7. Basic Operations on Graphs

Basic Operations on Graphs

- Deletion of edges
- Deletion of vertices
- Addition of edges
- Union
- Complement
- Join

Deletion of Edges

- If G = (V,E) is a graph and e 2 E one of tis edges, then G - e := (V,E – {e}) is a subgraph of G. In such a case we say that G-e is obtained from G by **deletion of edge** e.

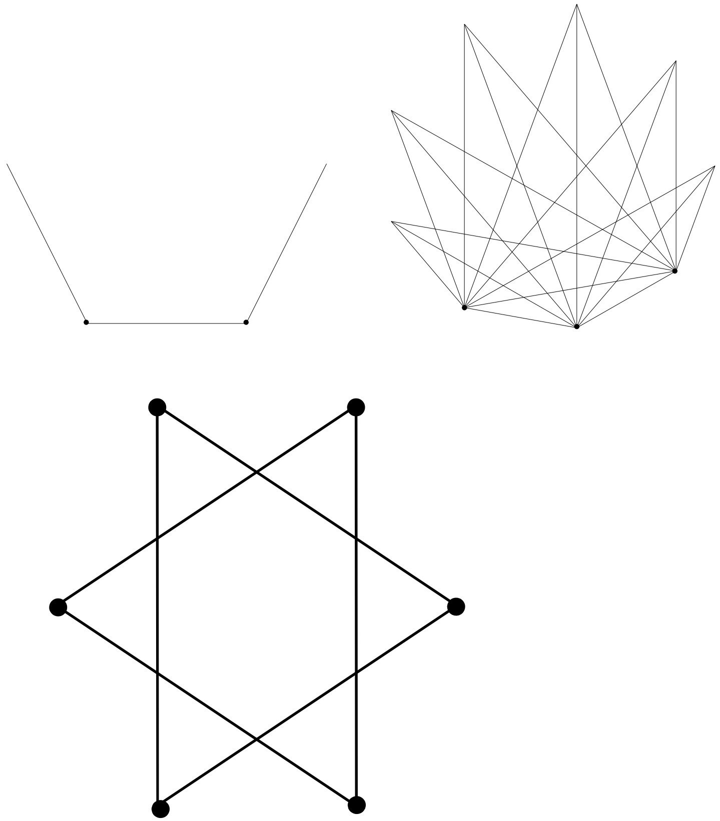
Deletion of Vertices

- Let x 2 V(G) be a vertex of graph G, then G - x is the subgraph obtained from G by removal of x grom V(G) and removal of all edges from E(G) having x as an endpoint. G – x is obtained from G by **deletion of vertex x.**

Edge Addition

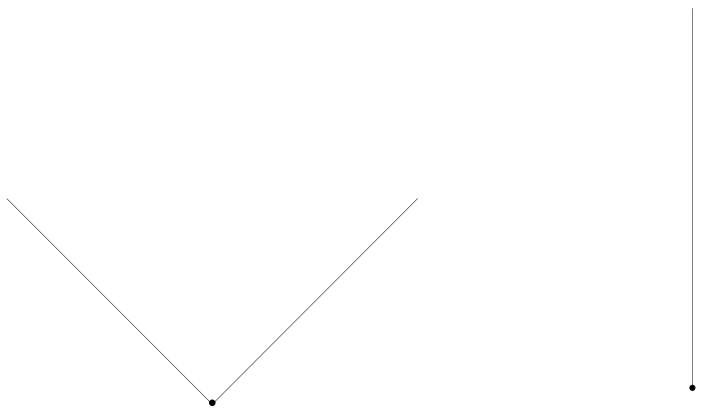
- Let G be a graph and (u,v) a pair of non-adjacent vertices. Let e = uv denot the new edge between u and v. By G’ = G + uv =   G + e we denote the graph obtained from G by addition of edge e. In other words:
    - V(G’) : = V(G),
    - E(G’) : = E(G) [{e}.

