# 1RSB in the ‘small-world’ spin glass

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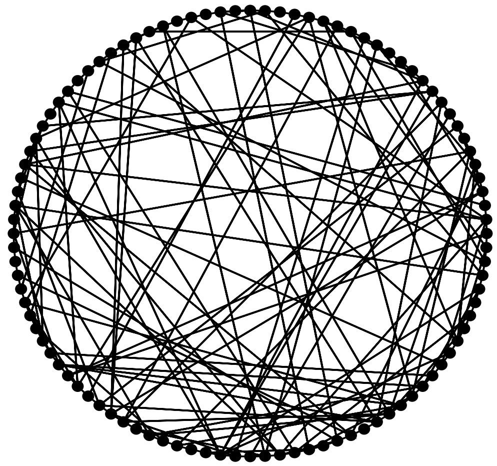
# Introduction and motivation



*In what sense small world?*
Ising spins interacting through

- Ferromagnetic short range interactions $J_0$ on a ring
- Frustrated long range interactions $J_l$ as “shortcuts”.

*Why bother?*

1 Spin glass study with finite connectivity some kind of geometry (“maximal tractable model”)

2 RKKY interations (short range ferro, long range oscillatory)

3 Technical motivations:
   - Extending previous RS work which used replicated transfer matrices
   - The cavity method gives a more intuitive interpretation of replica results
   - RSB exists at least in special cases of the model
   - Conjecture in RS paper on non-reentrance in the phase diagram
   - Possible consequences for message passing algorithms on small world type networks?

# Definitions

- Hamiltonian:

$$H = -J_0 \sum_{i=1}^{N} \sigma_i \sigma_{i+1} - \frac{1}{\langle k \rangle} \sum_{i,j=0}^{N} J_{ij} c_{ij} \sigma_i \sigma_j$$

where $\sigma_i \in \{-1, 1\}$ and $\sigma_{N+1} \equiv \sigma_1$.

- Random variables:

$$p(J_{ij}) \to J_{ij}$$
$$p(k_i) \to c_{ij} \in \{0, 1\}$$

through the constraint $\sum_{j=1}^{N} c_{ij} = k_i$.
Furthermore, $c_{ii} = 0$, $\langle k \rangle = \sum_k p(k) k$

- Goal: obtaining phase diagrams through calculation of observables
$m = \frac{1}{N} \sum_i \overline{\langle \sigma_i \rangle}$ and $q = \frac{1}{N} \sum_i \overline{\langle \sigma_i \rangle \langle \sigma_i \rangle}$

# Dealing with the randomness

Objective:

$$f = \lim_{N\to\infty} -\frac{1}{\beta N}\langle \log Z\rangle_{J,k}$$

where

$$Z = \sum_{\boldsymbol{\sigma}} e^{-\beta H(\boldsymbol{\sigma},\{J_{ij}\},\{k_i\})}$$

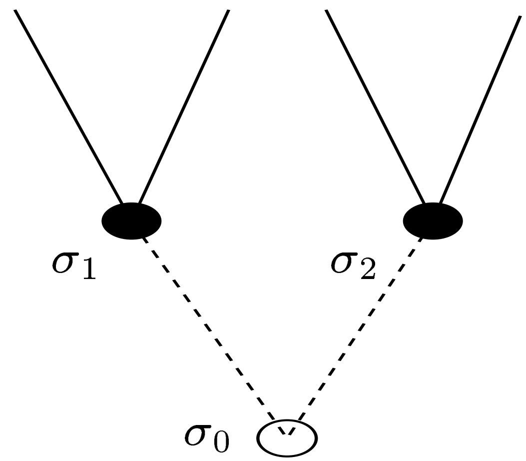
1 *Replica theory:*

$$f = \lim_{N\to\infty}\lim_{n\to 0}\frac{1}{\beta N}\log\langle Z^n\rangle_{J,k}$$

Solution (RS) using replicated transfer matrices
see [J. Phys. A. Math. Gen: **37** (2004) 6455-6475].

2 *Cavity method:* Local iteration equations with random disorder.

