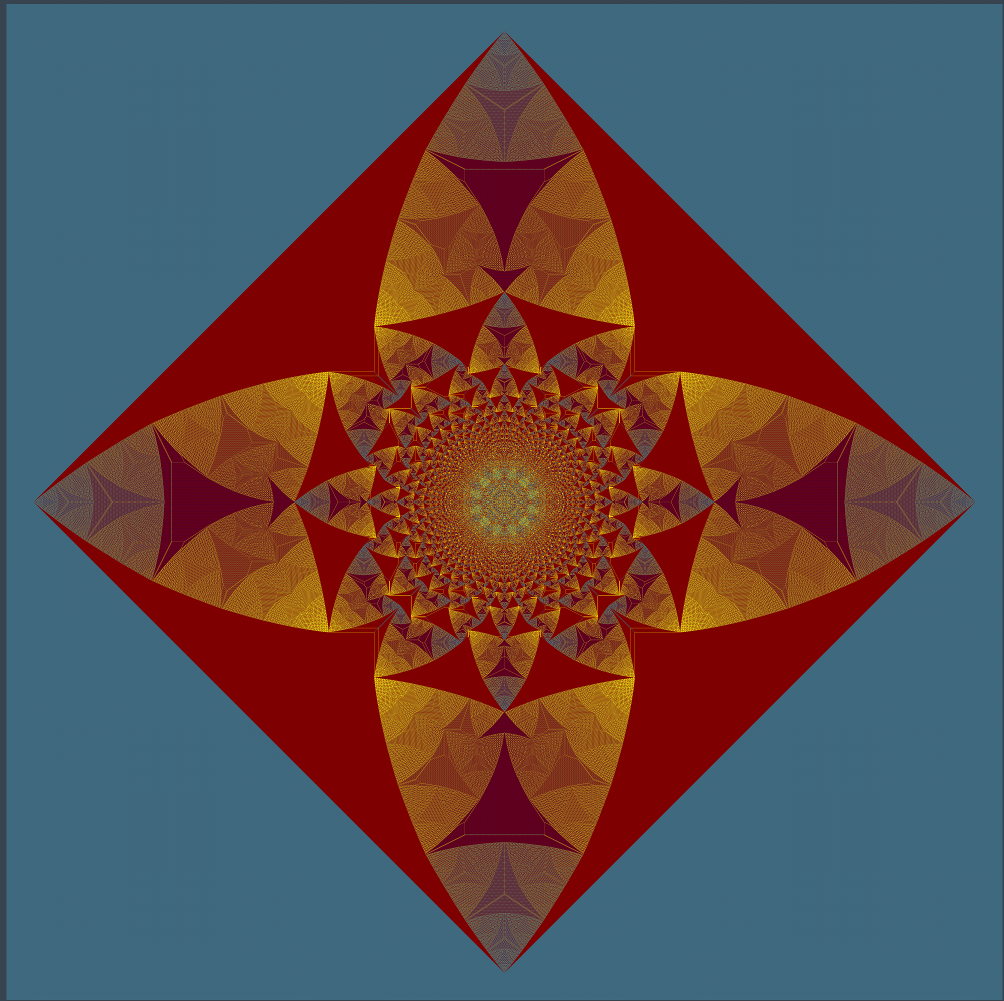
# Pattern Formation

Suppose we drop $N$ chips at the origin of the two-dimensional grid.
With background height 2.



# Pattern Formation

Suppose we drop $N$ chips at the origin of the two-dimensional grid.
With checkerboard background 1 / 3.



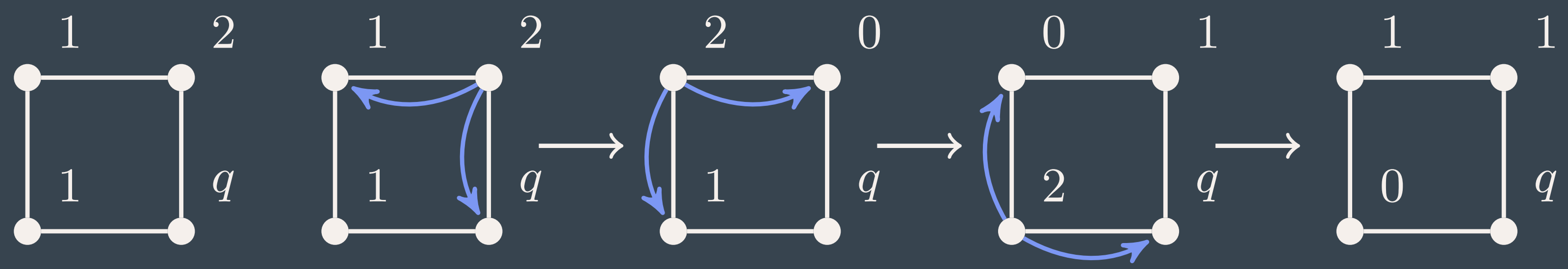
# The pulse in three dimensions

$N$ chips at the center of a large grid.

Video $\rightarrow$

# Finite Graphs with a Sink

All initial configurations terminate.



