Generalized Belief Propagation Receiver for Near-Optimal Detection of Two-Dimensional Channels with Memory

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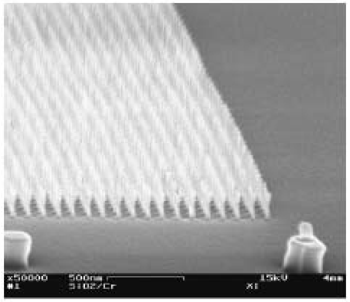
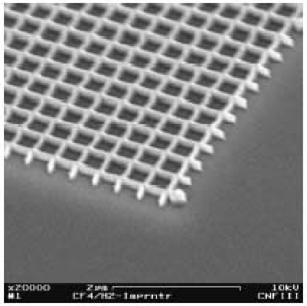
Examples of 2-D channels

- 2-D inter-symbol interference (ISI) channels
  - Magnetic and optical recording





Science enables:

6 million-million bits/square inch!

from: J.A. O'Sullivan, N. Singla, Y. Wu, R.S. Indeck

Examples of 2-D channels

- Multiple-access (MA) channels – Wyner's model
  - Cellular network's uplink.
  - Indoor Wireless LAN.

Outline

- Examples of 2-D channels

- System model
- Optimal detection

- 2-D channels as undirected graphical models
- Detection = Inference
  - Exact inference
  - Approximate inference
    - Belief propagation (BP)
    - Generalized belief propagation (GBP)

- Experimental results

Outline (cont.)

- Cluster variation method (CVM) for estimation of the information rate of 2D channels.
- Experimental results
- Why is GBP-based CVM correct in this case?

System model

Binary symbols

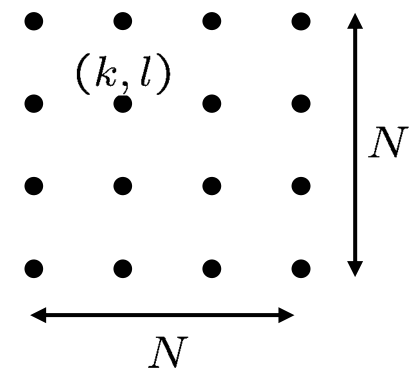
Interference intensity

$$y_{k,l} = d_{k,l} + v_{k,l} + \sum_{(i,j)\in<k,l>} \alpha_{i,j} d_{i,j} \quad \forall k,l = 1,\ldots,N$$

Observed samples

AWGN $\mathcal{N}(0,\sigma^2)$

Short range 2-D interference topology



In vector-matrix form

$$\mathbf{y}_{N^2\times 1} = \mathbf{S}_{N^2\times N^2}\mathbf{d}_{N^2\times 1} + \mathbf{v}_{N^2\times 1}$$

Interference band matrix

- 2-D channels differ in $\mathbf{S}$

System model (cont.)

- Objective
  - Recover $\mathbf{d}$ from $\mathbf{y}$, or
  - Compute $\Pr(\mathbf{x}|\mathbf{y})$, where $\mathbf{x}$ are possible values of $\mathbf{d}$.
  - Assume that d are i.i.d and equiprobable.

$$\Pr(\mathbf{x} \mid \mathbf{y}) \propto \exp(-\frac{1}{2\sigma^2}\|\mathbf{y} - \mathbf{S}\mathbf{x}\|^2)$$

- Omitting non-sufficient statistics terms

$$\Pr(\mathrm{x}|\mathrm{y}) \propto \exp\left(-\frac{1}{\sigma^2}\left(\sum_{(i>j)} R_{ij}x_ix_j - \sum_i h_ix_i\right)\right)$$

where

- $\mathrm{R} = \mathrm{S}^T\mathrm{S}$ interference cross-correlation matrix
- $\mathrm{h} = \mathrm{y}^T\mathrm{S}$ output vector of a filter matched to the interference structure

Optimal detection

- MAP decision

$$\hat{d}_i = \arg\max_{x_i} \Pr(x_i|\mathbf{y}) = \arg\max_{x_i} \sum_{\mathbf{x}\backslash x_i} \Pr(\mathbf{x}|\mathbf{y})$$

