What is an explicit bijection?

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StackExchange Search on MathOverflow…

mathoverflow

What is an explicit bijection in combinatorics?

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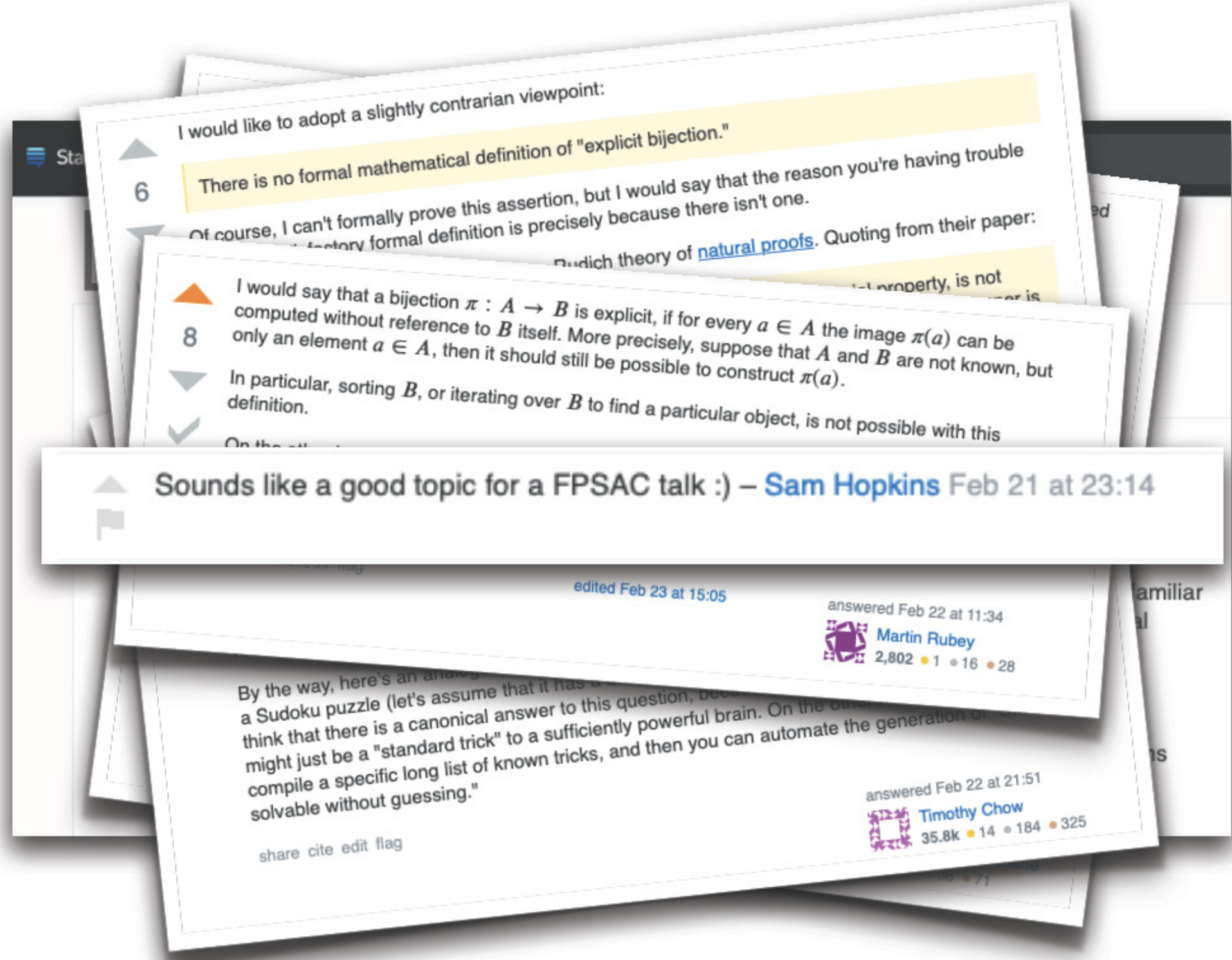
A standard way of demonstrating that two collections of combinatorial objects have the same cardinality is to exhibit a bijection between them. Browsing through some examples (here, there, yonder) quickly reveals that combinatorialists call such bijections *explicit*, presumably to differentiate them from other less palpable kinds of bijections. Wikipedia speaks of the method of bijective proof.