Graph Union Revisited



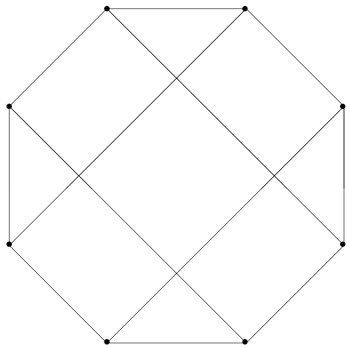
- If G and H are graphs we denote by G t H their disjoint union.
- Instead of G t G we write 2G.
- Generalization to nG, for an arbitrary positive integer n:
  - 0G := ;.
  - (n+1)G := nG t G
- Example:
  - Top row : C_6 t K_9
  - Bottom row: $2K_3$.

Graph Complement

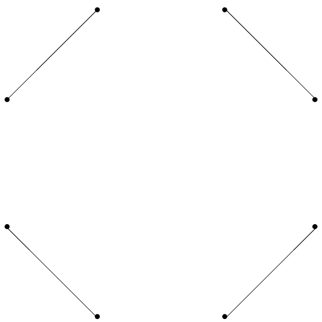


- The **graph complement** G^c of a simple graph G has $V(G^c) := V(G)$, but two vertices u and v are adjacent in G^c if and only if they are not adjacent in G.
- For instance C_4^c is isomorphic to $2K_2$.

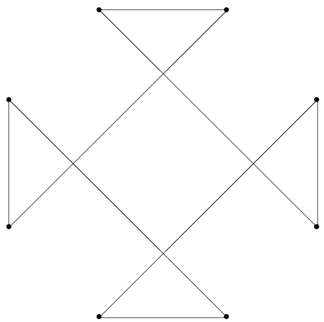
Graph Difference



G



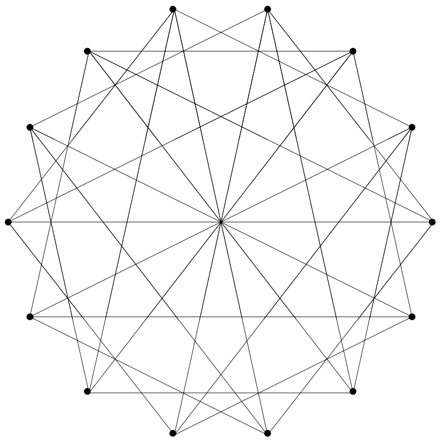
H



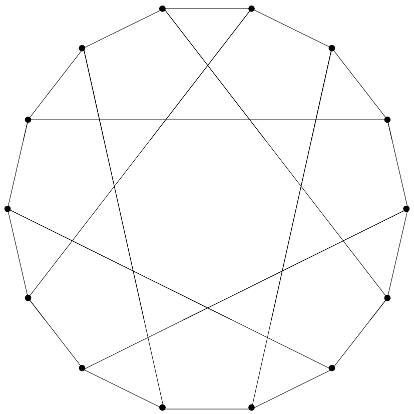
G\H

- If H is a spanning subgraph of G we may define graph difference G \H as follows:
- V(G\H) := V(G).
- E(G\H) := E(G)\E(H).

Bipartite Complement



X



X^b

- For a bipartite graph X (with a given biparitition) one can define a **bipartite complement** X^b**.** This is the graph difference of $K_{m,n}$ and X: $X^b = K_{m,n} \setminus X$.

Empty Graph Revisited.

- The word “empty graph” is used in two meanings.
- First Meaning: ;. No vertices, no edges.
- Second Meaning: $E_n := K_n^c .= nK_1$. There are n vertices, no edges.
- E_0 = ; = 0. G will be called the **void graph** or **zero graph**.

Graph Join

- **Join** of graphs G and H is denoted by G*H and defined as follows:
  - $G*H := (G^c$ t $H^c)^c$
- In particular, this means that $K_{m,n}$ is a join of two empty graphs E_n and E_m.

