

FPSAC2019, Ljubljana

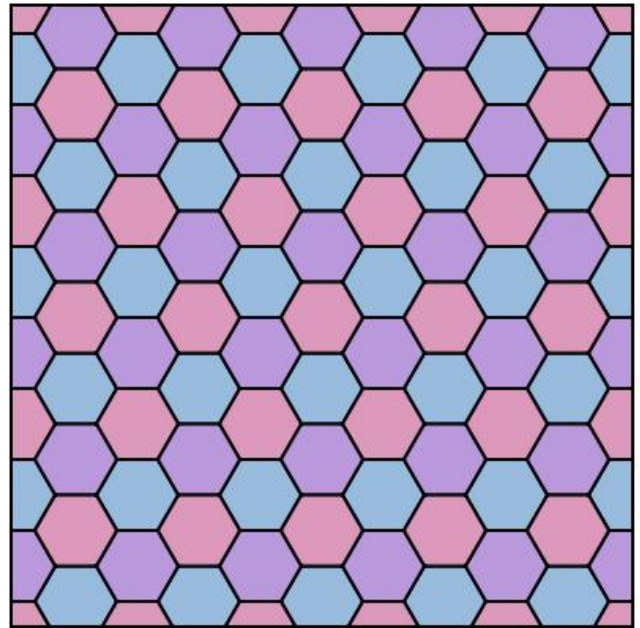
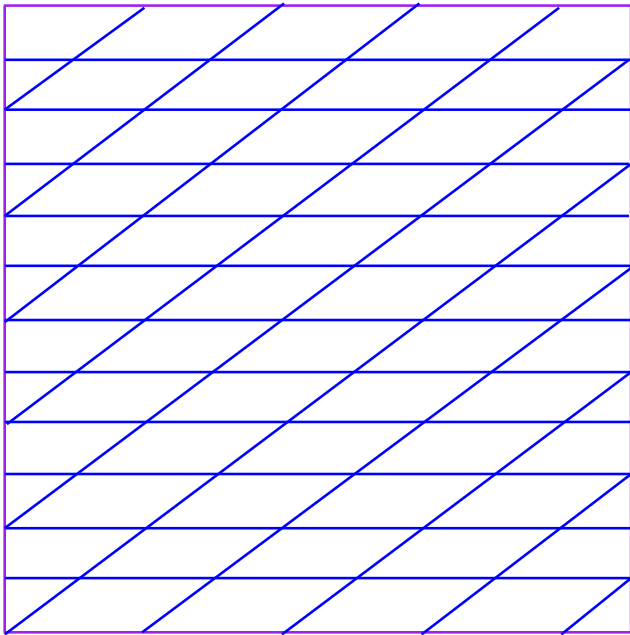
# Can You Pave the Plane with Identical Tiles?

*Chuanming Zong*

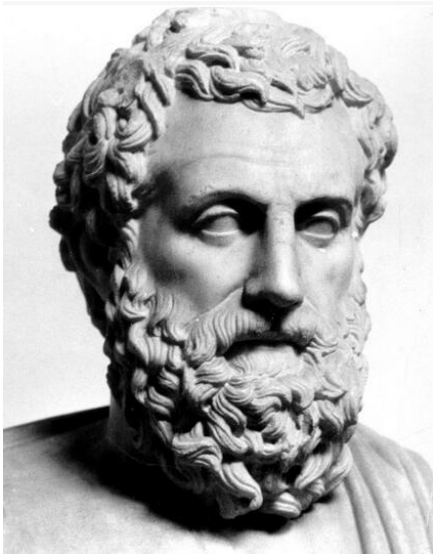
Tianjin University



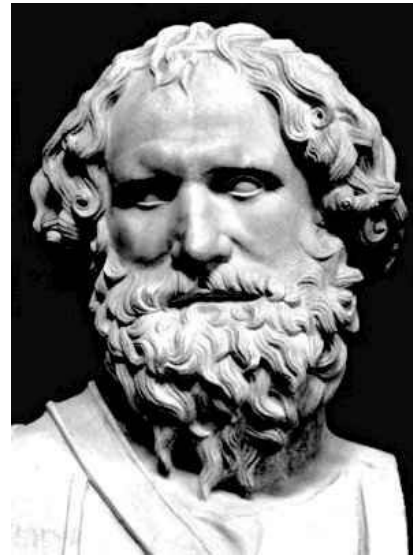
What kind of polygons can tile the whole plane?



Tiling is one of the ancient subjects in mathematics. Early studies can be traced back to Aristotle and Archimedes. There are thousands of papers about tilings. However, solutions to some fundamental problems are still missing.