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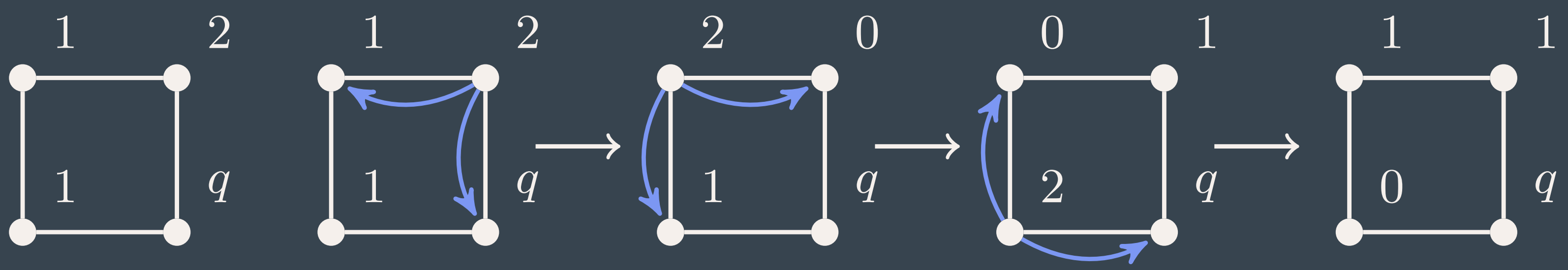
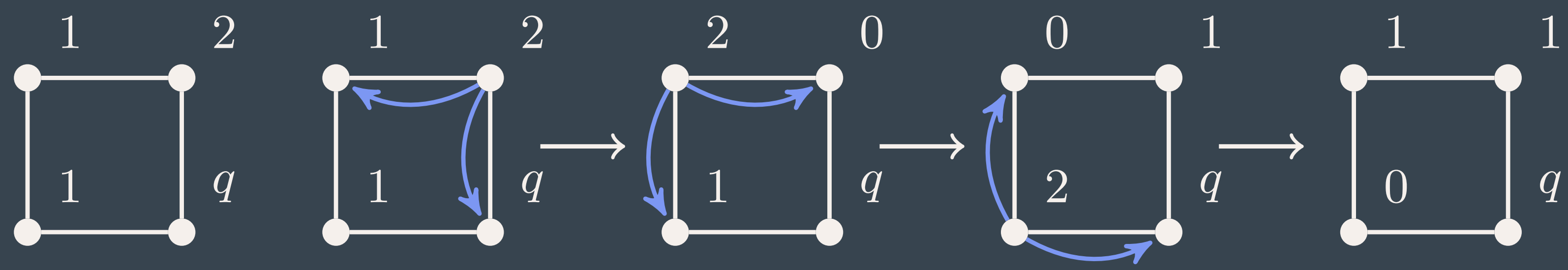



- Stable – No possible firings

# Finite Graphs with a Sink

All initial configurations terminate.




- Stable – No possible firings
- Critical – Stable + Reachable (results from a generic initial)

# Finite Graphs with a Sink

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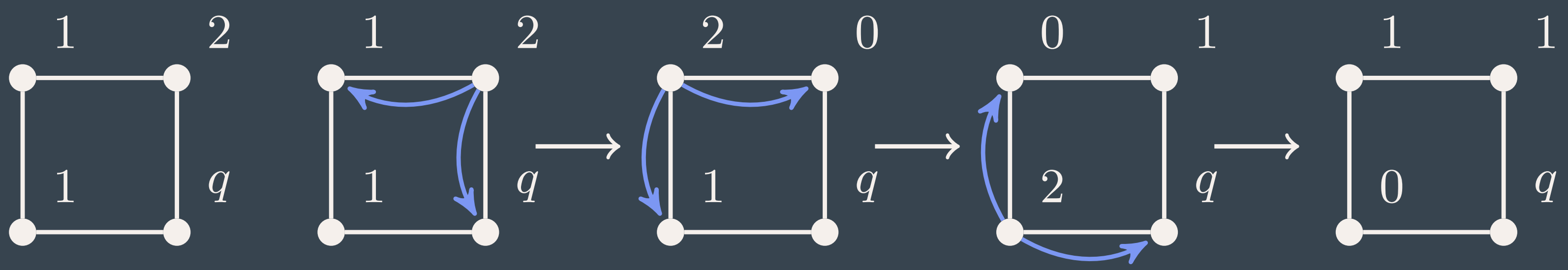
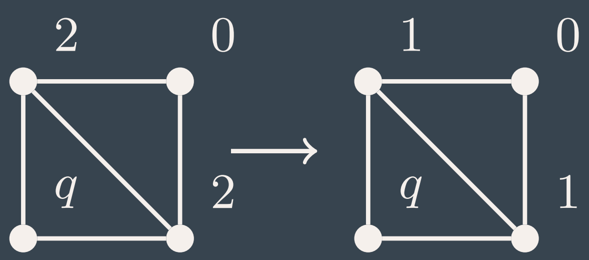



- Stable – No possible firings
- Critical – Stable + Reachable (results from a generic initial)
- Superstable – No possible group firings



# Criticals and Superstables

$$\boxed{\#\ \text{Criticals} = \#\ \text{Superstables} = \#\ \text{Spanning Trees}}$$