Exercises 7

- **N1.** Show that for any set F µ E(G) the graph G-F is well-defined.
- **N2.** Show that for any set X µ V(G) the graph G-X is well-defined.
- **N3.** Show that for any set X µ V(G) and any set F µ E(G) the graph G-X-F is well-defined.
- **N4.** Prove that H is a subgraph of G if and only if H is obtained from G by a succession of vertex and edge deletion.

8. Advanced Operations on Graphs

Cone and Suspension

- The join of G and K_1 is called the **cone** over G and is denoted by Cone(G) = $G*K_1$.
- The join $G*(2K_1)$ is called **suspension**.

Examples

- Any complete multipartite graph is a join of empty graphs.
- The cone Cone(C_n) is called a **pyramid** or **wheel** W_n.
- The **octahedral graph** is the suspension over C_4. It can be written in the form:
  - $O_3 = (2K_1)*(2K_1)*(2K_1)$.
- Construction can be generalized to:
  - $O_n = (2K_1)*(2K_1)* \dots *(2K_1)$

Subdivision

- Let e 2 E(G) be an edge of G. Let S(G,e) denote the graph obtained from G by replacing the edge e by a path of length 2 passing through a new vertex. Such an operation is called **subdivision** of the edge e..
- Let F be a subset of E(G), then S(G,F) denotes the graph obtained from the subdivision of each edge of F. In the case F = E, we drop the second argument and S(G) denotes the **subdivision graph** of G.
- Graph H is a **general subdivision** of graph G, if H is obtained from G by a sequence of edge subdivisions.

Graph Homeomorphism

- Graphs G and H are **homeomorphic**, if they have a common subdivision.
- Graph G is **topologically contained** in a graph K, if there exists a subgraph H of K, that is homeomorphic to G.

Matching

- Edges with no common endvertex are called **independent.** A set of pairwise independent edges is called a **matching.**

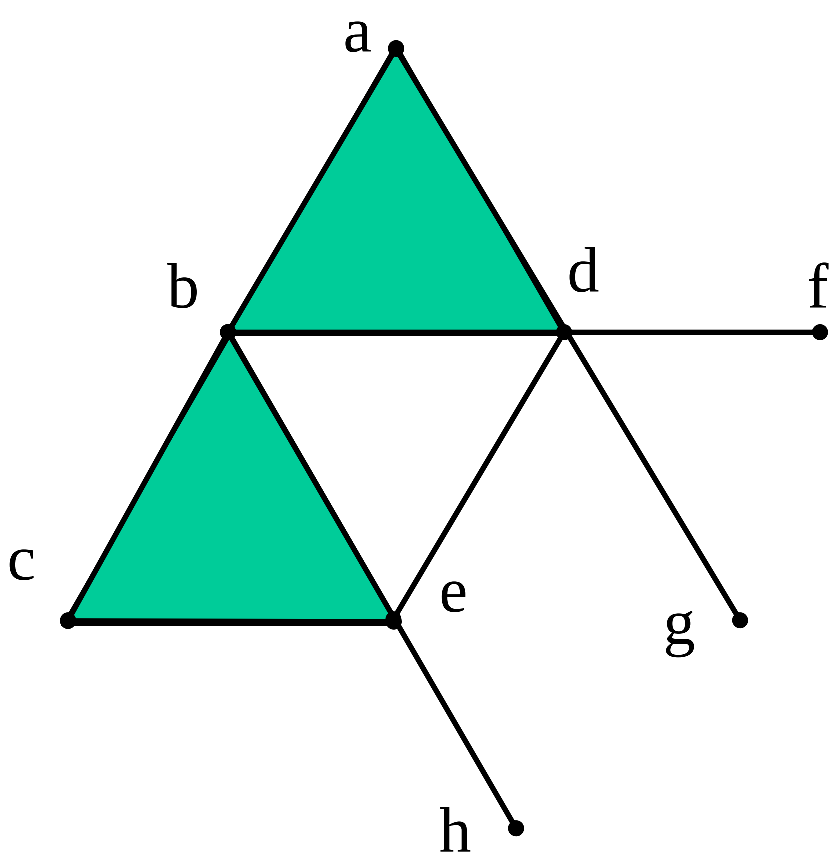
Maximal Matching

- A matching that cannot be augmented by adding new edges is called a **maximal matching.**