Aristotle, 384 BC – 322 BC



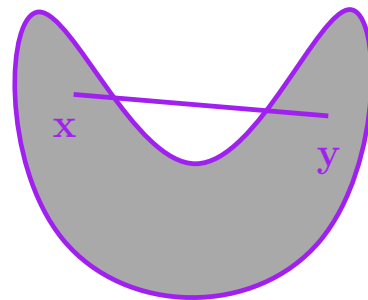
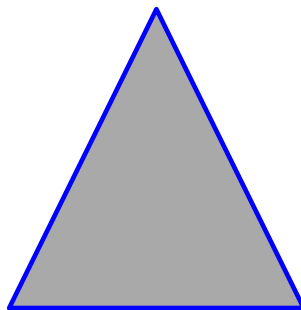
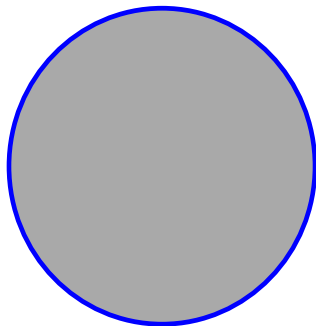
Archimedes, 287 BC – 212 BC

# Convex Bodies and Lattices

**Convex Body:** In the  $n$ -dimensional Euclidean space  $\mathbb{E}^n$ , we say a set  $K$  is convex if all the segments connecting two points of the set belong to the set. In other words,

$$\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} \in K$$

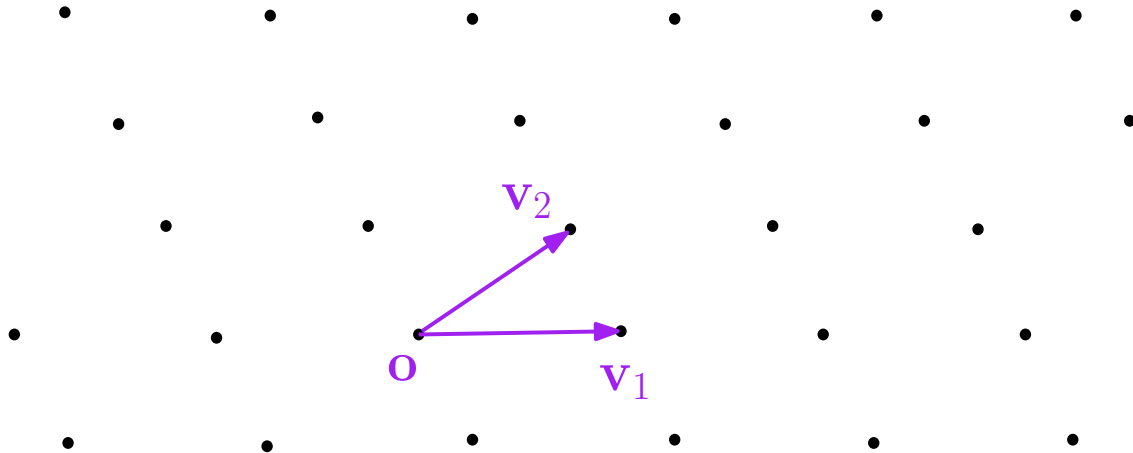
holds whenever both  $\mathbf{x}$  and  $\mathbf{y}$  belong to  $K$  and  $0 \leq \lambda \leq 1$ . We call  $K$  an  *$n$ -dimensional convex body* if it is convex, full-dimensional and compact.



**Lattice:** If  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are  $n$  independent vectors in  $\mathbb{E}^n$ , we call

$$\Lambda = \left\{ \sum_{i=1}^n z_i \mathbf{v}_i, \quad z_i \in \mathbb{Z} \right\}$$

an  $n$ -dimensional lattice.



## Lattice Tilings, Fedorov's Discovery

A family  $\mathfrak{F}$  of compact subsets of  $\mathbb{E}^n$  with non-empty interiors is a *tiling* of  $\mathbb{E}^n$  if their union is  $\mathbb{E}^n$  and any two distinct sets have disjoint interiors. The sets are called *tiles*.

To avoid complexity and confusion, in this talk we only deal with the tilings by identical convex polygon tiles. In particular, we call it a *congruent tiling* if all the tiles are congruent to each others, call it a *translative tiling* if all the tiles are translates of each others, and call it a *lattice tiling* if it is a translative tiling and all the translative vectors together is a lattice.

**Theorem 1 (Fedorov, 1885).** *A convex domain can form a lattice tiling of  $\mathbb{E}^2$  if and only if it is a parallelogram or a centrally symmetric hexagon; a convex body can form a lattice tiling in  $\mathbb{E}^3$  if and only if it is a parallelotope, a hexagonal prism, a rhombic dodecahedron, an elongated octahedron or a truncated octahedron.*

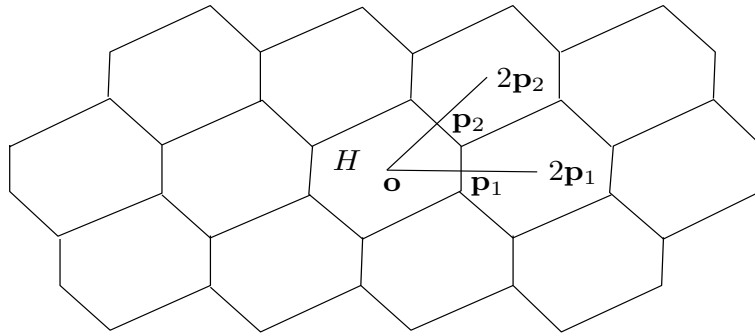


Figure 3.1

**Remark.** Of course, we can treat a parallelogram as a special centrally symmetric hexagon. Therefore, essentially there is only one type of convex polygons which can produce lattice tilings in the plane.

## Hilbert's Problem and Bieberbach's Question

Clearly, a lattice is an additive group in the space, and a lattice tile is a fundamental region of the group of motions.