- Duality. Critical $\longleftrightarrow$ Superstable (Dhar '90)
  Criticals = Recurrent states of Abelian Sandpile Model

- Tutte polynomials. (Merino '01)
  Stanley's $O$-conjecture for $h$-vectors of cographic matroids

- Bijections. Extended burning algorithm (Cori, Le Borgne '03)
  # Criticals with $t$ chips = # Spanning trees with external activity $t$

- Superstables of $K_n$ = Parking Functions
  (Superstables of $G$ = $G$-parking functions) (Postnikov, Shapiro '03)
  (Dhar, Majumdar '92, Biggs, Winkler '97, Chebikin, Pylyavskyy '04)

# Criticals and Superstables

Discrete Diffusion. Graph Laplacian $\Delta$. Firing site $i$:

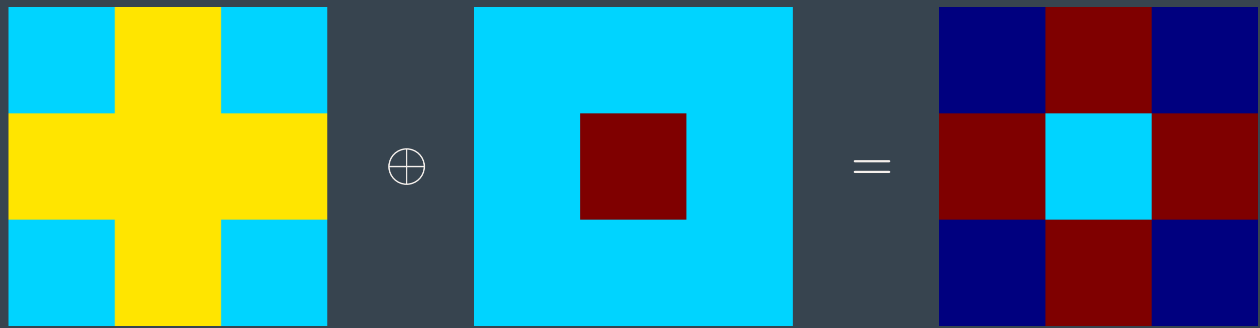
$$\mathbf{c} - \Delta e_i = \mathbf{c}'$$

- Laplacian potential functions. (Baker, Shokrieh '11)
  Superstables = Energy minimizers
- Extensions of chip-firing. Laplacian → M-matrix. (Guzman, K. '15, '16)
  Superstables = Integer points inside fundamental parallelepipeds.
- Coxeter groups. Cartan matrices (Benkhart, K., Reiner '18)
  Superstables = Miniscule dominant weights

# Sandpile Group $\mathcal{S}(G)$

- Group of critical configurations under sandpile addition:

$$a \oplus b = \text{stabilization of } (a + b)$$



- Chip configurations under firing equivalence:

$$\mathcal{S}(G) \cong \mathrm{coker}(\Delta_q) = \mathbb{Z}^{n-1}/\mathsf{im}\Delta_q$$

# Sandpile Group $\mathcal{S}(G)$

Graph invariant in the form of a finite abelian group,

$$|\mathcal{S}(G)| = \#\text{ spanning trees of } G$$

Group structure for various graph classes. Invariant factors. Smith Normal Form. (Lorenzini '91, Merris '92, Biggs '99, Wagner '00, Cori, Rossin '00, Reiner+ '02 '03 '12, Levine '09, Norine, Whalen '11)

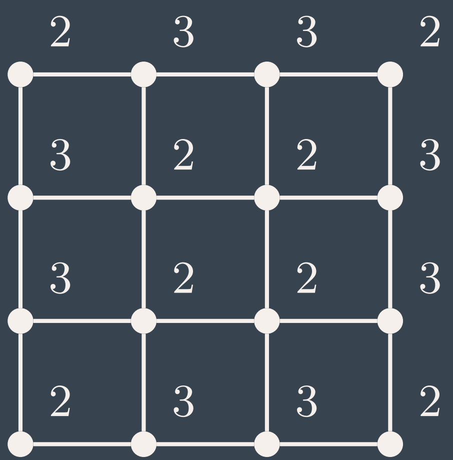
Structure of random graphs. (A type of) Cohen-Lenstra heuristic for the $p$-sylow subgroups of Sandpile groups. (Clancy, Leake, Payne '15; Wood '17)

Sandpile Torsors. (Wagner '00, Gioan '07, Bernardi '08, Holroyd, Levine, Meszaros, Peres, Propp, Wilson '08, Chan, Church, Grochow '15, Baker, Wang '17, Backman, Baker, Yuen '17, McDonough '18)

# Sandpile Group Identity

$\mathcal{S}(G)$ identity element? All 0s configuration is not critical.

