A statistical mechanics analysis of coded CDMA with regular LDPC codes

Toshiyuki Tanaka

Department of Electronics and Information Engineering

Tokyo Metropolitan University, Japan

David Saad

Neural Computing Research Group - Aston University

Birmingham, United Kingdom

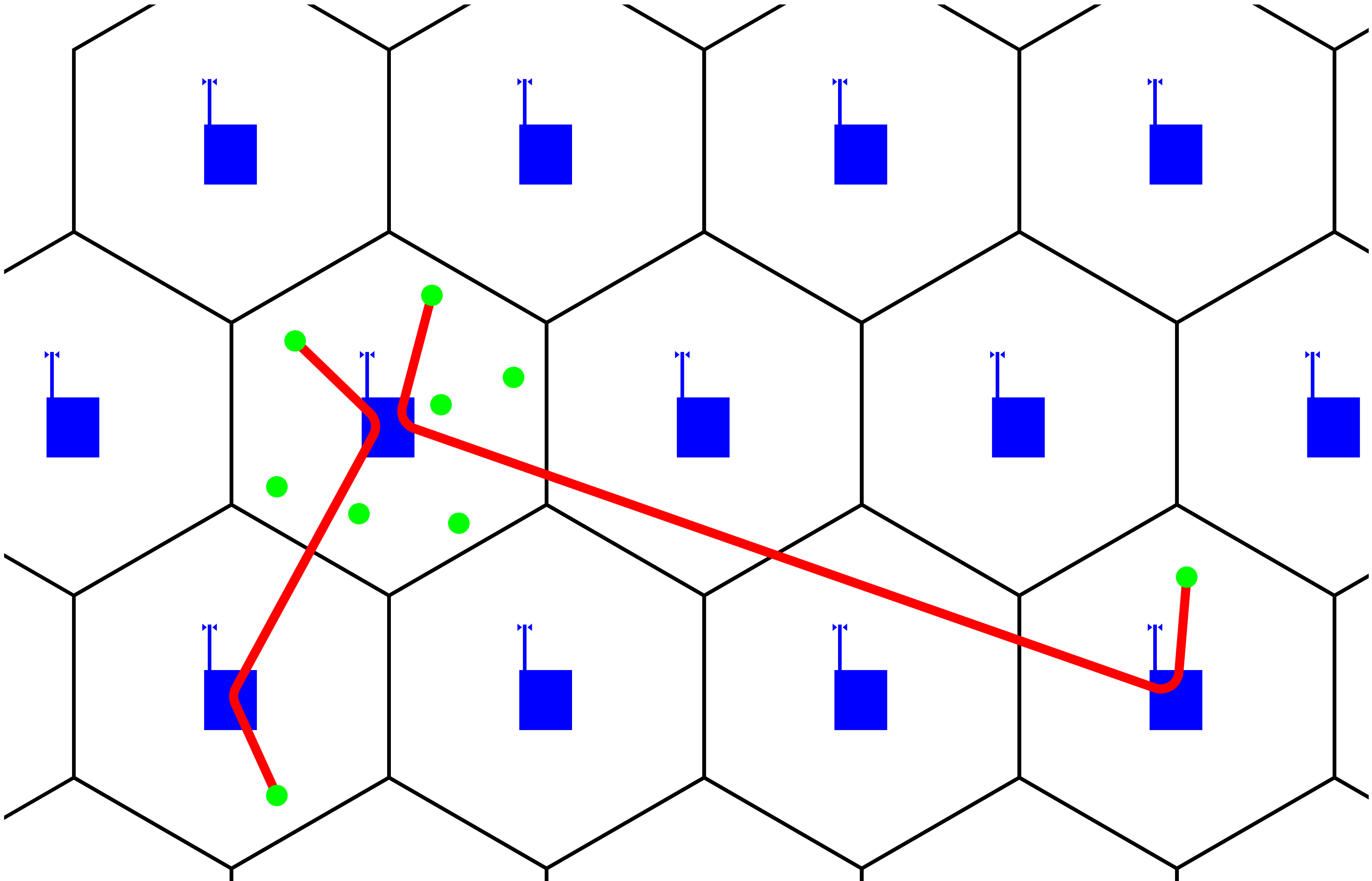
Tokyo Metropolitan University

Aston University

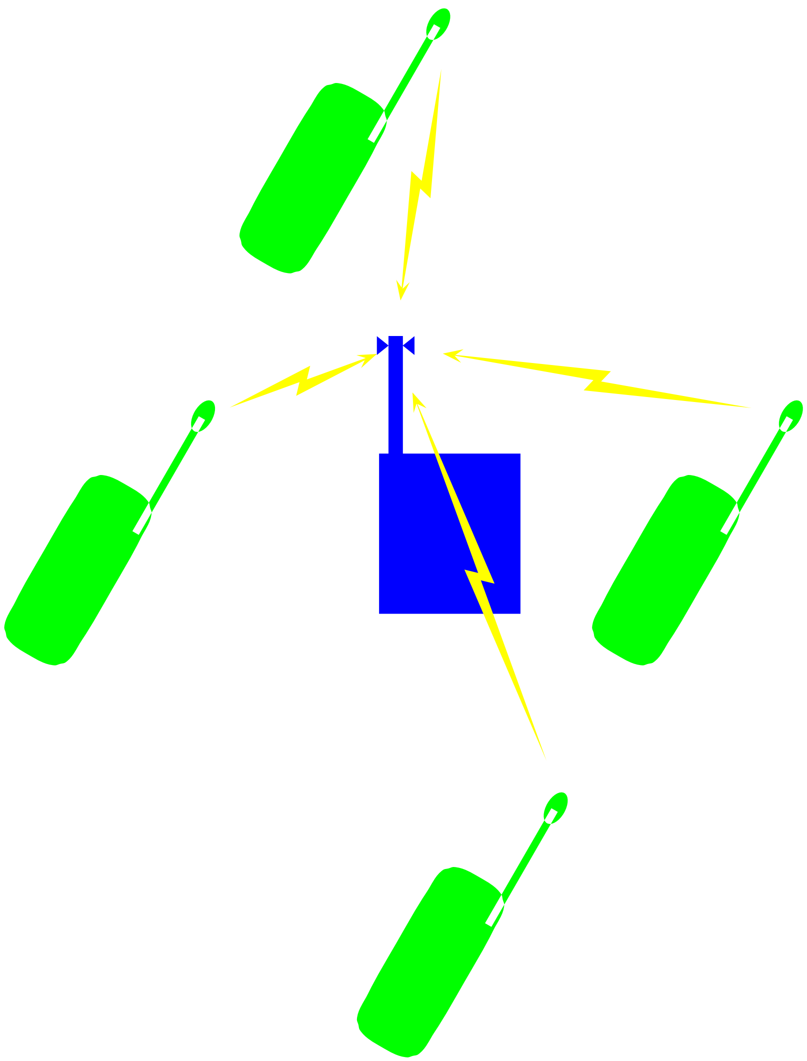
Outline