# The Cavity method on a random graph




- Basic (RS) assumption: All $k$ neighbours of spin $\sigma_0$ are only statistically dependent through spin $\sigma_0$ (no loop effects).
- In absence of spin $\sigma_0$, state of neighbouring spin $\sigma_j$ (then having $k-1$ neighbours) is characterized by

$$p(\sigma_j) \sim e^{\beta h_j \sigma_j}$$

- Upon linking spin $\sigma_0$ with $k-1$ neighbours, one may define

$$Z(\sigma_0) = \sum_{\sigma_1,\cdots,\sigma_{k-1}} \exp\left\{\beta\left[\sigma_0 \sum_{l=1}^{k-1} J_l\sigma_l + \sum_{l=1}^{k-1} h_l\sigma_l\right]\right\}$$

$$= \frac{\exp\left\{\beta\sigma_0 \sum_{l=1}^{k-1} u(J_l, h_l)\right\}}{c(\{J_l\}, \{h_l\})}$$

with

$$u(J_l, h_l) = \frac{1}{\beta}\tanh^{-1}[\tanh(\beta J_l)\tanh(\beta h_l)]$$

- It follows that

$$h_0 = \sum_{l=1}^{k-1} u(J_l, h_l)$$

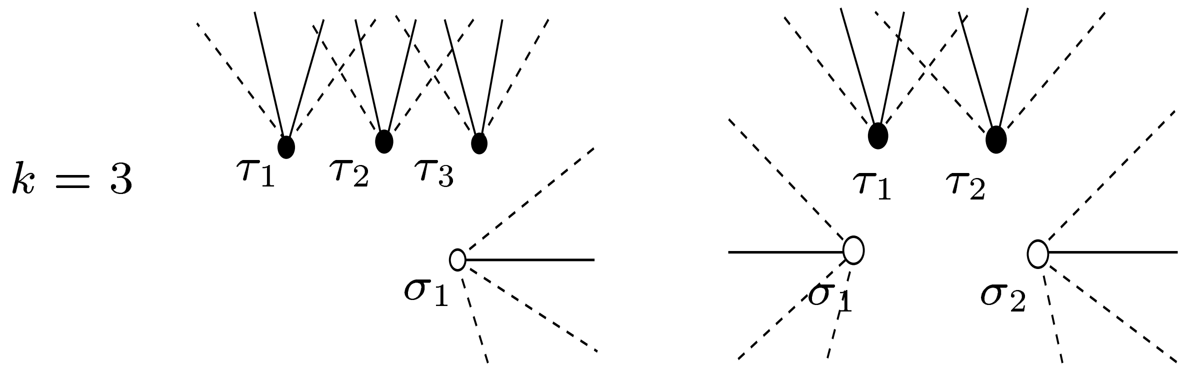
- The iterative equation of cavity field distributions for fixed connectivity $k$

$$W(h) = \int \prod_{l=1}^{k-1} [dJ_l p(J_l) dh_l W(h_l)]\delta\left[h - \sum_{l=1}^{k-1} u(J_l, h_l)\right]$$

- Interpretation of $h_0$ and $u_l$: they parametrize messages $\mu_{0\to(0j)}(\sigma_0) \sim e^{\beta h_0 \sigma_0}$, $\mu_{(l0)\to 0}(\sigma_0) \sim e^{\beta u_l \sigma_0}$. Consequently, $W(h)$ parametrizes the distribution of messages over the whole graph, i.e., describing statistics of belief propagation over random instances of graphs.
- $W(h)$ is the main order parameter of the RS theory, from which all macroscopic observables follow.

# Cavity method for the 'small-world' lattice

- Two types of cavity spins, either missing a long-range ($\tau$-spin), or a short-range bond ($\sigma$-spin).
- Two types of cavity fields, $h$ (coupling to $\tau$-spin) and $x$ (coupling to $\sigma$-spin).