In 1900, Hilbert proposed a list of mathematical problems in his ICM lecture in Paris. As a generalized inverse of Fedorov's discovery, he proposed the following problem as a part of his 18th problem:

*A fundamental region of each group of motions, together with the congruent regions arising from the group, evidently fills up space completely. The question arises: whether polyhedra also exist which do not appear as fundamental regions of groups of motions, by means of which nevertheless by a suitable juxtaposition of congruent copies a complete filling up of all space is possible.*



Hilbert proposed his problem in the space, perhaps he believed that there is no such domain in the plane.

When Reinhardt started his Doctoral thesis at Frankfurt am Main in 1910s, Bieberbach suggested him to determine all the convex domains which can form congruent tilings in the plane and so that to verify that Hilbert's problem indeed has positive answer in the plane.

**Problem 1.** *To determine all the two-dimensional convex tiles.*

**Remark.** In 1954, Venkov proved that if an  $n$ -dimensional convex body can produce a translative tiling, then it is a *parallelohedron*. However, to classify all of the parallelohedra is a very challenging job and our knowledge about it is limited to dimensions five or less.

## Classification of the Two-dimensional Tiles

It can be easily shown that a convex tile must be a polytope. Then, by *Euler's formula* it can be shown that the number of the edges of a two-dimensional convex tile is at most six.

Apparently, two identical triangles can make a parallelogram and two identical quadrilaterals can make a centrally symmetric hexagon. Thus, by Fedorov's theorem, identical triangles or quadrilaterals can always tile the plane nicely.

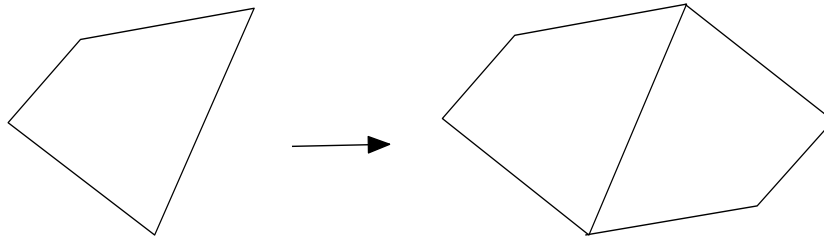


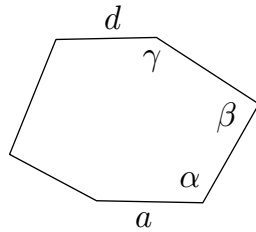
Figure 3.2

Then, Bieberbach's problem can be reformulated as:

*What kind of convex pentagons or hexagons can tile the plane?*

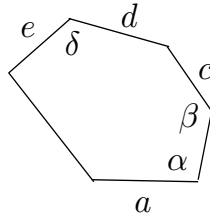
For the hexagon case, in 1918 Reinhardt claimed the following solution.

**Theorem 2.** *A convex hexagon can tile the whole plane if and only if it is one of the following three types:*



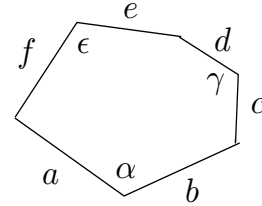
$$a = d$$

$$\alpha + \beta + \gamma = 2\pi$$



$$a = d \quad c = e$$

$$\alpha + \beta + \delta = 2\pi$$



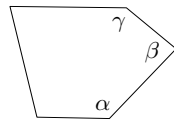
$$a = b \quad c = d \quad e = f$$

$$\alpha = \gamma = \epsilon = 2\pi/3$$

Figure 3.3

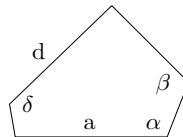
The story of hunting the pentagon tiles is very dramatic! In 1918, Reinhardt discovered five types of pentagon tiles and believed that the list was complete. Half a century later, Kershner discovered three new types of pentagon tiles and claimed the completeness of the extended list. Unfortunately, six types of new two-dimensional convex tiles were discovered in 1970s and 1980s by James, Rice and Stein, respectively. However, this is still not the end of the story. In 2017, Mann, Mcloud-Mann and Derau reported another one!!!