- Why CDMA?
- Combining LDPC and CDMA
- Transition points
- Decoding schemes
- Summary and future research

Overview: Mobile communication system



Multiple access (1)

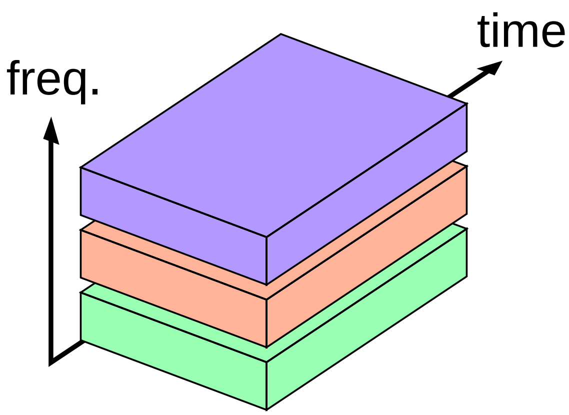


Simultaneous communication of (uncoordinated) users to the same base station

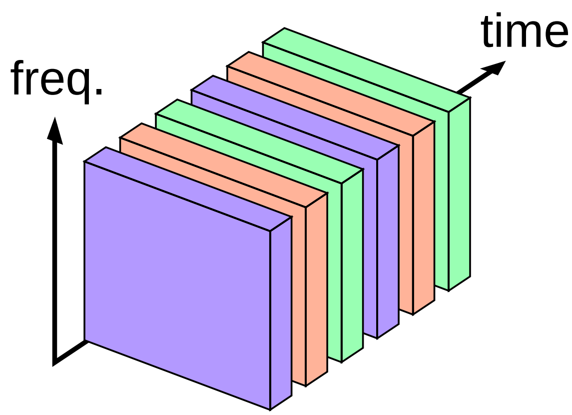
- **Frequency-division (FDMA)**
- **Time-division (TDMA)**
- **Code-division (CDMA)**