It seems that we have here a typical example of an informal mathematical notion that is quite familiar to most mathematicians, however it is difficult to pin down a proper and satisfying mathematical definition. I asked the local combinatorialists and did not really get a good answer.

Question: *What is a proper mathematical definition of an explicit bijection?*

Often we ask for an explicit bijection between two *families* of combinatorial objects, i.e., bijections $b_n : A_n \to B_n$, one for each $n \in \mathbb{N}$. Here $(A_n)_n$ and $(B_n)_n$ are two families of combinatorial objects, parametrized by n. The parameter need not be a single number.


I would like to adopt a slightly contrarian viewpoint:
6
There is no formal mathematical definition of "explicit bijection."
Of course, I can't formally prove this assertion, but I would say that the reason you're having trouble
factory formal definition is precisely because there isn't one.
Rudich theory of natural proofs. Quoting from their paper:
property, is not
I would say that a bijection $\pi : A \to B$ is explicit, if for every $a \in A$ the image $\pi(a)$ can be
8
computed without reference to B itself. More precisely, suppose that A and B are not known, but
only an element $a \in A$, then it should still be possible to construct $\pi(a)$.
In particular, sorting B, or iterating over B to find a particular object, is not possible with this
definition.
On the
Sounds like a good topic for a FPSAC talk :) – Sam Hopkins Feb 21 at 23:14
edited Feb 23 at 15:05
answered Feb 22 at 11:34
Martin Rubey
2,802 • 1 • 16 • 28
familiar
By the way, here's an
a Sudoku puzzle (let's assume that it has
think that there is a canonical answer to this question, bec
might just be a "standard trick" to a sufficiently powerful brain. On the oth
compile a specific long list of known tricks, and then you can automate the generation of
solvable without guessing."
share cite edit flag
answered Feb 22 at 21:51
Timothy Chow
35.8k • 14 • 184 • 325

An explicit bijection $f : A \to B$ is ...

- ... computable in polynomial time.
- ... a natural isomorphism.
- ... computed without reference to B.
- ... given without prior knowledge that $A \cong B$.

An explicit bijection $f : A \to B$ is ...

additional property or structure

- ... computable in polynomial time.
- ... a natural isomorphism.
- ... computed without reference to B.
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An explicit bijection $f : A \to B$ is ...

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the way it is constructed

FOL & ZFC

$$\bot \quad \top \quad \land \quad \lor \quad \Rightarrow \quad \forall \quad \exists$$