Problem - intractable

2-D channels as undirected graphical models

- p($\mathbf{x}|\mathbf{y}$) defines an undirected graphical model

$$\Pr(\mathbf{x}|\mathbf{y}) \propto \prod_{(i>j)} \psi_{ij}(x_i, x_j) \prod_i \phi_i(x_i, h_i)$$

where the potentials

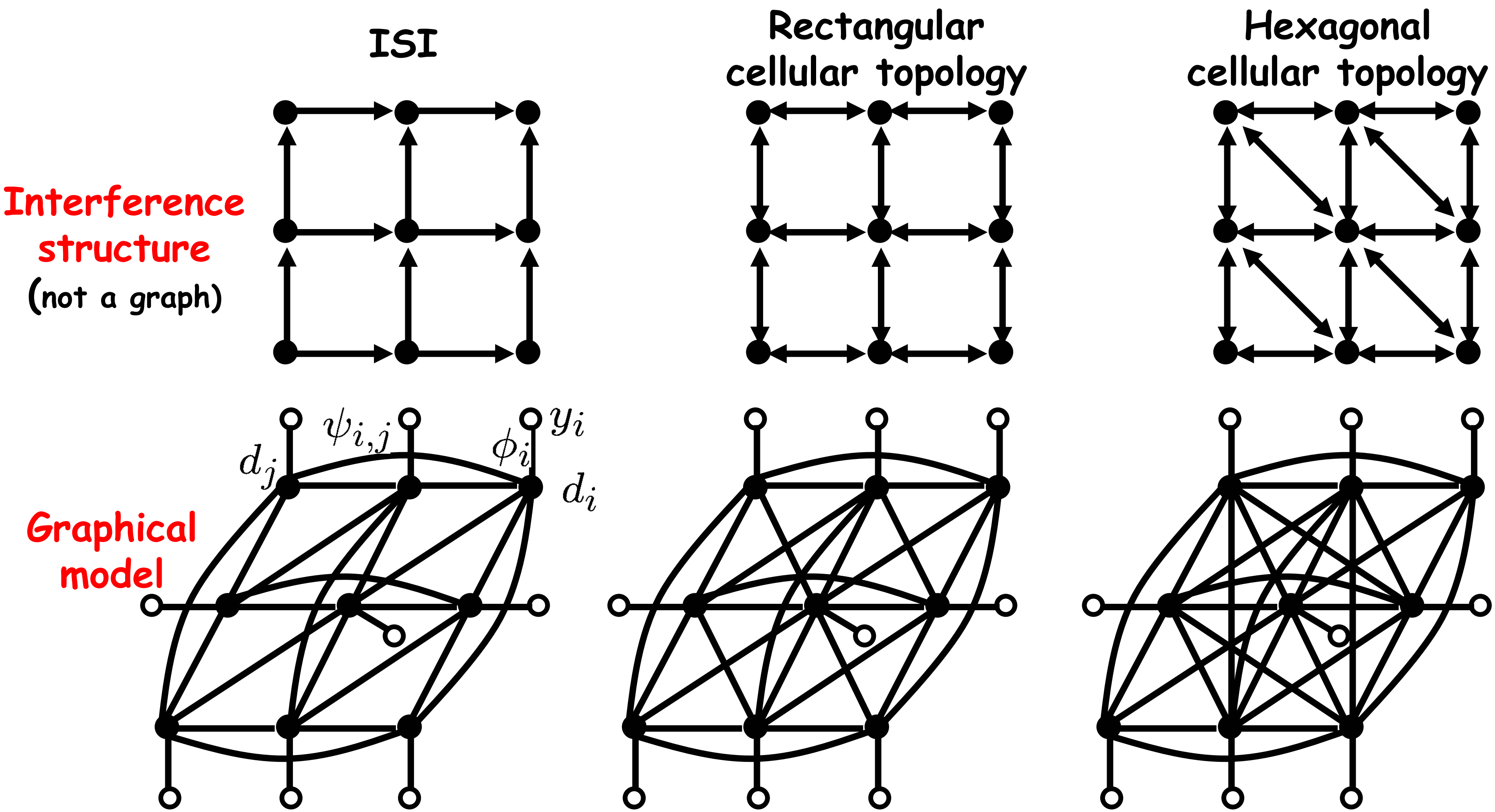
- Compatibility function

$$\psi_{ij}(x_i, x_j) = \exp\left(-\frac{R_{ij} x_i x_j}{\sigma^2}\right)$$

- Evidence, or local likelihood

$$\phi_i(x_i, y_i) = \exp\left(\frac{h_i x_i}{\sigma^2}\right)$$

Examples of 2-D channel representations



Exact inference – junction tree

- Junction tree complexity
  - Exponential in the size of the largest clique
    - For NxN grid-like graphs: N x memory depth ν

- Conclusion
  - We must resort to approximate inference methods

Approximate inference: belief propagation

- Belief propagation often yields good approximations when the cycles in the graph are long
- However – 2-D channels contain many short cycles
- Empirical results show
  - Belief propagation fails to converge
  - When it converges its approximation is poor