Multiple access (2)

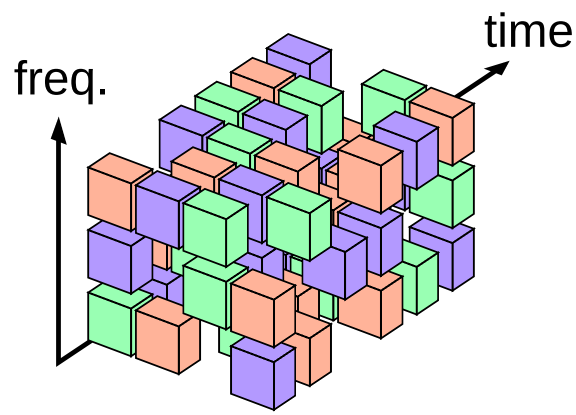
How a channel is divided ... ?



Frequency-division
(FDMA)

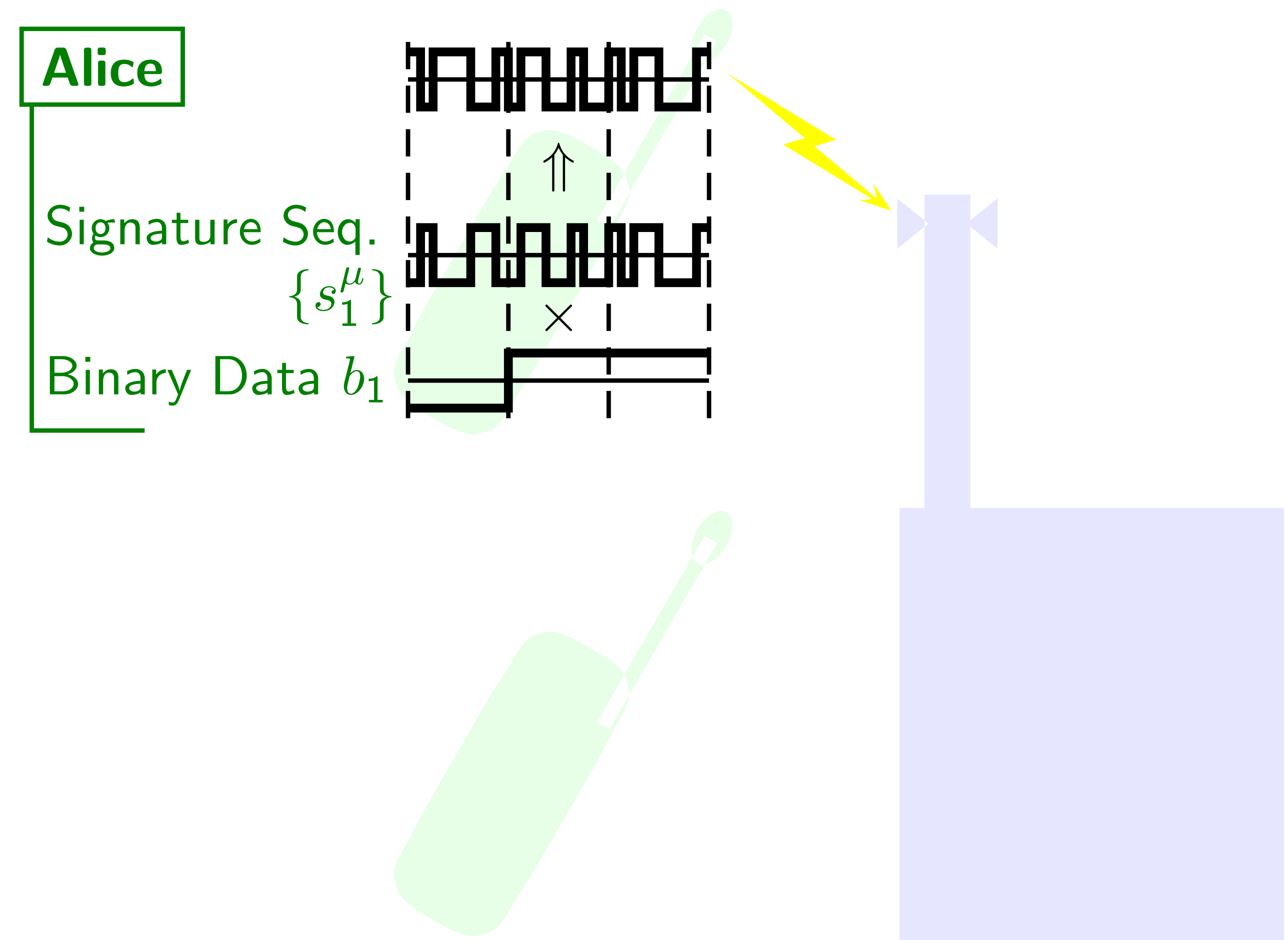


Time-division
(TDMA)