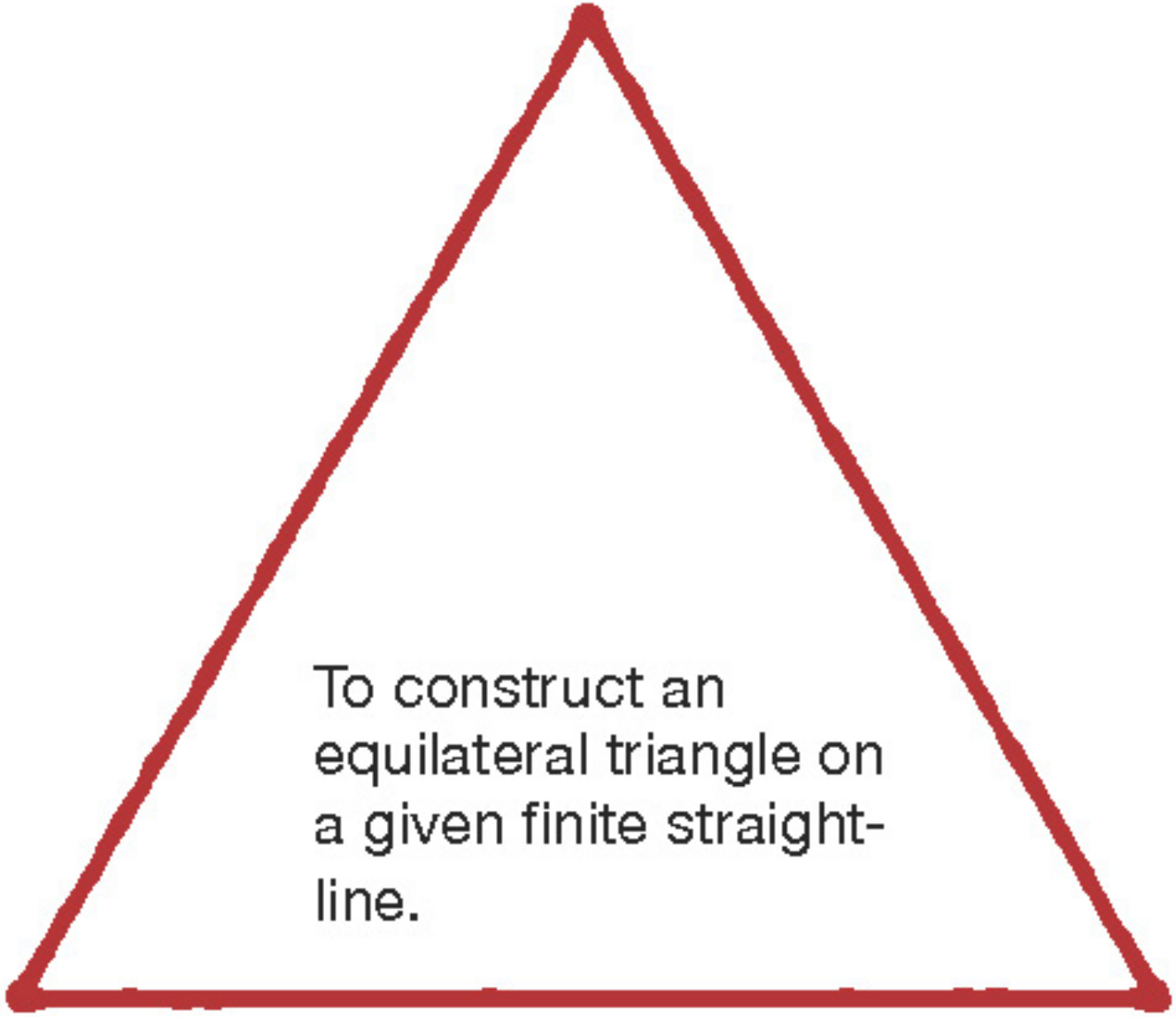
$$\in$$

FOL & ZFC

$$\bot \quad \top \quad \wedge \quad \vee \quad \Rightarrow \quad \forall \quad \exists$$

$$\in \quad \subseteq \quad \varnothing \quad \cap \quad \cup \quad \wp$$

' Επὶ τῆς δοθείσης
εὐθείας πεπερασμένης
τρίγωνον ἰσόπλευρον
συστήσασθαι


To construct an
equilateral triangle on
a given finite straight-
line.

First isomorphism theorem

$$G / \ker \phi \cong \operatorname{im} \phi$$

First isomorphism theorem

$\exists\, \theta : G / \ker \phi \to \operatorname{im} \phi$. θ iso

First isomorphism theorem

$$\exists\, \theta : G / \ker \phi \to \operatorname{im} \phi \;.\; \theta \text{ iso}$$

Proof: Consider the map $\theta : x\,(\ker \phi) \mapsto \phi\, x$.

What would Euclid do?