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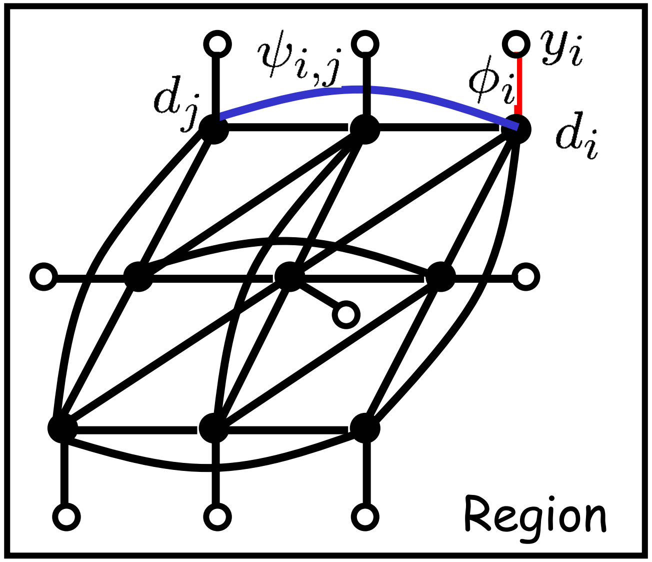
Generalized belief propagation (GBP)

[J. S. Yedidia, W. T. Freeman and Y. Weiss]

- BP <-> Bethe approximation to the free energy
- GBP <-> More complicated free energy approximations
  - Several ways for creating free energy approximations
    - We use the Kikuchi approximation, i.e., the cluster variation method
- GBP is a global name for a family of message passing algorithms
  - Several ways for passing messages
    - We use the 'two-way algorithm'.

Generalized belief propagation (GBP)

- Rational - choose the basic region to encompass the shortest loops
- We have nearest neighbor and next-nearest neighbor interactions, thus a 3x3 nodes basic cluster is a natural choice
- E.g. ISI



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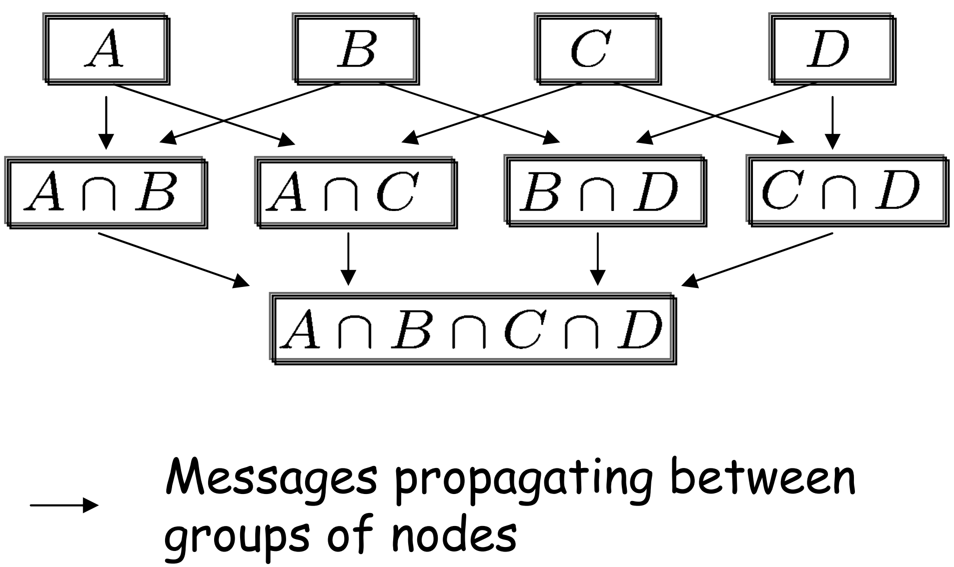
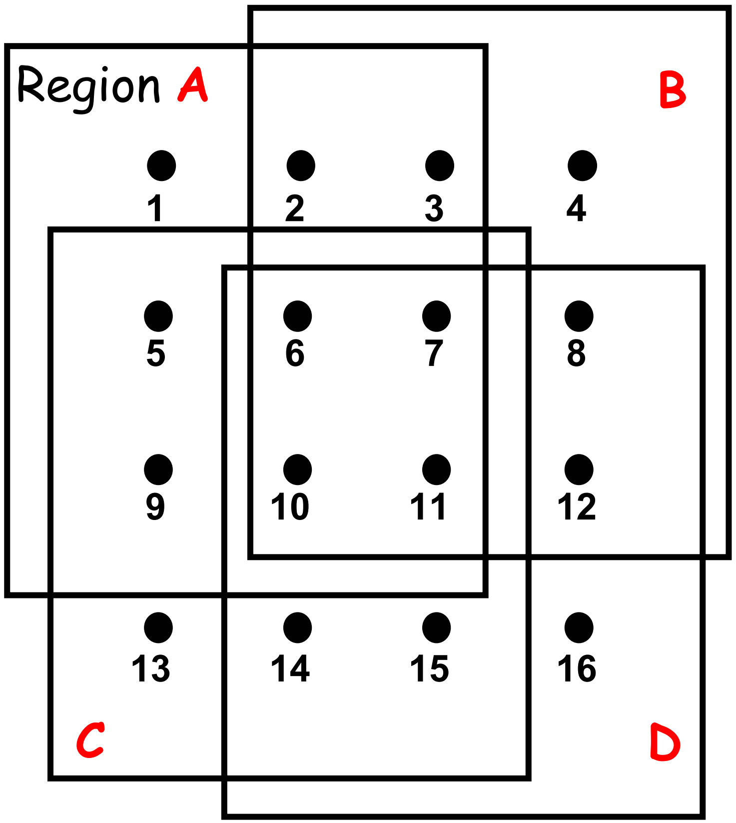
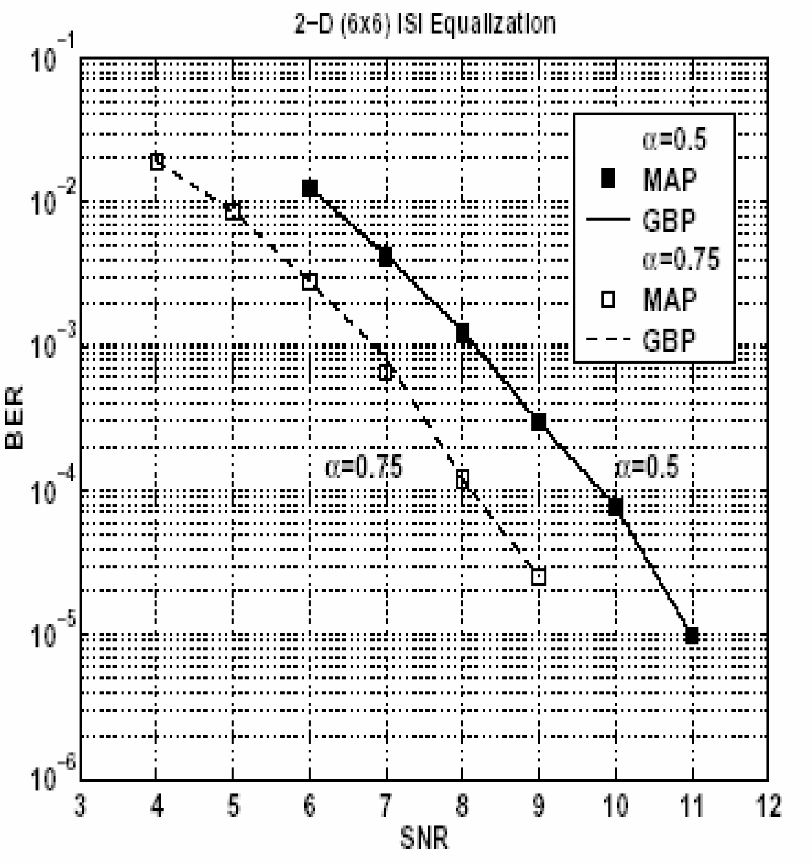
Generalized belief propagation (GBP)