$G$ = grid with sink along the boundary.




| | | | |
|---|---|---|---|
| 2 | 3 | 3 | 2 |
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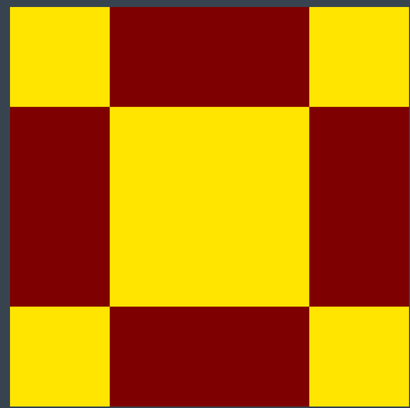

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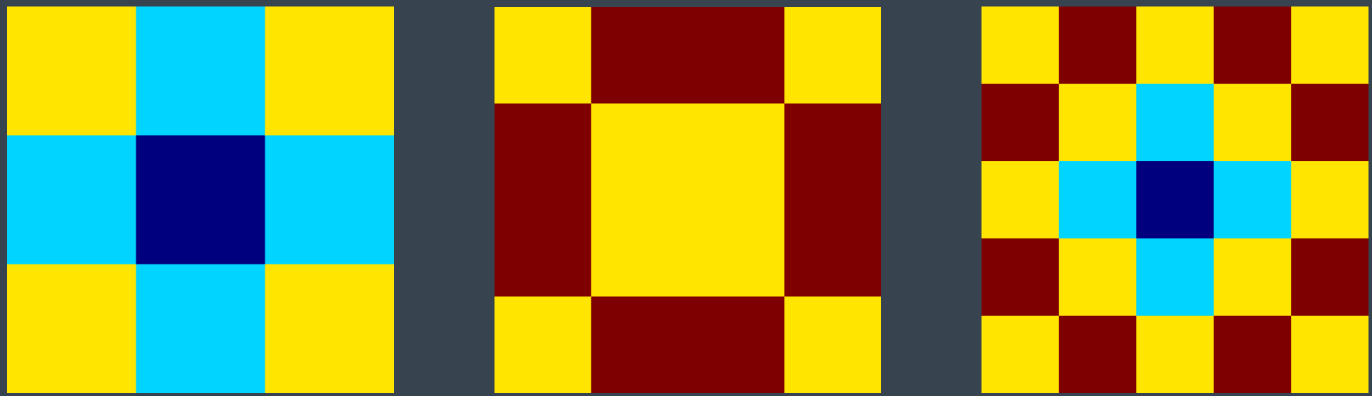
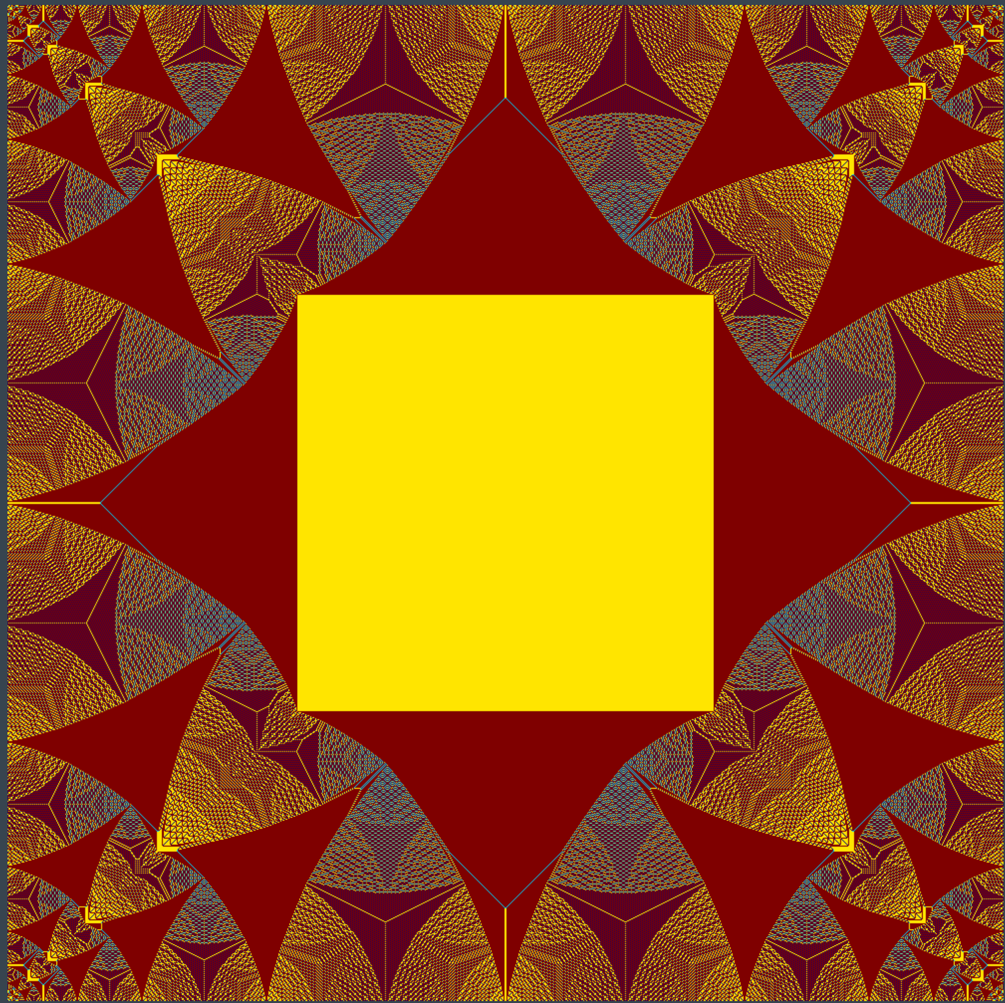


Identity elements for $3 \times 3$, $4 \times 4$, and $5 \times 5$ grids.