- Two types of iterations: Adding to the graph $\sigma_0$ (left) or $\tau_0$ (right).

$$x_0 = u(J_0, x_1) + \sum_{l=1}^{k} u(J_l, h_l)$$
$$h_0 = u(J_0, x_1) + u(J_0, x_2) + \sum_{l=1}^{k-1} u(J_l, h_l)$$

- Coupled distribution iterations

$$\begin{aligned}
\Phi(x) &= \int dx' \Phi(x') \prod_{l=1}^{k} [dJ_l p(J_l) dh_l W(h_l)] \\
&\quad \times \delta[x - u(J_0, x') - \sum_{l=1}^{k} u(J_l, h_l)] \\
W(h) &= \int dx dx' \Phi(x) \Phi(x') \prod_{l=1}^{k-1} [dJ_l p(J_l) dh_l W(h_l)] \\
&\quad \times \delta[h - u(J_0, x) - u(J_0, x') - \sum_{l=1}^{k-1} u(J_l, h_l)]
\end{aligned}$$

- ‘Real effective field’ distribution:

$$R(H) = \int dx dx' \Phi(x)\Phi(x') \prod_{l=1}^{k} [dJ_l p(J_l) dh_l W(h_l)]$$

$$\times \delta[H - u(J_0, x) - u(J_0, x') - \sum_{l=1}^{k} u(J_l, h_l)]$$

- Observables:

$$m = \int dH R(H) \tanh(\beta H)$$
$$q = \int dH R(H) \tanh^2(\beta H)$$

- Other observables are found by defining appropriate graph operations from which they are derived.
- Resulting equations equivalent to replicated transfer matrix results

# Bifurcation conditions and RS phase diagrams

- Bifurcation conditions for the first moments of $W(h)$ determine second order phase transitions from paramagnetic phase:

  Paramagnetic ($m = 0, q = 0$) to Ferromagnetic ($m \neq 0, q \neq 0$):

$$\left[ \langle k \rangle (e^{2\beta J_0} - 1) + \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} \right] \left\langle \tanh \left( \frac{\beta J}{\langle k \rangle} \right) \right\rangle_J = 1$$

  Paramagnetic to Spin Glass ($m = 0, q \neq 0$):

$$\left[ 2 \langle k \rangle \sinh^2(\beta J_0) + \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} \right] \left\langle \tanh^2 \left( \frac{\beta J}{\langle k \rangle} \right) \right\rangle_J = 1$$

# One step RSB

- RS assumption: cavity fields characterize global minimum of free energy before *and* after graph iteration with corresponding free energy shift $\Delta F$.
- 1RSB (Mezard & Parisi) Several local minima are considered characterized by $h^\alpha$ ($\alpha$ labels pure state). Corresponding free energy shifts $\Delta F^\alpha$ reshuffle order in $F^\alpha$.
- Different pure states $\alpha \in \{1, \ldots, \mathcal{M}\}$

$$W^\alpha = \frac{\exp(-\beta F^\alpha)}{\sum_\gamma \exp(-\beta F^\gamma)}$$

- Basic self-consistent ansatz

$$\rho(F) = \exp(\beta\mu(F - F^{ref}))$$

- Order parameter functions site-dependent and factorize over pure states at each site

$$P(\mathbf{h}) = \frac{1}{N} \sum_i \prod_{\alpha=1}^{\mathcal{M}} P_i(h^\alpha)$$

$$Q(\boldsymbol{x}) = \frac{1}{N} \sum_i \prod_{\alpha=1}^{\mathcal{M}} Q_i(x^\alpha)$$

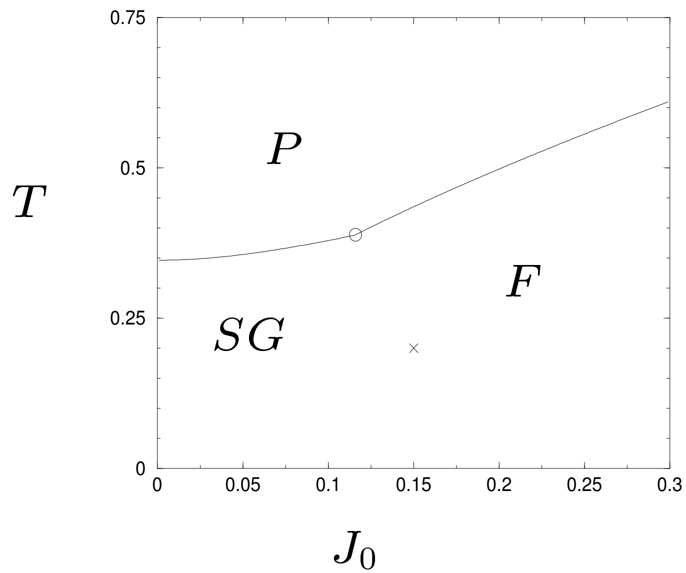
- Advanced population dynamics algorithm, involving $\mathcal{N}$ populations of $\mathcal{M}$ fields.
- All observables are evaluated at value of $\mu$ for which