**Theorem 3.** *A convex pentagon can tile the whole plane if it belongs to one of the following fifteen types:*



$$\alpha + \beta + \gamma = 2\pi$$

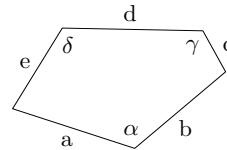
K. Reinhardt



$$a = d$$

$$\alpha + \beta + \delta = 2\pi$$

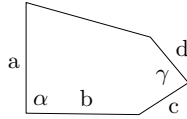
K. Reinhardt



$$a = b \quad d = c + e$$

$$\alpha = \gamma = \delta = 2\pi/3$$

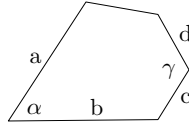
K. Reinhardt



$$a = b \quad c = d$$

$$\alpha = \gamma = \pi/2$$

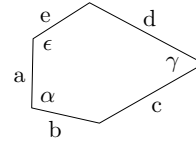
K. Reinhardt



$$a = b \quad c = d$$

$$\alpha = \gamma/2 = \pi/3$$

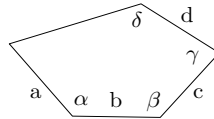
K. Reinhardt



$$a = b = e \quad c = d$$

$$\alpha = 2\gamma \quad \gamma + \epsilon = \pi$$

R.B. Kershner

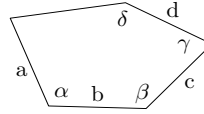


$$a = b = c = d$$

$$\alpha + 2\delta = 2\pi$$

$$2\beta + \gamma = 2\pi$$

R.B. Kershner

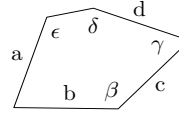


$$a = b = c = d$$

$$2\alpha + \beta = 2\pi$$

$$\gamma + 2\delta = 2\pi$$

R.B. Kershner

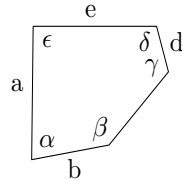


$$a = b = c = d$$

$$\beta + 2\epsilon = 2\pi$$

$$\gamma + 2\delta = 2\pi$$

M. Rice

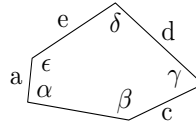


$$a = e = b + d$$

$$\epsilon = \pi/2 \quad 2\beta - \delta = \pi$$

$$2\gamma + \delta = 2\pi \quad \alpha + \delta = \pi$$

R. James

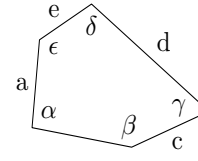


$$2a + c = d = e$$

$$\alpha = \pi/2 \quad \gamma + \epsilon = \pi$$

$$2\beta + \gamma = 2\pi$$

M. Rice

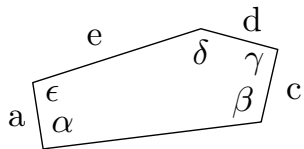


$$2a = c + e = d$$

$$\alpha = \pi/2 \quad \gamma + \epsilon = \pi$$

$$2\beta + \gamma = 2\pi$$

M. Rice

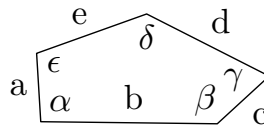


$$c = d \quad 2c = e$$

$$\beta = \epsilon = \pi - \delta/2$$

$$\alpha = \gamma = \pi/2$$

M. Rice



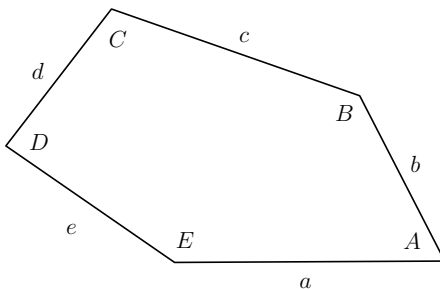
$$2a = 2c = d$$

$$\alpha = \pi/2 \quad 2\beta + \gamma = 2\pi$$

$$\gamma + \epsilon = \pi \quad \alpha + \beta + \delta = 2\pi$$

R. Stein

Figure 3.4



$$A = 60^\circ \quad B = 135^\circ \quad C = 105^\circ \quad D = 90^\circ \quad E = 150^\circ$$

$$a = 1 \quad b = 1/2 \quad c = 1/\sqrt{2}(\sqrt{3} - 1) \quad d = 1/2 \quad e = 1/2$$

Mann, Mcloud-Mann and Derau

**Remark.** In fact, the pentagons of type six (discovered by Kershner) are counterexamples to Hilbert's problems in the plane, although Kershner himself did not realize this fact.

**Remark.** The Mathematical Association of America paved a lobby floor in Washington D. C. with one of Rice's tiles.



**Theorem 4 (Rao, 2017).** *The list in Theorem 3 is complete.*

**Proof Idea.** First, by studying the angle condition, a computer algorithm reduces to 371 possible types of pentagons tiles. Then, checking them further by computer and eliminating the impossible cases.

**Remark.** Rao's paper has not published yet. Some experts are still checking his proof.



## Multiple Tilings

Assume that  $\mathfrak{F} = \{K_1, K_2, K_3, \dots\}$  is a family of convex bodies in  $\mathbb{E}^n$  and  $k$  is a positive integer. We call  $\mathfrak{F}$  a *k-fold tiling* of  $\mathbb{E}^n$  if every point  $\mathbf{x} \in \mathbb{E}^n$  belongs to at least  $k$  of these convex bodies and every point  $\mathbf{x} \in \mathbb{E}^n$  belongs to at most  $k$  of the  $\text{int}(K_i)$ . In other words, a *k-fold tiling* of  $\mathbb{E}^n$  is both a *k-fold packing* and a *k-fold covering* in  $\mathbb{E}^n$ .

In particular, we call a *k-fold tiling* of  $\mathbb{E}^n$  a *k-fold congruent tiling*, a *k-fold translative tiling*, or a *k-fold lattice tiling* if all  $K_i$  are congruent to  $K_1$ , all  $K_i$  are translates of  $K_1$ , or all  $K_i$  are translates of  $K_1$  and the translative vectors form a lattice in  $\mathbb{E}^n$ , respectively. In these particular cases, we call  $K_1$  a *k-fold congruent tile*, a *k-fold translative tile* or a *k-fold lattice tile*, respectively.