# Sandpile Group Identity

$G = 1000 \times 1000$ grid with sink along the boundary



(Dhar '95, Le Borgne, Rossin '02)

# Sandpile Group and Divisors on Curves

- Divisors on Curves (Graph as a Riemann surface)
  (Bacher, de la Harpe, Nagnibeda '97, Kotani, Sunada '00, Lorenzini '89)

| Curves | Graphs |
|---|---|
| Divisor $D$ | Chip configuration $\mathbf{c}$ |
| $\deg(D)$ | $\mathrm{wt}(\mathbf{c})$ |
| Canonical $K$ | $\mathbf{c}_{\max} - \mathbf{1}$ |
| Effective $D$ | $\mathbf{c} \geq 0$ |
| Linearly equivalent | Firing equivalent |
| Divisor class | Firing class |
| $q$-reduced | Superstable |
| Picard group / Jacobian | Sandpile group |

# Riemann–Roch Theorem

The *rank* of a divisor $r(D)$:

- If $D$ is not equivalent to any effective divisor then

$$r(D) = -1.$$

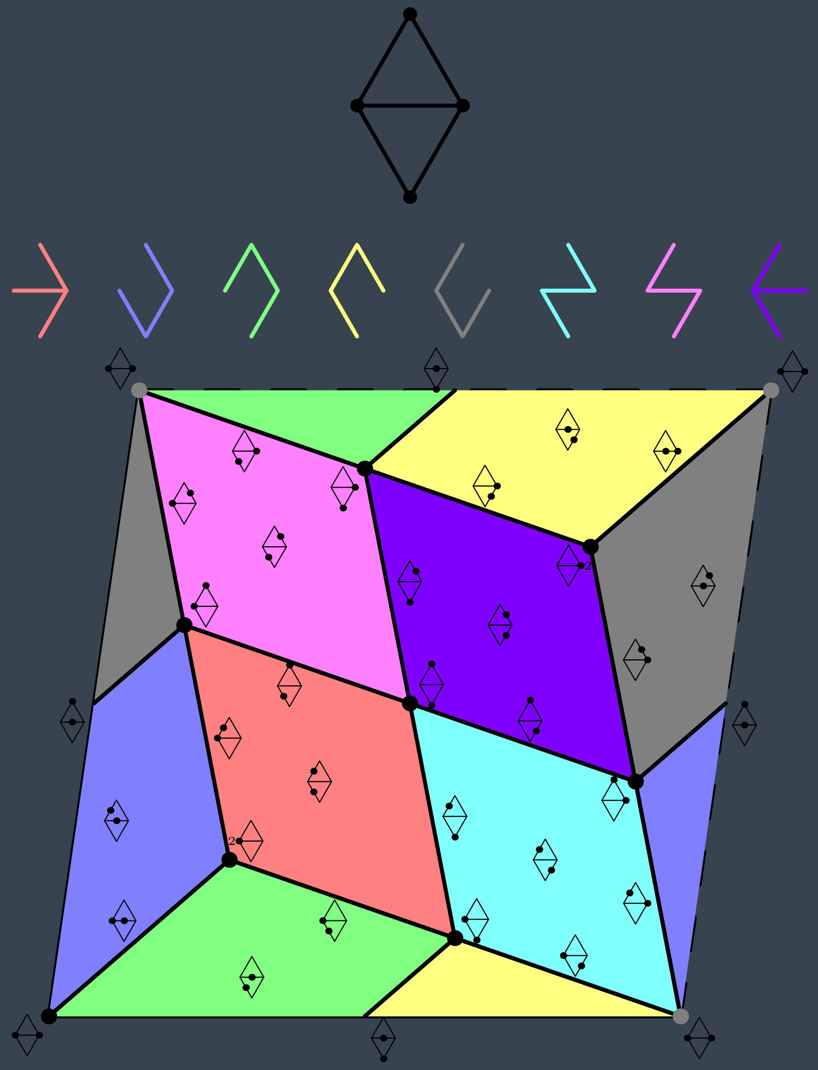
- $r(D) \geq k$ if and only if for any removal of $k$ chips from $D$, the resulting divisor is still equivalent to an effective divisor.

**Theorem:** (Baker, Norine '07) Let $G$ be a finite graph, $D$ a divisor on $G$ and $K$ the canonical divisor on $G$, then

$$r(D) - r(K - D) = \deg(D) + 1 - g.$$

# Divisors on Curves

- Abel–Jacobi Theory
- Riemann–Roch Theorem
- Clifford's Theorem
- Torelli's Theorem
- Max Rank Conjecture
- Tropical Geometry



Decomposition of Picard torus by break divisors.