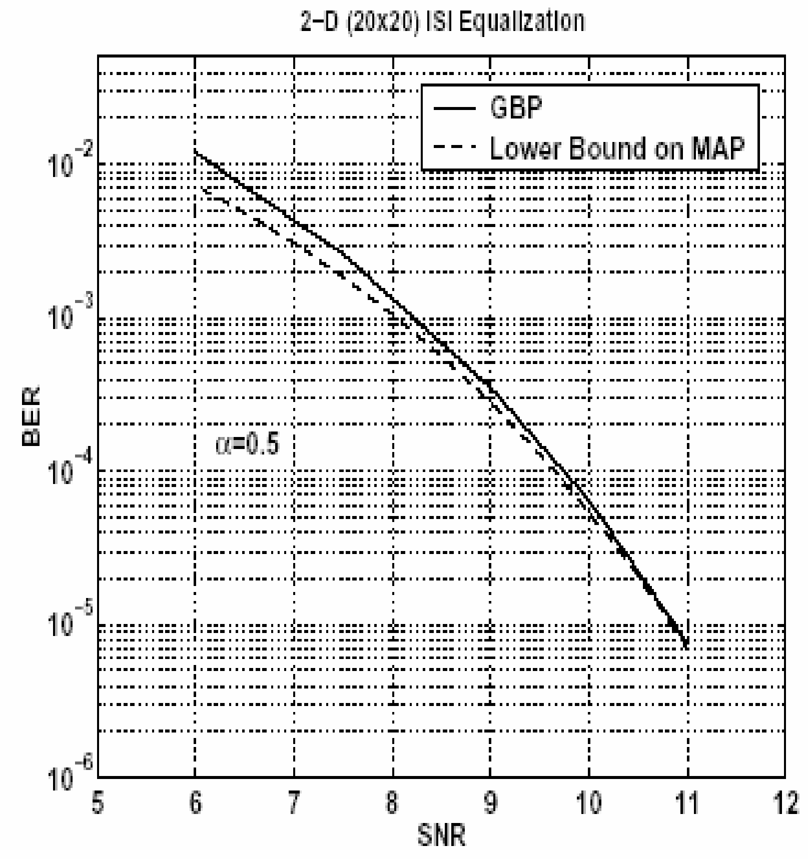
- In GBP messages are propagating between groups (regions) of nodes
- Exact inference is performed within each group