To construct an isomorphism
$G / \ker \phi \cong \operatorname{im} \phi$

Solution: Consider the map $\theta : x(\ker \phi) \mapsto \phi x$.

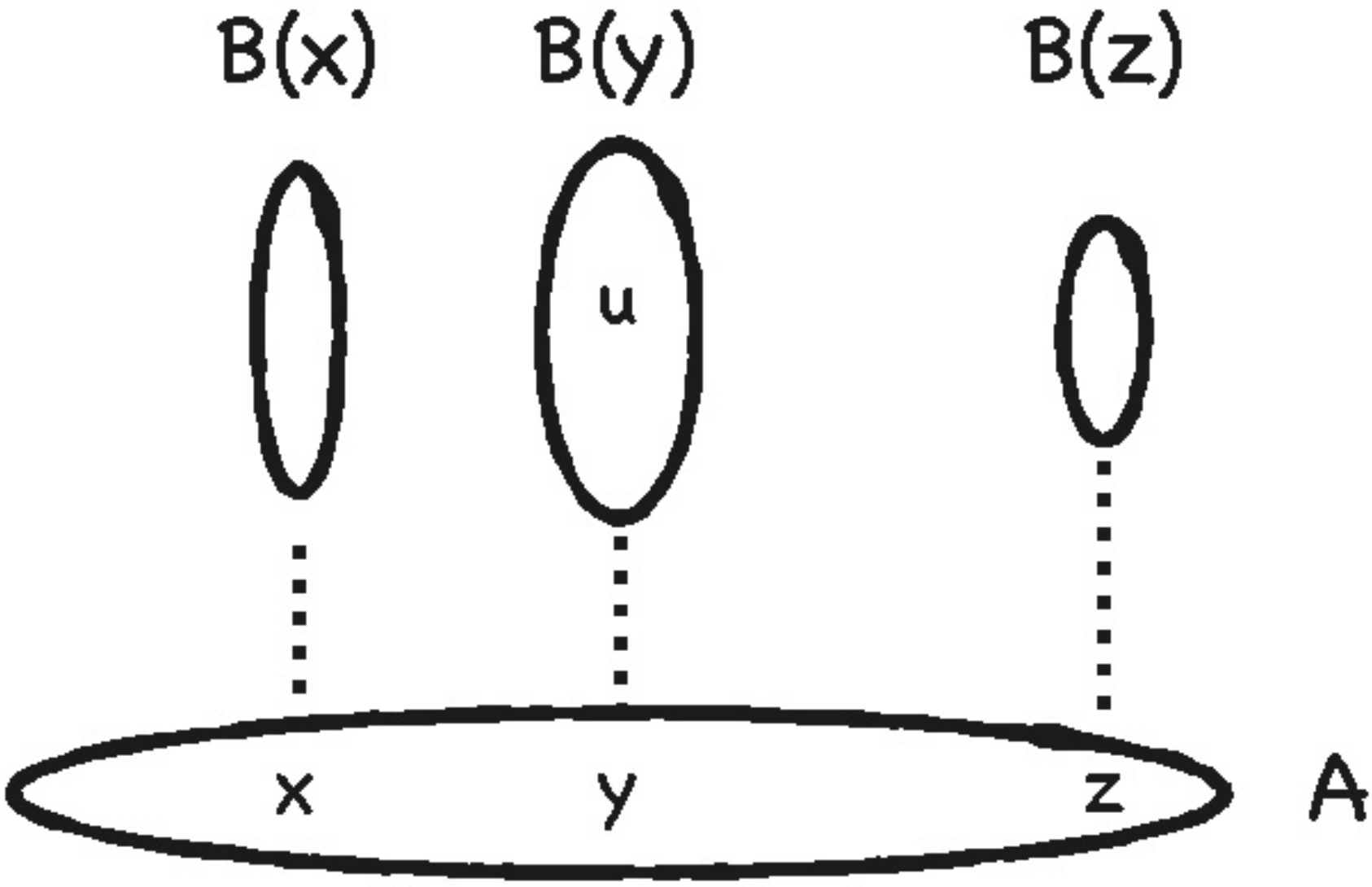
Type theory

$$t : A$$

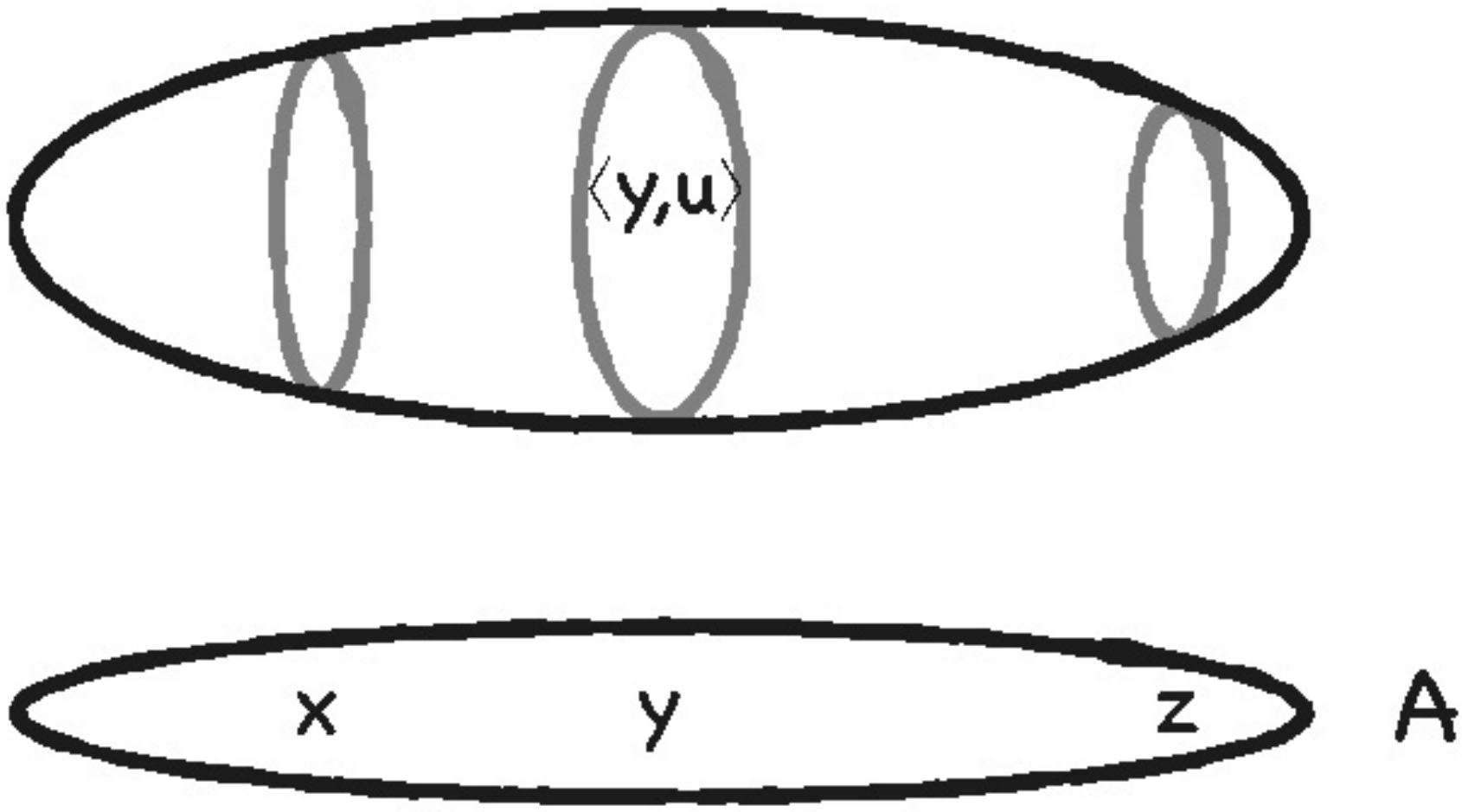
Type theoretic constructions

Type	Element
1	$\langle\rangle$
0	
$\mathbb{N}$	0, succ n
$\mathcal{U}$	A, B
$A \times B$	$\langle a, b \rangle$
$A + B$	$\iota_1(a)$, $\iota_2(b)$
$A \to B$	$x \mapsto e(x)$
$\prod (x : A) . B(x)$	$x \mapsto e(x)$
$\sum (x : A) . B(x)$	$\langle a, b \rangle$