(An, Baker, Kuperberg, Shokrieh '14)

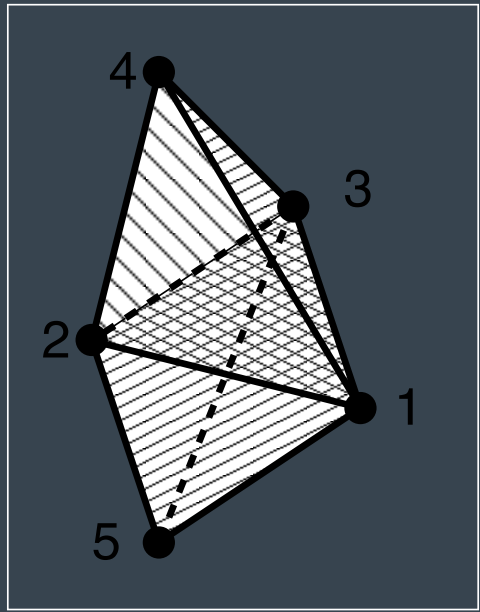
# Chip-Firing in Higher Dimensions

- Algebraic (Duval, K., Martin '11, '14)

  Combinatorial Laplacian
  (Hodge Laplacian)

  Higher dimensional spanning trees
  (simplicial matroids)

  Sandpile group $\mathcal{S}(G)$
  (family of group invariants)




Flow on edges
Reroute across incident faces

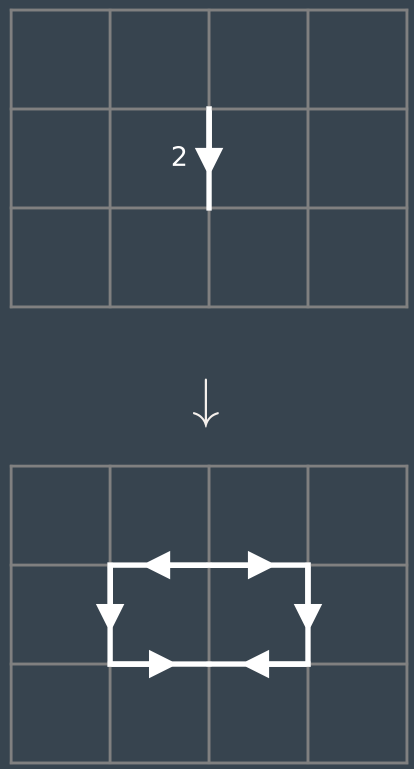
# Chip-Firing in Higher Dimensions

- Dynamic (Felzenszwalb, K. '19)

  Does the process stop?

  Order of firings?

  Pattern Formation?