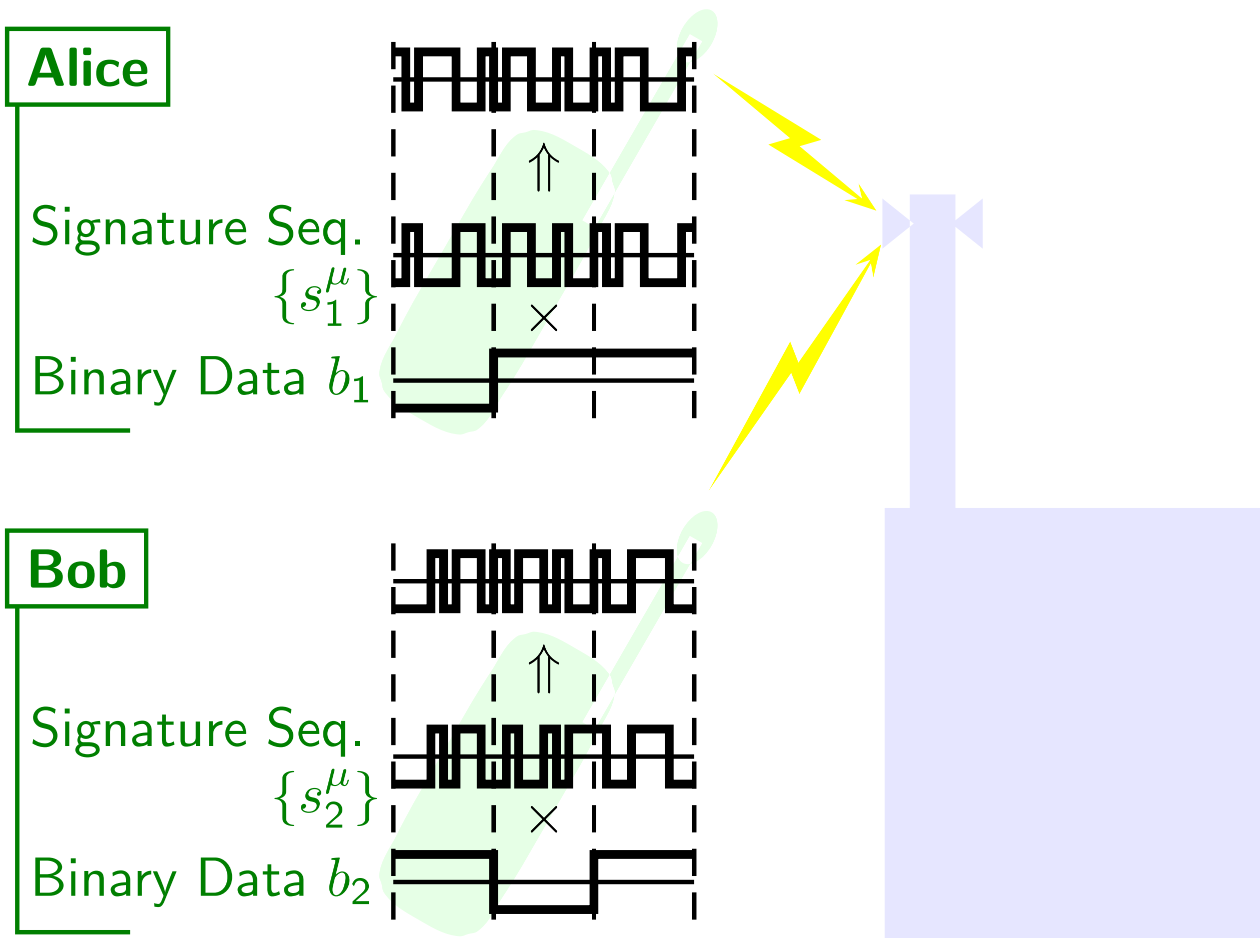


Code-division
(CDMA)