Results: ISI equalization





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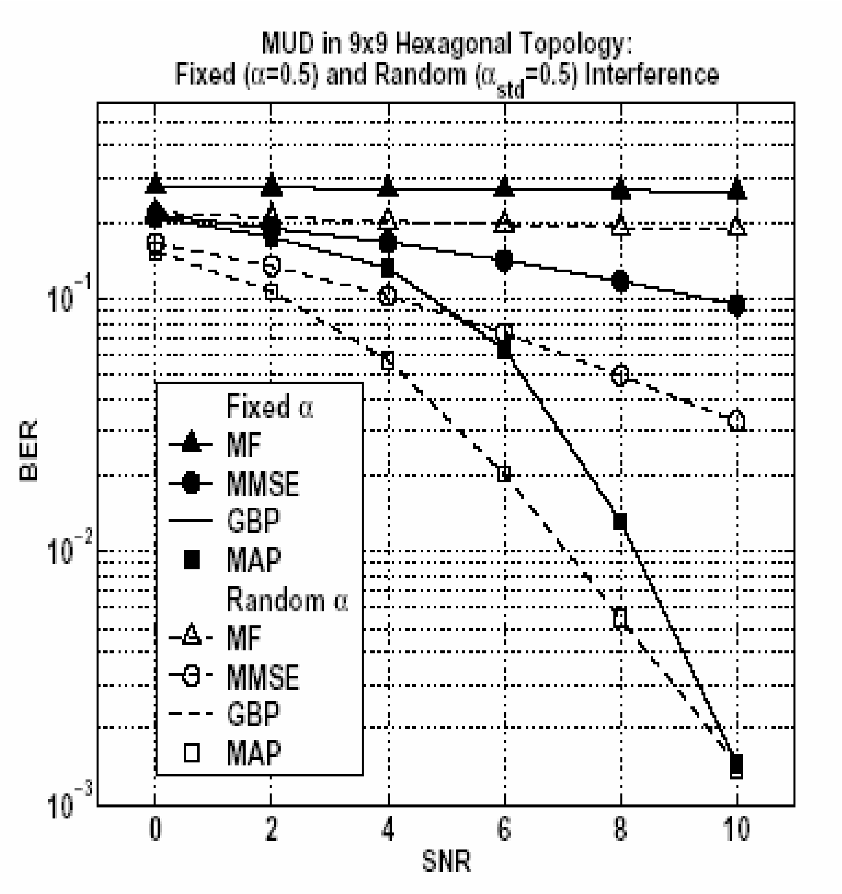
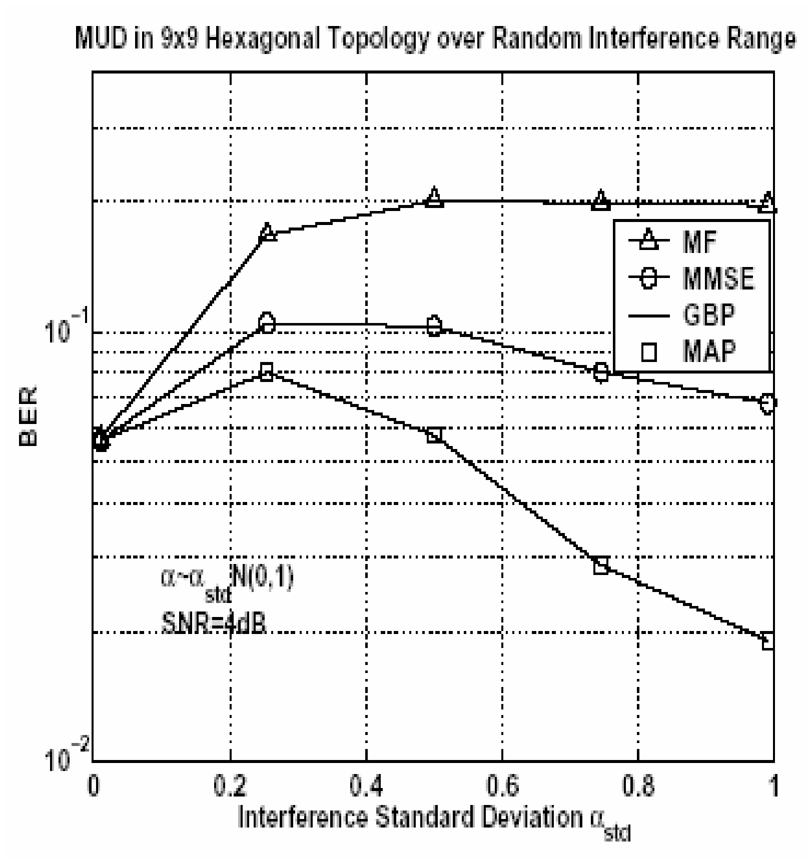
Results:
hexagonal topology cellular network