$$\frac{\partial f}{\partial \mu} = 0$$

Equations are recovered exactly within replica theory using a one step RSB ansatz a la Monasson.

# Numerical results

$$p(k) = \delta_{k,6},$$
$$p(J) = \frac{5}{8}\delta(J-1) + \frac{3}{8}(J+1)$$



RS:

$$f = -0.3561$$

$$q = 0.5789$$

1RSB:

$\mathcal{N} = 2000$ and $\mathcal{M} = 1000$

$$\mu = 0.32 \pm 0.01.$$

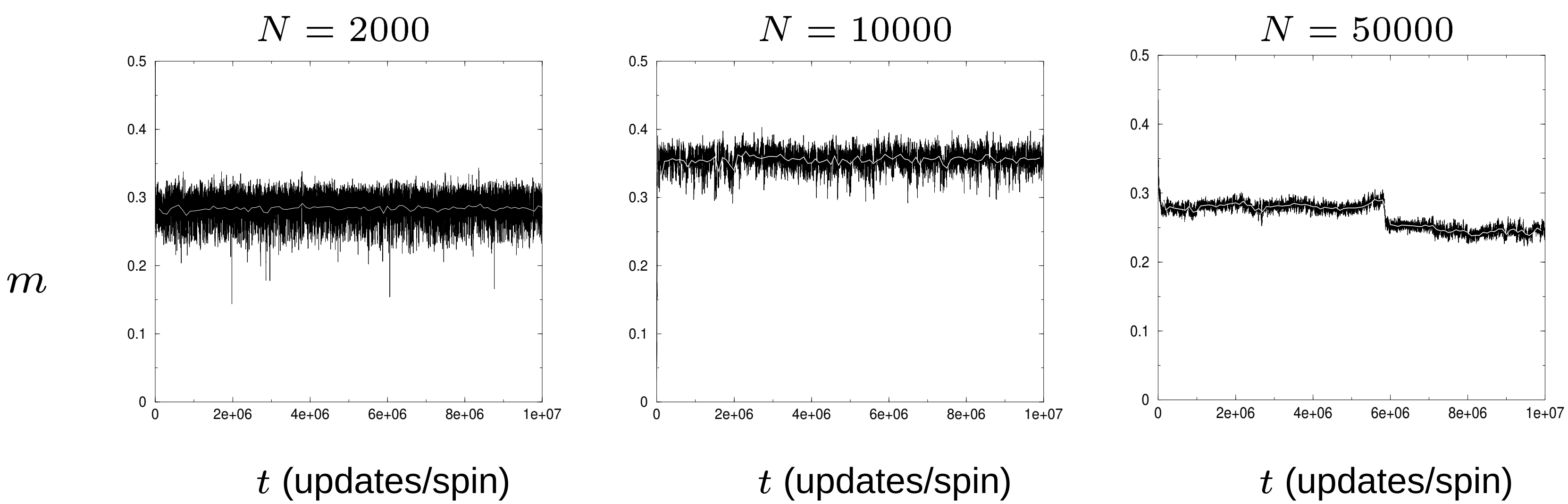
$$f = -0.3557 \pm 0.0001$$

$$q_0 = 0.397 \pm 0.003$$

$$q_1 = 0.673 \pm 0.003$$

$$m = 0.2 \pm 0.05$$

# Simulations




Unfortunately: large finite size effects, long equilibration times.. Bad statistics. However: nonzero magnetization.

# Conclusions and outlook

- Small world type graph can be studied with the cavity method, giving the same results as the replica method
- RSB occurs in various regions of the phase diagram, and can be detected within this framework.
- Extension to next-nearest neighbour interactions possible, though numerically expensive