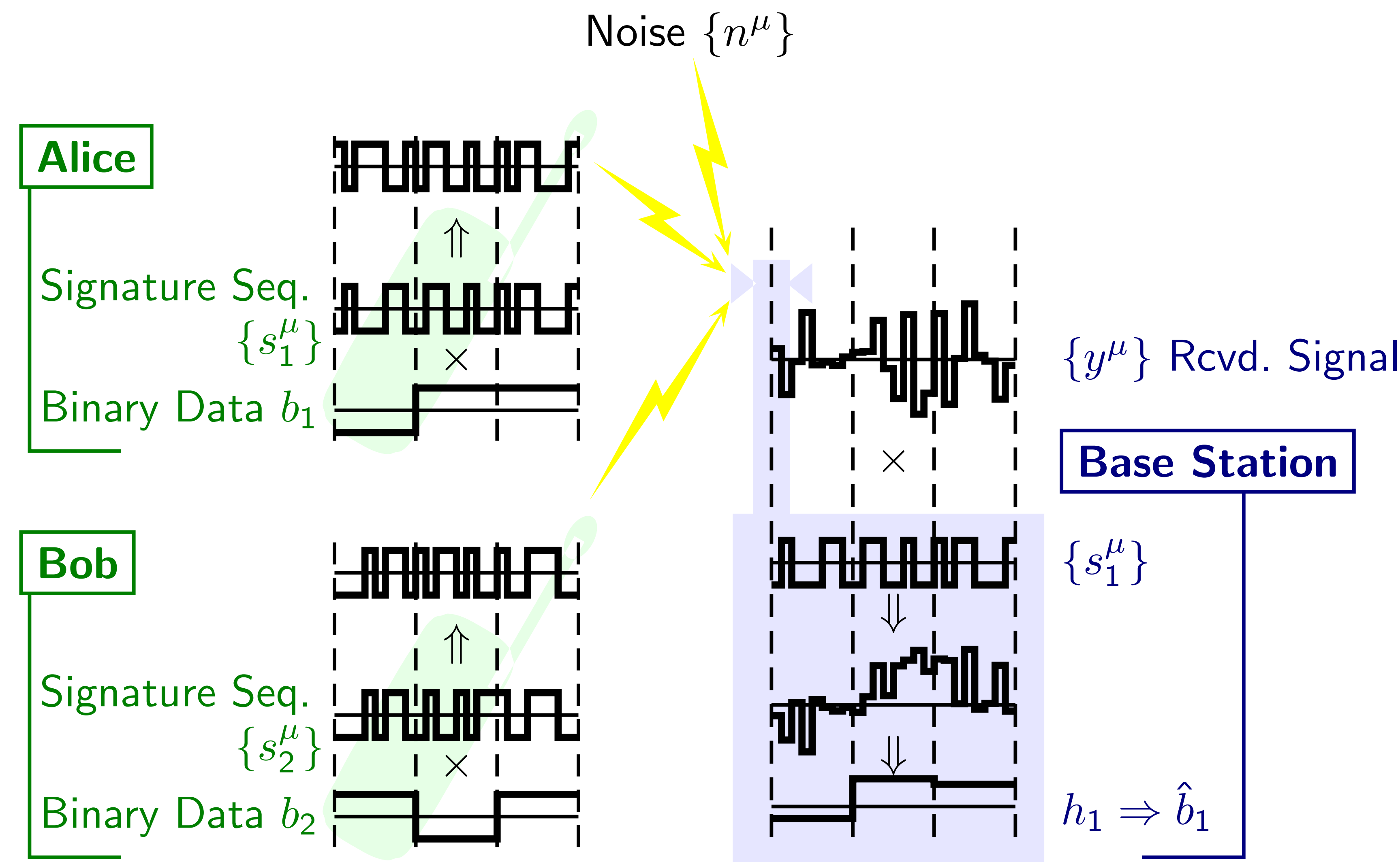
CDMA: Principle



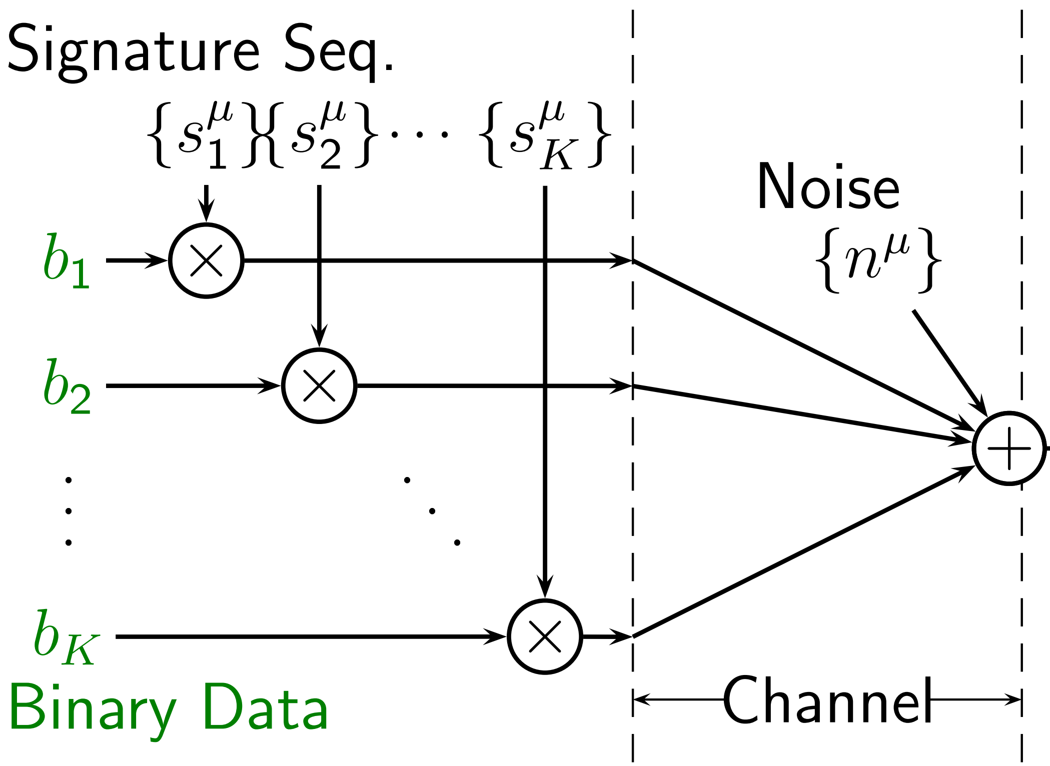
CDMA: Principle



CDMA: Principle



CDMA channel (K users)



$$y^\mu = \frac{1}{\sqrt{G}} \sum_{k=1}^{K} s_k^\mu b_k + n^\mu$$

$$(\mu = 1, \ldots, G)$$

CDMA detection problem

To estimate $b_1, \ldots, b_K$ from $y_1, \ldots, y_G$

Remark 1: Relationship with Perceptron

Detection Problem: To estimate $\boldsymbol{b} = (b_1, \ldots, b_K)^T$ from $(\boldsymbol{s}^1, y_1), \ldots, (\boldsymbol{s}^G, y_G)$, where $\boldsymbol{s}^\mu \equiv (s_1^\mu, \ldots, s_K^\mu)^T$ and

$$y_\mu = \frac{1}{\sqrt{G}} \boldsymbol{b} \cdot \boldsymbol{s}^\mu + n^\mu \quad (\mu = 1, \ldots, G)$$

$\Rightarrow$ Problem equivalent to **Learning of Binary-weight Linear Perceptron w/ Additive Output Noise**