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Conclusions - ITW

- Practically optimal error performance using a fully tractable message passing scheme
  - Consistent both over SNR and interference range
  - The marginal beliefs well approximate the a-posteriori probabilities (APP)
  - superior to other sub-optimal receivers
- A real-life application in which GBP>>BP

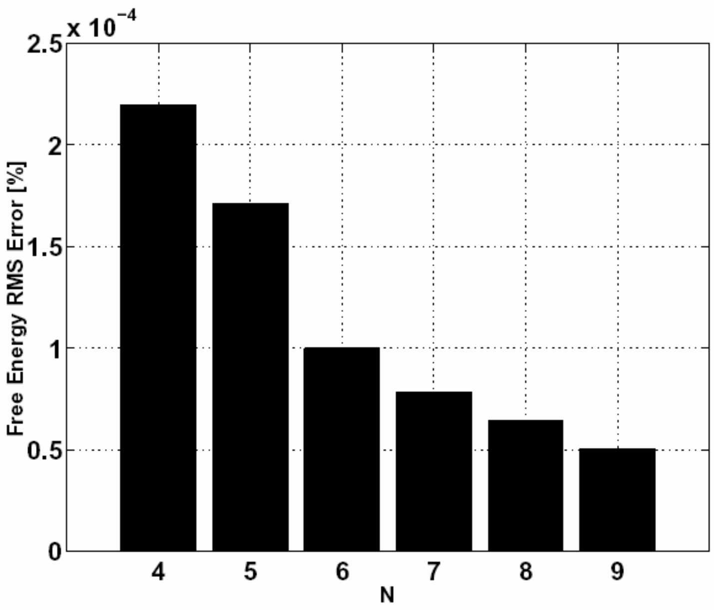
Current work

- Joint work with Shlomo Shamai (Shitz), Technion.
- Cluster variation method (CVM) for the estimation of the information rate of 2D channels.
- Experimental results.
- Why is GBP-based CVM correct in this case?

Approximate free energy

NxN system

Estimate the exact and approximate free energies per site



Hexagonal cellular network, SNR=0dB, alpha=0.5

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The connection between the free energy and the information rate

Assume a system of size $N^2=k$

$$x^k \equiv x_1,\dots,x_k \qquad y^k \equiv y_1,\dots,y_k$$

are stationary random processes.

Information rate:

$$I(x;y) = h(y) - h(y\,|\,x)$$

$$h(y) \equiv \lim_{k\to\infty} h(y^k)/k \qquad h(y\,|\,x) \equiv \lim_{k\to\infty} h(y^k\,|\,x^k)/k$$

are differential entropy rates

As far as we know, only bounds exist for I(x;y) in the 2D case

The connection between the free energy and the information rate (cont.)

h(y|x): $$h(y \mid x) \square \lim_{k\to\infty} h(y^k \mid x^k)/k = \lim_{k\to\infty} h(v^k)/k$$

$$= \frac{1}{2}(1+\log 2\pi\sigma^2)$$

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h(y): Assuming a stationary ergodic process, using the Shannon-McMillan-Breiman theorem

$$-\frac{1}{k}\log(p(y^k)) \to h(y)$$ with probability one.

The connection between free energy and symmetric information rate

Estimating $p(y^k)$:

Assume $x_1,\ldots,x_k$ are i.i.d. and equiprobable.

$$p(y^k) = \frac{1}{2^k}\sum p(y^k \mid x^k) = \frac{1}{2^k}(2\pi\sigma^2)^{-k/2} \underbrace{\sum \exp\left(-\frac{1}{2\sigma^2}\left\|y^k - Sx^k\right\|^2\right)}_{Z}$$