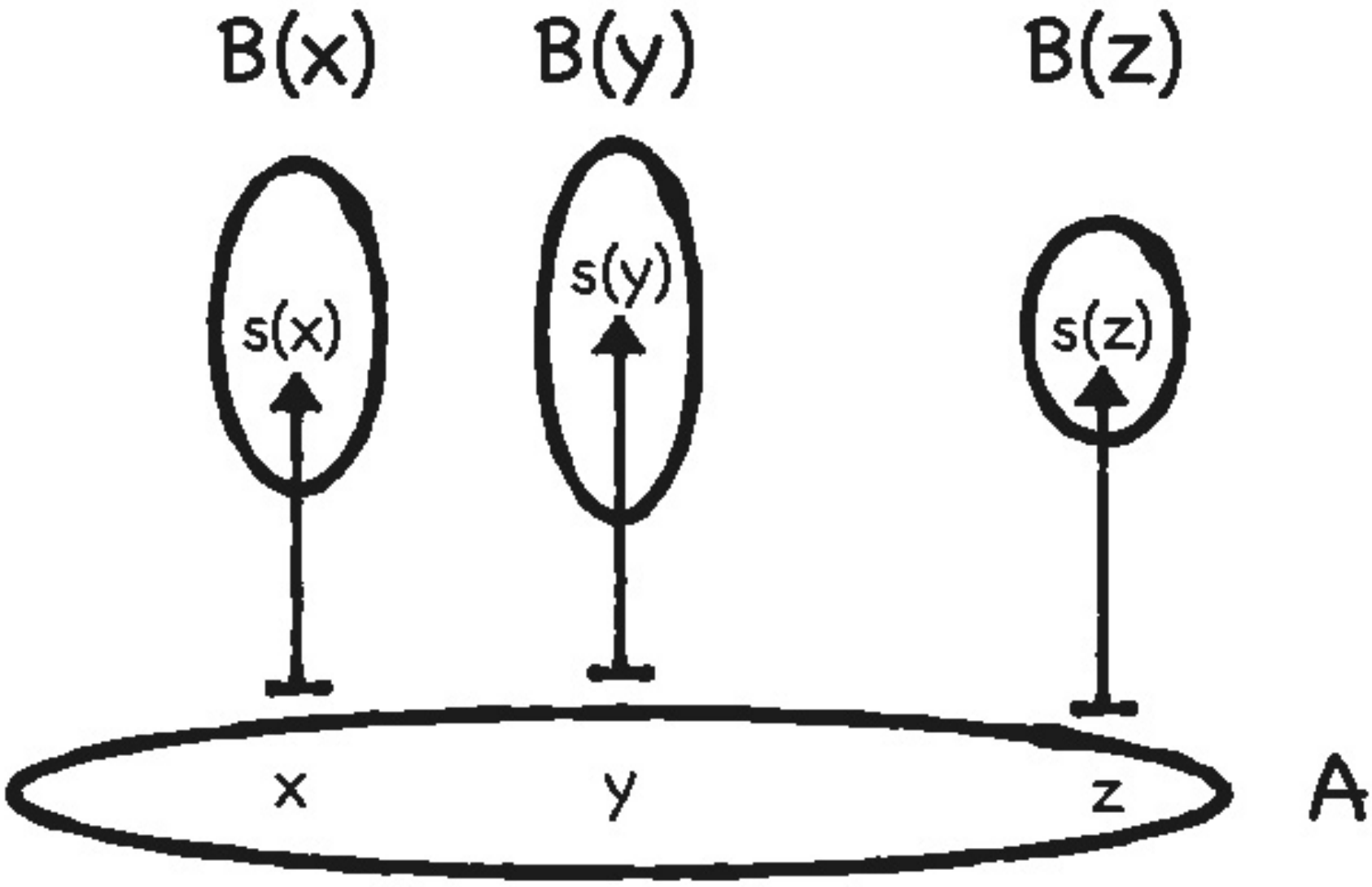
Dependent type B over A



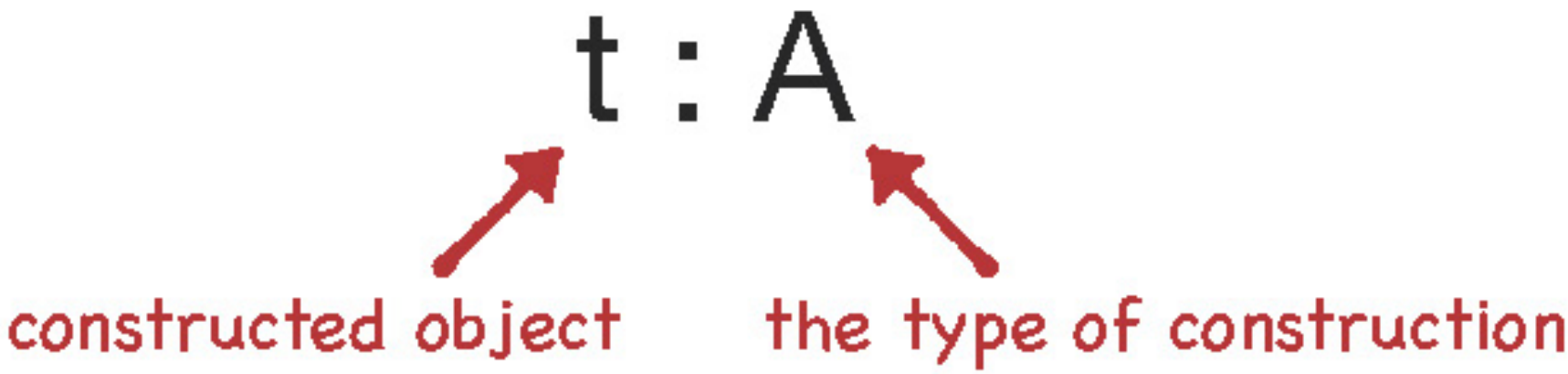
$$\Sigma(x : A).B(x)$$



Π(x : A).B(x)



Where is logic?



Where is logic?

The Curry-Howard correspondence

Logic	Types	Element/proof
$\top$	1	$\langle\rangle$
$\bot$	0	
$A \wedge B$	$A \times B$	$\langle a, b \rangle$
$A \vee B$	$A + B$	$\iota_1(a), \iota_2(b)$
$A \Rightarrow B$	$A \to B$	$x \mapsto e(x)$
$\forall\, x \in A\,.\, B(x)$	$\prod (x : A)\,.\, B(x)$	$x \mapsto e(x)$
$\exists\, x \in A\,.\, B(x)$	$\sum (x : A)\,.\, B(x)$	$\langle a, b \rangle$

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$$p : \exists\, x \in A \,.\, B(x)$$

cannot extract an element of A from p

$$p : \Sigma\, (x : A) \,.\, B(x)$$

may project an element of A from p

Propositional truncation

Type	Element
$\lVert A \rVert$	$\lvert a \rvert$

"A quotiented by A × A"

"the (only) equivalence class"

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$\exists\, x \in A\,.\, B(x)$	$\|\sum (x : A)\,.\, B(x)\|$	$\|\langle a, b\rangle\|$