Analysis: Bayesian framework

- Prior: $p(\boldsymbol{b})$
- Channel chr.: $p(\boldsymbol{y}|\boldsymbol{b})$
  as defined by pdf of noise $n^\mu = y^\mu - G^{-1/2}\boldsymbol{b}\cdot\boldsymbol{s}^\mu$
- Posterior:

$$p(\boldsymbol{b}|\boldsymbol{y}) = \frac{p(\boldsymbol{y}|\boldsymbol{b})p(\boldsymbol{b})}{p(\boldsymbol{y})}, \quad p(\boldsymbol{y}) = \sum_{\boldsymbol{b}'} p(\boldsymbol{y}|\boldsymbol{b}')p(\boldsymbol{b}')$$

Bayes decision theory

- Loss fn.:

$$L_k(\hat{\boldsymbol{b}}) = 1 - \delta_{\hat{b}_k b_k}$$

$\boldsymbol{b}$: True Info. Data; $\hat{\boldsymbol{b}}$: Estimate

$\rightarrow$ **Optimum decision rule**

(in the sense of minimising expected loss)

$$\hat{b}_k = \arg \max_{b_k \in \{-1,1\}} p(b_k|\boldsymbol{y}), \quad p(b_k|\boldsymbol{y}) = \sum_{\boldsymbol{b} \backslash b_k} p(\boldsymbol{b}|\boldsymbol{y})$$

Expected loss — Bit error rate (BER)

$$P_b \equiv E\big(1 - \delta_{\hat{b}_k b_k}\big) \quad \text{(same for all users due to symmetry)}$$

Statistical-mechanical analysis

- **Large-system limit**:
  K, $G \to \infty$ with load $\beta \equiv K/G = O(1)$
- **Random spreading**: s_k^μ: i.i.d.; mean=0, variance=1

Replica analysis

Objective: To evaluate Shannon entropy of $\boldsymbol{y}$ per user:

$$\mathcal{I} = -\lim_{K\to\infty} \frac{1}{K} \left\langle \int p_0(\boldsymbol{y}) \log p(\boldsymbol{y})\, d\boldsymbol{y} \right\rangle_s$$

- $p_0(\boldsymbol{y})$: True distribution
- $p(\boldsymbol{y})$: Postulated distribution by receiver

Replica method:

$$\mathcal{I} = -\lim_{K\to\infty} \left\{ \lim_{n\to 0} \frac{\partial}{\partial n} \left[\frac{1}{K} \left\langle \int p_0(\boldsymbol{y}) \left[p(\boldsymbol{y})\right]^n d\boldsymbol{y} \right\rangle_s \right] \right\}$$

Binary uniform prior, AWGN

Assumptions:

- Binary uniform prior

$$p(\boldsymbol{b}) = \text{const. over } \{-1,\, 1\}^K$$

- Additive White Gaussian Noise Channel

$$n^\mu \sim N(0,\, \sigma_0^2), \quad \text{i.i.d.}$$

Binary uniform prior, AWGN

Prior: $p(\boldsymbol{b}) = \text{const. over } \{-1,\, 1\}^K$

Conditional:

$$p(\boldsymbol{y} \mid \boldsymbol{b}) = \prod_{\mu=1}^{G} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left[-\frac{(y^\mu - G^{-1/2}\sum_{k=1}^{K} s_k^\mu b_k)^2}{2\sigma_0^2}\right]$$

$\longrightarrow$ **Posterior**:

$$p(\boldsymbol{b}|\boldsymbol{y}) = Z^{-1} \exp\left(-\sigma_0^{-2} H(\boldsymbol{b})\right)$$

Remark 2: Relationship with Hopfield models

$\longrightarrow$ **Posterior**:

$$p(\boldsymbol{b}|\boldsymbol{y}) = Z^{-1} \exp\left(-\sigma_0^{-2} H(\boldsymbol{b})\right)$$

$$H(\boldsymbol{b}) \equiv \frac{1}{2}\boldsymbol{b}^T W \boldsymbol{b} - \boldsymbol{h}^T \boldsymbol{b}$$

$W = (w_{ij}), \quad w_{ij} = \frac{1}{G}\sum_{\mu} s_i^{\mu} s_j^{\mu}$: Correlation Mtx. of Signature Seq.