$$-\frac{1}{k}\log(p(y^k)) = \log(2) + \log\sqrt{2\pi\sigma^2} + F/k$$

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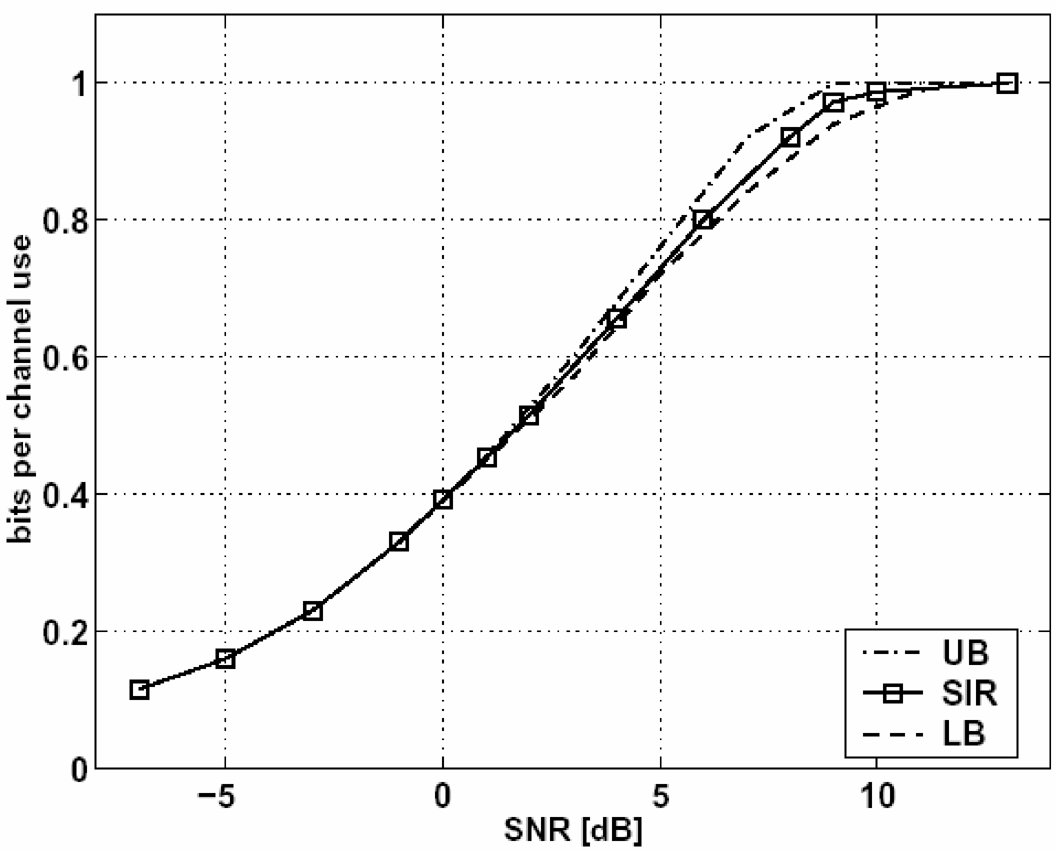
The connection between free energy and the symmetric information rate (cont.)

Thus,

$$\begin{aligned} I(x;y) &= h(y) - h(y \mid x) \\ &= (\log(2) + \log\sqrt{2\pi\sigma^2} + F/k) - (1/2 + \log\sqrt{2\pi\sigma^2}) \\ &= \log 2 - 1/2 + F/k \end{aligned}$$

Idea: Use the CVM of a large system to approximate the free energy F.

Experimental results ISI



Upper and lower bounds – Chen and Siegel, "On the symmetric information rate of two-dimensional finite state ISI channels", ITW 2003.