Perfect Matching

- **Proposition:** Let M be a matching of a graph G on n vertices. Then $|M| \cdot n/2$.
- A matching M with $|M| = n/2$ is called a **perfect matching.**

Abstract Simplicial Complex

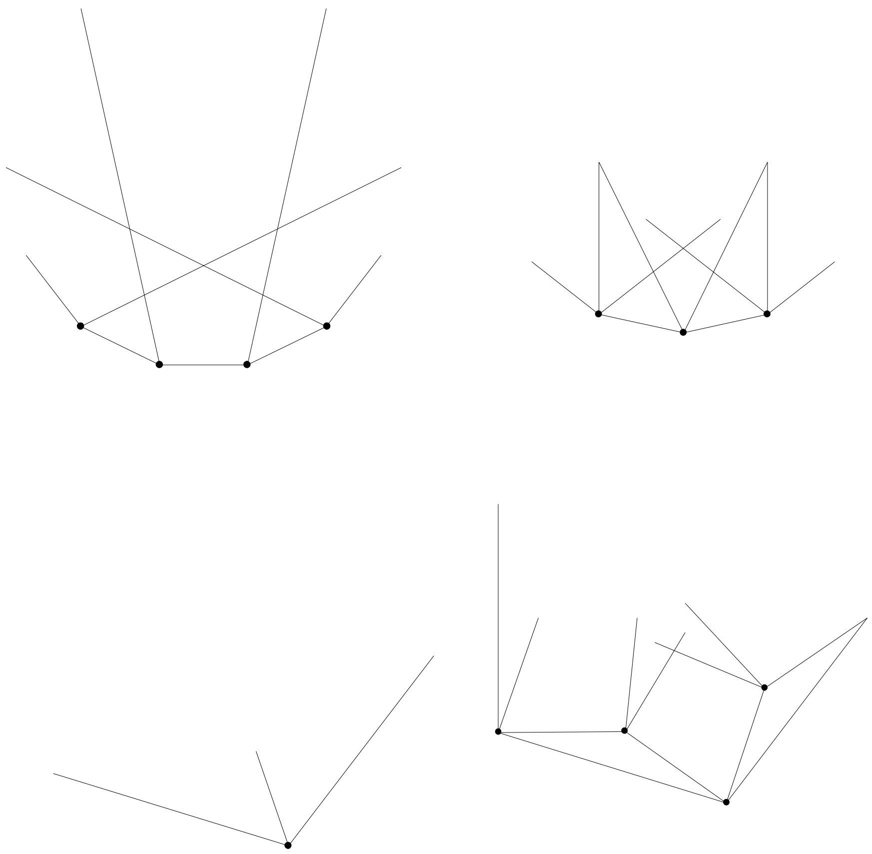


- K µ P(S) is an abstract simplicial complex if for each σ 2 K and each τ µ σ it follows that τ 2 K.
- On the left:
- K = {;, a, b, c, d, e, f, g, h, ab, ad, abd, bc, be, bce, bd, ce, df, dg, de, eh}

Line Graph L(G)

- Two edges with a common end-vertex are **incident**. Incidence is a binary relation on the edge set E(G).
- Line graph L(G) has the vertex set E(G), while the edges of L(G) are determined by the incidence of edges in G.

Examples



- The top row depicts the Heawood graph and its fourvalent linegraph.
- The bottom row depicts the Petersen graph and its line graph.