Furtwängler (1936) and Robinson (1979) first studied multiple lattice tilings by cubes. It was proved by Bolle, Gravin, Robins and Shiryaev that a translative  $k$ -tile must be centrally symmetric with centrally symmetric facets.

Let  $P$  denote an  $n$ -dimensional centrally symmetric convex polytope, let  $\tau(P)$  denote the smallest integer  $k$  such that  $P$  is a translative  $k$ -tile, and let  $\tau^*(P)$  denote the smallest integer  $k$  such that  $P$  is a lattice  $k$ -tile. For convenience, we define  $\tau(P) = \infty$  if  $P$  can not form translative tiling of any multiplicity. Clearly, for every convex polytope we have

$$\tau(P) \leq \tau^*(P).$$

**Problem 2.** *When  $D$  runs over all convex polygons, can  $\tau^*(D)$  (or  $\tau(D)$ ) take every positive integer?*

**Problem 3.** *Is  $\tau(D)$  always identical with  $\tau^*(D)$ ?*

## Multiple Lattice Tilings

In 1994, Bolle studied the two-dimensional lattice multiple tilings. He proved the following criterion:

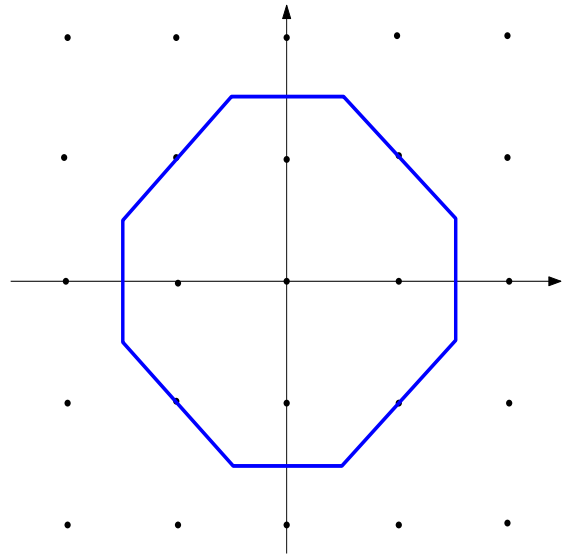
**Lemma 1.** *A convex polygon is a  $k$ -fold lattice tile for a lattice  $\Lambda$  and some positive integer  $k$  if and only if the following conditions are satisfied:*

- *It is centrally symmetric.*
- *When it is centered at the origin, in the relative interior of each edge  $G$  there is a point of  $\frac{1}{2}\Lambda$ .*
- *If the midpoint of  $G$  is not in  $\frac{1}{2}\Lambda$ , then  $G$  is a lattice vector of  $\Lambda$ .*

In 2013, Gravin, Robins and Shiryaev discovered the following example.

**Example 1.** Let  $\Lambda$  denote the two-dimensional integer lattice  $\mathbb{Z}^2$  and let  $D_8$  denote the polygon with vertices  $\mathbf{v}_1 = (\frac{1}{2}, -\frac{3}{2})$ ,  $\mathbf{v}_2 = (\frac{3}{2}, -\frac{1}{2})$ ,  $\mathbf{v}_3 = (\frac{3}{2}, \frac{1}{2})$ ,  $\mathbf{v}_4 = (\frac{1}{2}, \frac{3}{2})$ ,  $\mathbf{v}_5 = -\mathbf{v}_1$ ,  $\mathbf{v}_6 = -\mathbf{v}_2$ ,  $\mathbf{v}_7 = -\mathbf{v}_3$  and  $\mathbf{v}_8 = -\mathbf{v}_4$ . Then  $D_8 + \Lambda$  is a seven-fold lattice tiling of  $\mathbb{E}^2$ . Consequently, we have

$$\tau^*(D_8) \leq 7.$$



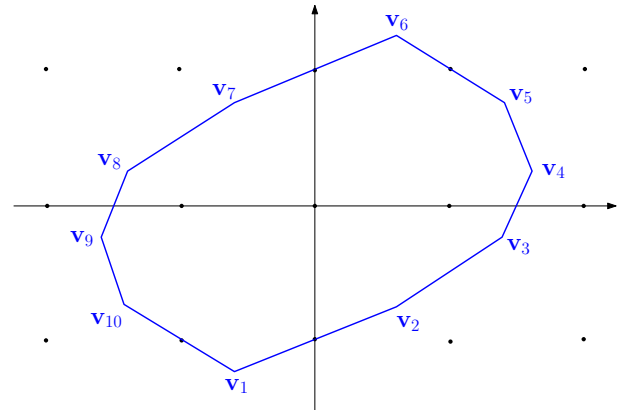
In 2017, Yang and Zong proved the following theorem:

**Theorem 5 (Yang and Zong).** *If  $D$  is a two-dimensional convex domain which is neither a parallelogram nor a centrally symmetric hexagon, then we have*

$$\tau^*(D) \geq 5,$$

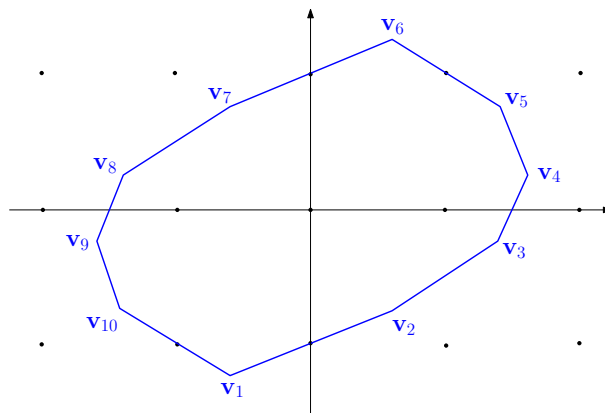
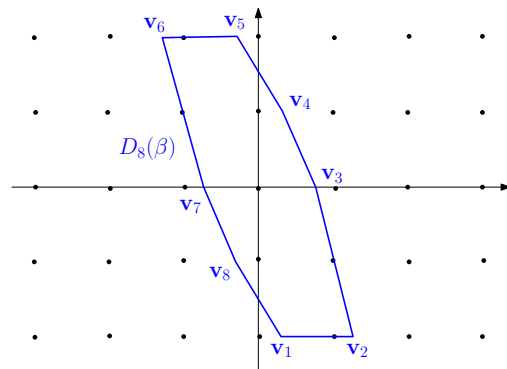
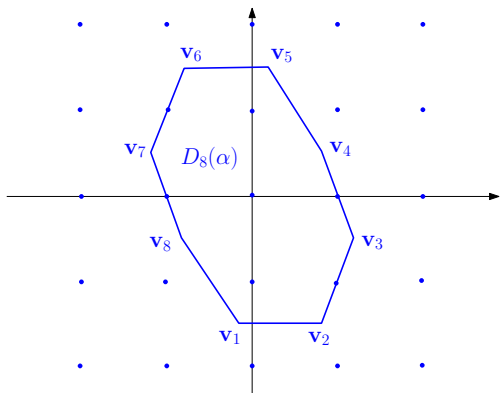
*where the equality holds at some particular decagons.*

**Remark.** The lower bound follows easily from the known results about multiple packings. However, the equality example is unexpected.



Even more unexpected, by studying lattice polygons, all the five-fold lattice tiles can be nicely characterized. They are two classes of octagons and one class of decagons, besides parallelograms and centrally symmetric hexagons.

**Theorem 6 (Zong).** *A convex domain can form a five-fold lattice tiling of the Euclidean plane if and only if it is a parallelogram, a centrally symmetric hexagon, under a suitable affine linear transformation, a centrally symmetric octagon with vertices  $\mathbf{v}_1 = (-\alpha, -\frac{3}{2})$ ,  $\mathbf{v}_2 = (1 - \alpha, -\frac{3}{2})$ ,  $\mathbf{v}_3 = (1 + \alpha, -\frac{1}{2})$ ,  $\mathbf{v}_4 = (1 - \alpha, \frac{1}{2})$ ,  $\mathbf{v}_5 = -\mathbf{v}_1$ ,  $\mathbf{v}_6 = -\mathbf{v}_2$ ,  $\mathbf{v}_7 = -\mathbf{v}_3$  and  $\mathbf{v}_8 = -\mathbf{v}_4$ , where  $0 < \alpha < \frac{1}{4}$ , or with vertices  $\mathbf{v}_1 = (\beta, -2)$ ,  $\mathbf{v}_2 = (1 + \beta, -2)$ ,  $\mathbf{v}_3 = (1 - \beta, 0)$ ,  $\mathbf{v}_4 = (\beta, 1)$ ,  $\mathbf{v}_5 = -\mathbf{v}_1$ ,  $\mathbf{v}_6 = -\mathbf{v}_2$ ,  $\mathbf{v}_7 = -\mathbf{v}_3$ ,  $\mathbf{v}_8 = -\mathbf{v}_4$ , where  $\frac{1}{4} < \beta < \frac{1}{3}$ , or a centrally symmetric decagon with  $\mathbf{u}_1 = (0, 1)$ ,  $\mathbf{u}_2 = (1, 1)$ ,  $\mathbf{u}_3 = (\frac{3}{2}, \frac{1}{2})$ ,  $\mathbf{u}_4 = (\frac{3}{2}, 0)$ ,  $\mathbf{u}_5 = (1, -\frac{1}{2})$ ,  $\mathbf{u}_6 = -\mathbf{u}_1$ ,  $\mathbf{u}_7 = -\mathbf{u}_2$ ,  $\mathbf{u}_8 = -\mathbf{u}_3$ ,  $\mathbf{u}_9 = -\mathbf{u}_4$  and  $\mathbf{u}_{10} = -\mathbf{u}_5$  as the middle points of its edges.*



## Multiple Translative Tilings

By introducing and studying adjacent wheels, in 2017 Yang and Zong were able to generalize the results of previous section to the translative case.