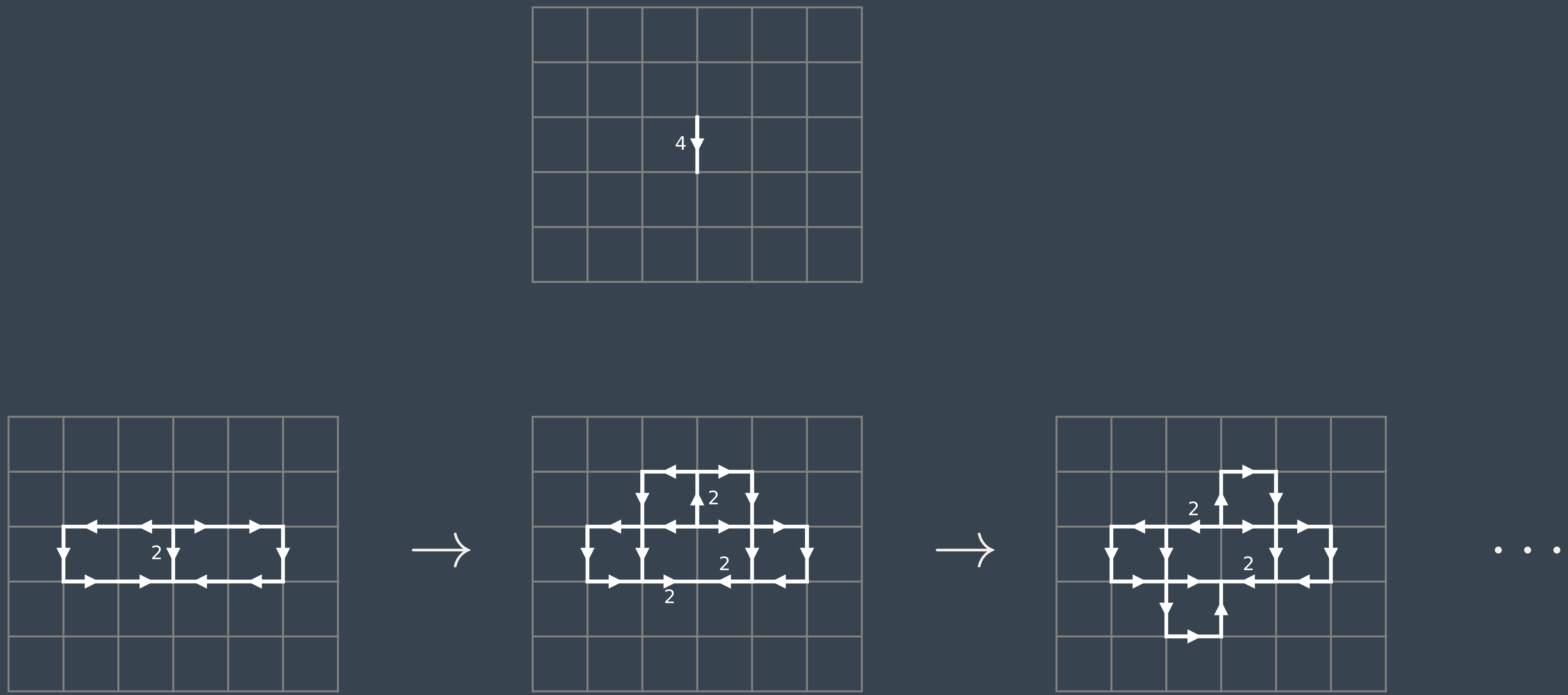



- Labeled chip-firing
  (Hopkins, McConville, Propp '17)

- Root system chip-firing
  (Galashin, Hopkins, McConville, Postnikov '18)

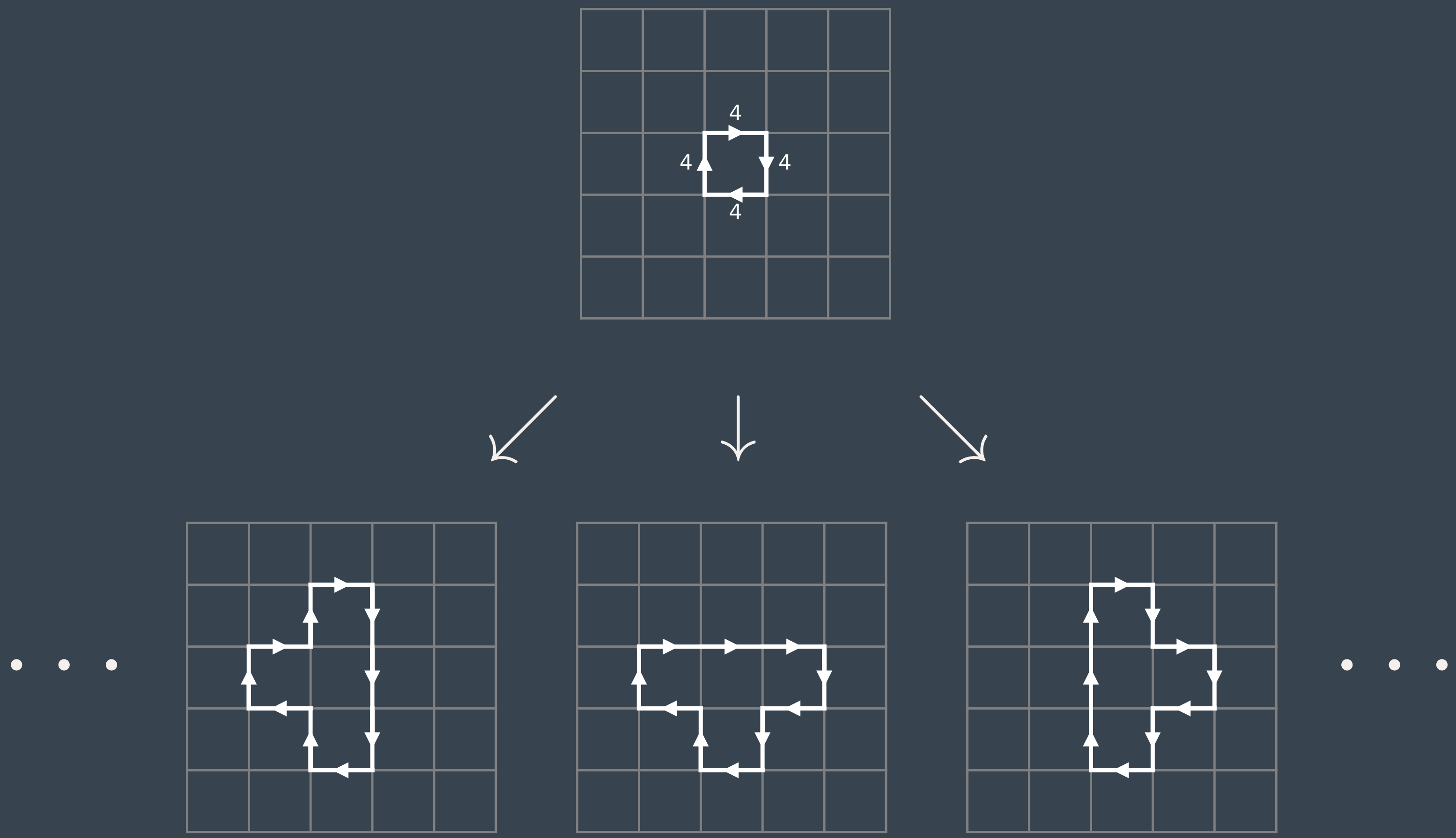
# Chip-Firing in Higher Dimensions

- A non-terminating example:



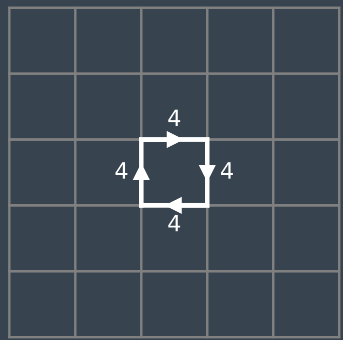
# Chip-Firing in Higher Dimensions

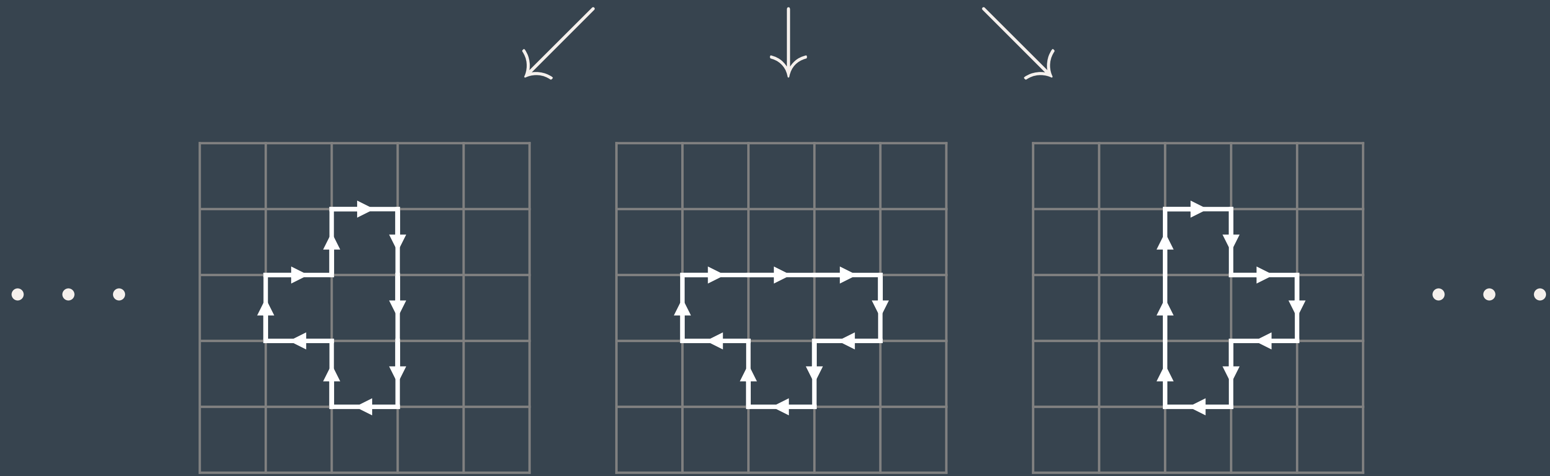
- Conservative flows (circulations) terminate:



# Chip-Firing in Higher Dimensions

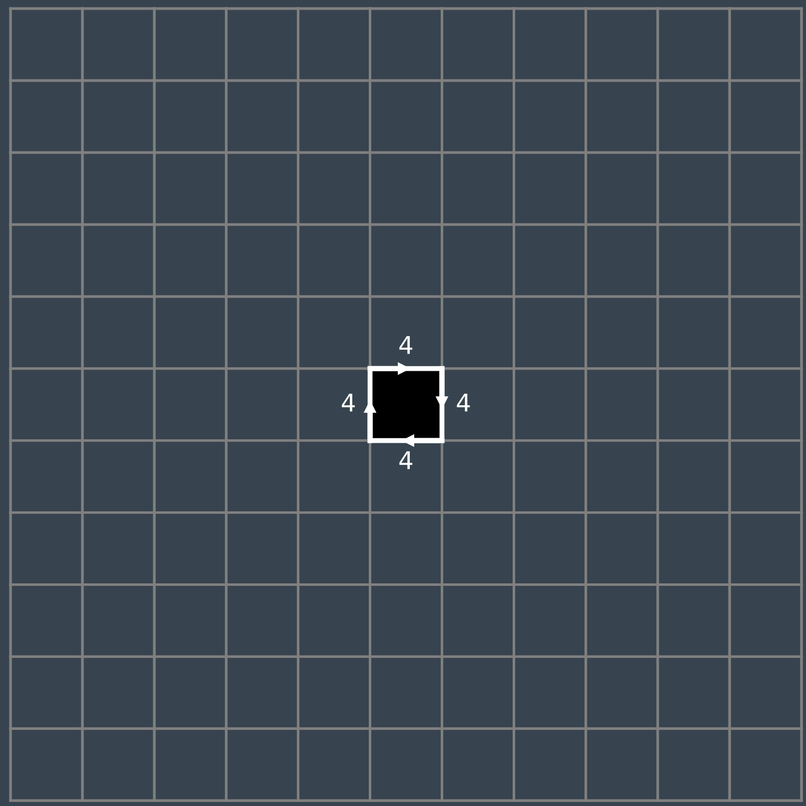
- Order matters!



# Chip-Firing in Higher Dimensions

- Remove a face from the grid:
  (Topological Constraint)



# Chip-Firing in Higher Dimensions