Equality type

Type	Element
$t =_A u$	“path from t to u”

Equality type

Type	Element
$t =_A u$	"path from t to u"
$A =_U B$	"A is equivalent to B"

Univalence axiom:

$(A =_U B) \simeq (A \simeq B)$

"equality is equivalent to equivalence"

$$\Pi\ (n : N)\ .\ \Sigma\ (p : N)\ .\ (n < p) \times \text{prime}(p)$$

Map that takes a number n to $\langle p, \langle q, r \rangle \rangle$ such that p is a number, q proves $n < p$, and r proves prime(p).

For every number to construct a prime larger than it.

$$\Pi\ (n : N)\ .\ \Sigma\ (p : N)\ .\ (n < p) \times \text{prime}(p)$$

Map that takes a number n to $\langle p, \langle q, r \rangle\rangle$ such that p is a number, q proves $n < p$, and r proves prime(p).

For every number to construct a prime larger than it.

$$\Pi\ (n : N)\ .\ \|\Sigma\ (p : N)\ .\ (n < p) \times \text{prime}(p)\|$$

Map that takes a number n to an (abstracted) proof showing that there is a prime larger than n.

For every number there exists a prime larger than it.

Explicit bijection

$$\Sigma\,(f : A \to B)\ \Sigma\,(g : B \to A)\ .\ (f \circ g = id_B) \times (g \circ f = id_A)$$

A quadruple $\langle f, \langle g, \langle p, q \rangle \rangle \rangle$ such that p is a path from $f \circ g$ to id_B and q a path from $g \circ f$ to id_A.

To construct a bijection from A to B.

Non-explicit bijection

$$\|\Sigma\,(f : A \to B)\,\Sigma\,(g : B \to A)\,.\,(f \circ g = \mathrm{id}_B) \times (g \circ f = \mathrm{id}_A)\|$$

A proof $|\langle f, \langle g, \langle p, q\rangle\rangle\rangle|$ such that p is a path from $f \circ g$ to id_B and q a path from $g \circ f$ to id_A.

There exists a bijection from A to B.

$$[n] := \Sigma\ (k : N)\ .\ k < n$$

The numbers 0,1, ..., n-1.

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$$\Sigma\ (A : U)\ \Sigma\ (n : N)\ .\ A \cong [n]$$

Type A with a number n and isomorphism $A \cong [n]$.

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Type A with a number n and isomorphism $A \cong [n]$.

$$\mathrm{Fin} := \Sigma\ (A : U)\ \|\Sigma\ (n : N)\ .\ A \cong [n]\|$$

Type A such that there exists $A \cong [n]$.

$$[n] := \Sigma\ (k : N)\ .\ k < n$$

The numbers 0,1, ..., n-1.

$$\Sigma\ (A : U)\ \Sigma\ (n : N)\ .\ A \cong [n]$$

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Type A such that there exists $A \cong [n]$.

$$\mathrm{Fin} \to U$$

Combinatorial spices.

$$[n] \to A$$

Ordered n-tuple of elements in A

$$[n] \to A$$

Ordered n-tuple of elements in A

$$\mathrm{Fin}_n := \Sigma\ (A : U)\ \| A \cong \mathrm{Fin}(n) \|$$

Type with n elements.

$$\Sigma\ (B : \mathrm{Fin}_n)\ .\ B \to A$$

Unordered n-tuples of elements in A

$$[n] \to A$$

Ordered n-tuple of elements in A

$$\mathrm{Fin}_n := \Sigma\,(A : U)\ \|A \cong \mathrm{Fin}(n)\|$$

Type with n elements.

$$\Sigma\,(B : \mathrm{Fin}_n)\ .\ B \to A$$

Unordered n-tuples of elements in A

$$\mathrm{Fin}_2$$

The infinite-dimensional real-projective space $\mathbb{RP}^\infty$