Let  $X^{\mathbf{v}}$  denote the subset of  $X$  consisting of all points  $\mathbf{x}_i$  such that

$$\mathbf{v} \in \partial(P_{2m}) + \mathbf{x}_i.$$

Since  $P_{2m} + X$  is a multiple tiling, the set  $X^{\mathbf{v}}$  can be divided into disjoint subsets  $X_1^{\mathbf{v}}, X_2^{\mathbf{v}}, \dots, X_t^{\mathbf{v}}$  such that the translates in  $P_{2m} + X_j^{\mathbf{v}}$  can be re-enumerated as  $P_{2m} + \mathbf{x}_1^j, P_{2m} + \mathbf{x}_2^j, \dots, P_{2m} + \mathbf{x}_{s_j}^j$  satisfying the following conditions:

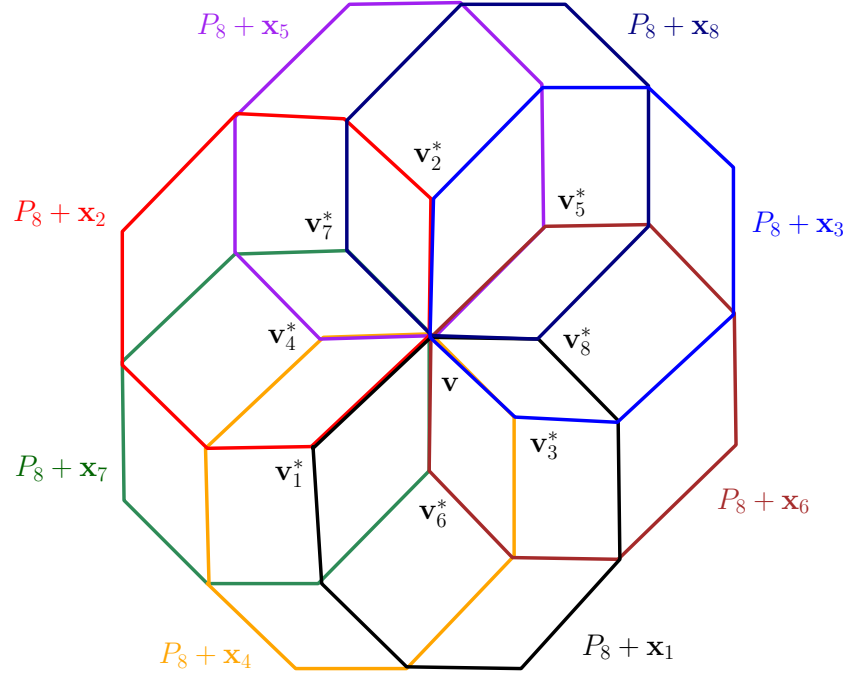
1.  $\mathbf{v} \in \partial(P_{2m}) + \mathbf{x}_i^j$  holds for all  $i = 1, 2, \dots, s_j$ .



2. Let  $\angle_i^j$  denote the inner angle of  $P_{2m} + \mathbf{x}_i^j$  at  $\mathbf{v}$  with two half-line edges  $L_{i,1}^j$  and  $L_{i,2}^j$  such that  $L_{i,1}^j$ ,  $\mathbf{x}_i^j - \mathbf{v}$  and  $L_{i,2}^j$  are in clock order. Then, the inner angles join properly as

$$L_{i,2}^j = L_{i+1,1}^j$$

holds for all  $i = 1, 2, \dots, s_j$ , where  $L_{s_j+1,1}^j = L_{1,1}^j$ .



For convenience, we call such a sequence  $P_{2m} + \mathbf{x}_1^j, P_{2m} + \mathbf{x}_2^j, \dots, P_{2m} + \mathbf{x}_{s_j}^j$  an *adjacent wheel* at  $\mathbf{v}$ . It is easy to see that

$$\sum_{i=1}^{s_j} \angle_i^j = 2w_j \cdot \pi$$

hold for positive integers  $w_j$ . Then we define

$$\phi(\mathbf{v}) = \sum_{j=1}^t w_j = \frac{1}{2\pi} \sum_{j=1}^t \sum_{i=1}^{s_j} \angle_i^j$$

and

$$\varphi(\mathbf{v}) = \#\{\mathbf{x}_i : \mathbf{x}_i \in X, \mathbf{v} \in \text{int}(P_{2m}) + \mathbf{x}_i\}.$$

Clearly, if  $P_{2m} + X$  is a  $\tau(P_{2m})$ -fold translative tiling of  $\mathbb{E}^2$ , then

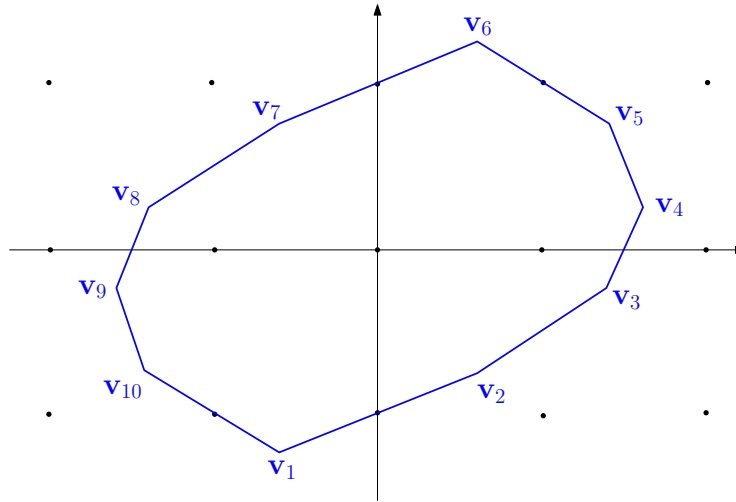
$$\tau(P_{2m}) = \varphi(\mathbf{v}) + \phi(\mathbf{v}) \tag{2}$$

holds for all  $\mathbf{v} \in V + X$ . By detailed analysis based on (2), Yang and Zong obtained the following results.