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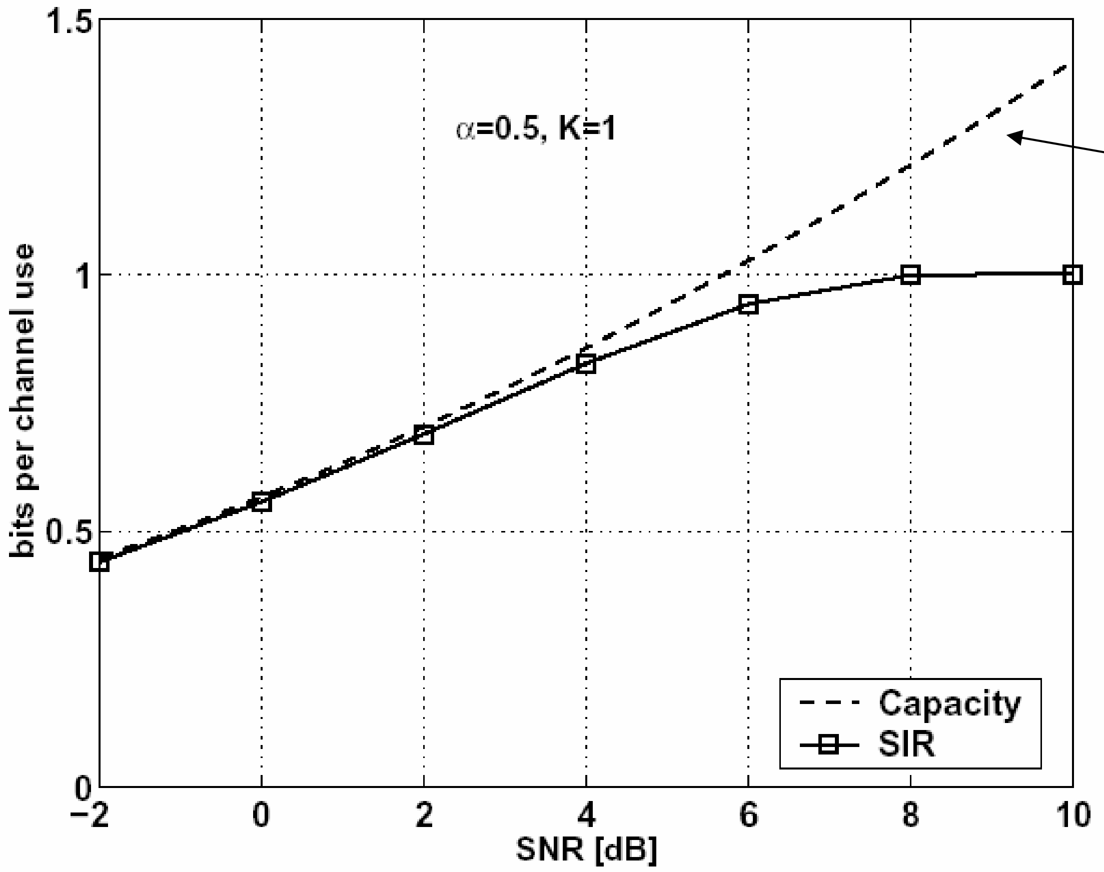
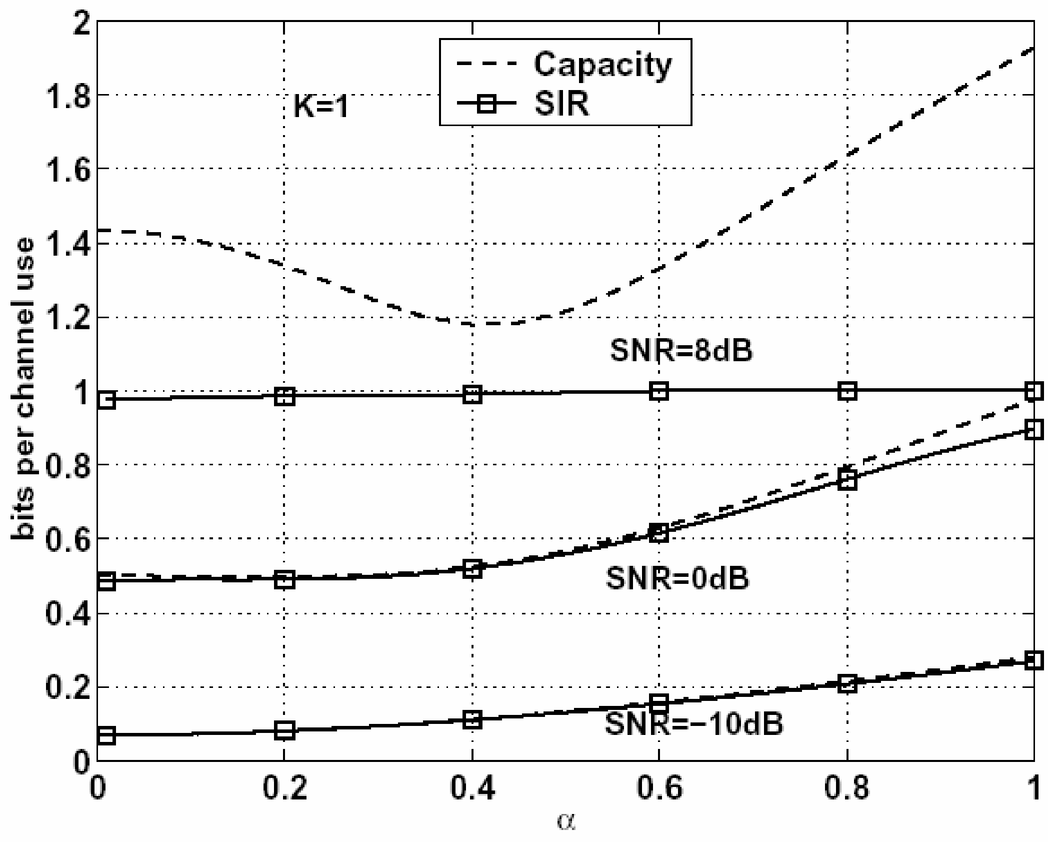
Experimental results: Wyner's model



Wyner's result

Experimental results: Wyner's model



Why do GBP-based CVM serve so remarkably?

- Currently we do not have a rigorous answer.
- Maybe the following empirical evidence may shed light on this issue:
  - The CVM results are not exact.
  - When does not it work? In a different setting : homogeneous antiferromagnetic interactions with random fields.
  - GBP converges to the same solution under different initial conditions (convexity of the Kikuchi free energy?).
  - 'Local' estimates may also perform well.

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Local estimates and GBP

Count the number of sites for which $|h_i| > \sum_j |R_{ij}|$

Local exact inference: neighborhood size $\pm 2, \pm 3$