**Theorem 7 (Yang and Zong).** *If  $D$  is a two-dimensional convex domain which is neither a parallelogram nor a centrally symmetric hexagon, then we have*

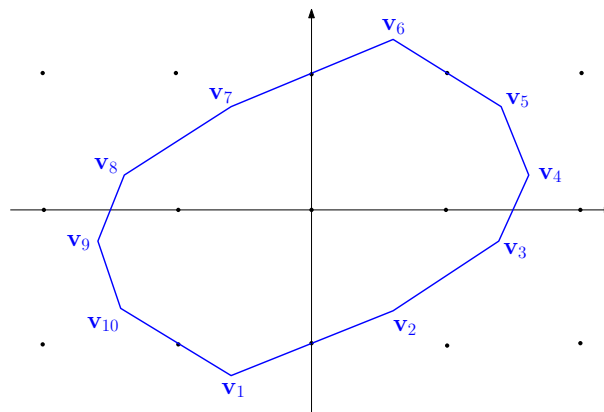
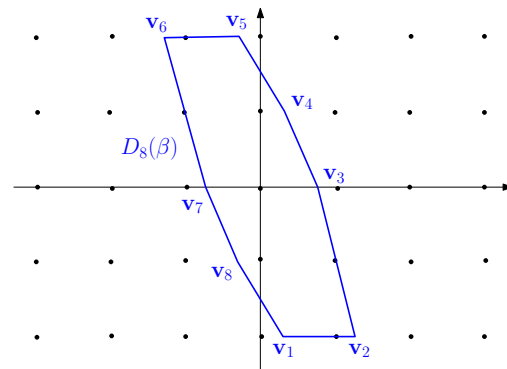
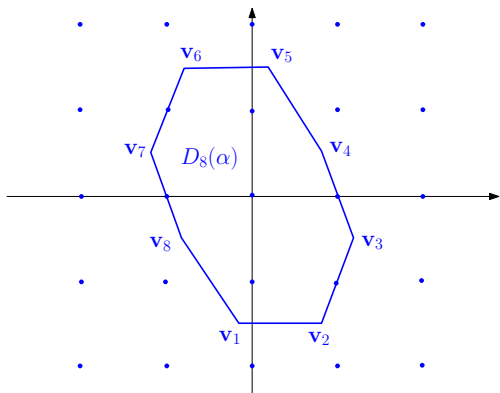
$$\tau(D) \geq 5,$$

*where the equality holds if  $D$  is some particular centrally symmetric octagon or some particular centrally symmetric decagon.*



**Theorem 8 (Yang and Zong).** *A convex domain can form a five-fold translative tiling of the Euclidean plane if and only if it is a parallelogram, a centrally symmetric hexagon, under a suitable affine linear transformation, a centrally symmetric octagon with vertices  $\mathbf{v}_1 = \left(\frac{3}{2} - \frac{5\alpha}{4}, -2\right)$ ,  $\mathbf{v}_2 = \left(-\frac{1}{2} - \frac{5\alpha}{4}, -2\right)$ ,  $\mathbf{v}_3 = \left(\frac{\alpha}{4} - \frac{3}{2}, 0\right)$ ,  $\mathbf{v}_4 = \left(\frac{\alpha}{4} - \frac{3}{2}, 1\right)$ ,  $\mathbf{v}_5 = -\mathbf{v}_1$ ,  $\mathbf{v}_6 = -\mathbf{v}_2$ ,  $\mathbf{v}_7 = -\mathbf{v}_3$  and  $\mathbf{v}_8 = -\mathbf{v}_4$ , where  $0 < \alpha < \frac{2}{3}$ , or with vertices  $\mathbf{v}_1 = (2 - \beta, -3)$ ,  $\mathbf{v}_2 = (-\beta, -3)$ ,  $\mathbf{v}_3 = (-2, -1)$ ,  $\mathbf{v}_4 = (-2, 1)$ ,  $\mathbf{v}_5 = -\mathbf{v}_1$ ,  $\mathbf{v}_6 = -\mathbf{v}_2$ ,  $\mathbf{v}_7 = -\mathbf{v}_3$  and  $\mathbf{v}_8 = -\mathbf{v}_4$ , where  $0 < \beta \leq 1$ , or a centrally symmetric decagon with  $\mathbf{u}_1 = (0, 1)$ ,  $\mathbf{u}_2 = (1, 1)$ ,  $\mathbf{u}_3 = \left(\frac{3}{2}, \frac{1}{2}\right)$ ,  $\mathbf{u}_4 = \left(\frac{3}{2}, 0\right)$ ,  $\mathbf{u}_5 = \left(1, -\frac{1}{2}\right)$ ,  $\mathbf{u}_6 = -\mathbf{u}_1$ ,  $\mathbf{u}_7 = -\mathbf{u}_2$ ,  $\mathbf{u}_8 = -\mathbf{u}_3$ ,  $\mathbf{u}_9 = -\mathbf{u}_4$  and  $\mathbf{u}_{10} = -\mathbf{u}_5$  as the middle points of its edges.*

**Remark.** In fact, it can be easily verified that every two-dimensional five-fold translative tile is a five-fold lattice tile.



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*Thank you very much!*