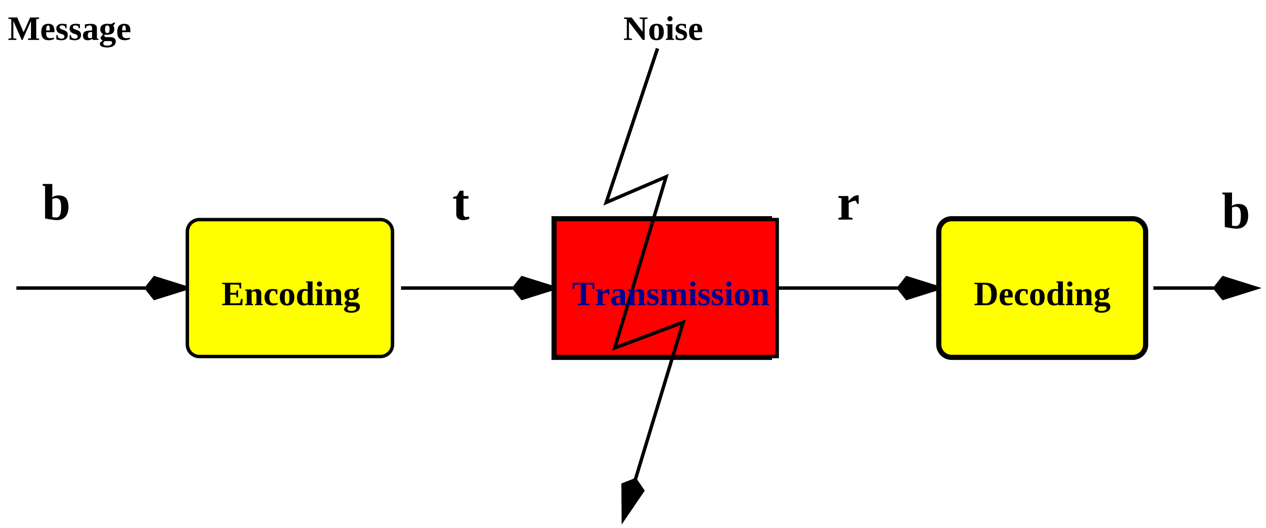
$\boldsymbol{h} = (h_k), \quad h_k = \frac{1}{\sqrt{G}}\sum_{\mu} s_k^{\mu} y^{\mu}$: Matched-Filter Output Vector

$\Rightarrow$ Ising spin systems - Hopfield models

(Miyajima et al., 1993; Kechriotis & Manolakos, 1996)

Error correcting codes

- The message is an N dimensional binary vector $\boldsymbol{b}$
- Encoded to M dimensional binary vector $\boldsymbol{t}$; transmitted through a noisy channel; code rate $R = N/M$
- Received message $\boldsymbol{r} = \boldsymbol{t} + \boldsymbol{n}$ is decoded to retrieve $\boldsymbol{b}$



Regular LDPC Codes

- Gallager's code (1962) – Construct two sparse matrices A and B of dimensionality $(M-N)\times N$ and $(M-N)\times(M-N)$ respectively.

- The matrix $\mathcal{A} = [\boldsymbol{A} \mid \boldsymbol{B}]$ has L non-zero (unit) elements per row and C per column.

N, M-N, M-N, C, L

$$A = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & & \ddots \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & & & \ddots \end{pmatrix}$$

Gallager's Code

Encoding
$$\underbrace{\boldsymbol{t}}_{\in\{0,1\}^M} = \boldsymbol{G}^T \underbrace{\boldsymbol{b}}_{\in\{0,1\}^N} \pmod 2$$

Generator matrix – $\boldsymbol{G} = [\boldsymbol{I} \mid \boldsymbol{B}^{-1}\boldsymbol{A}] \pmod 2$

The received vector – $\boldsymbol{r} = \boldsymbol{G}^T\boldsymbol{b} + \boldsymbol{n} \pmod 2$

Decoding
$$\boldsymbol{z} = \mathcal{A}\boldsymbol{r} = \underbrace{\mathcal{A}\boldsymbol{G}^T\boldsymbol{b}}_{=0} + \mathcal{A}\boldsymbol{n} \pmod 2 ,$$

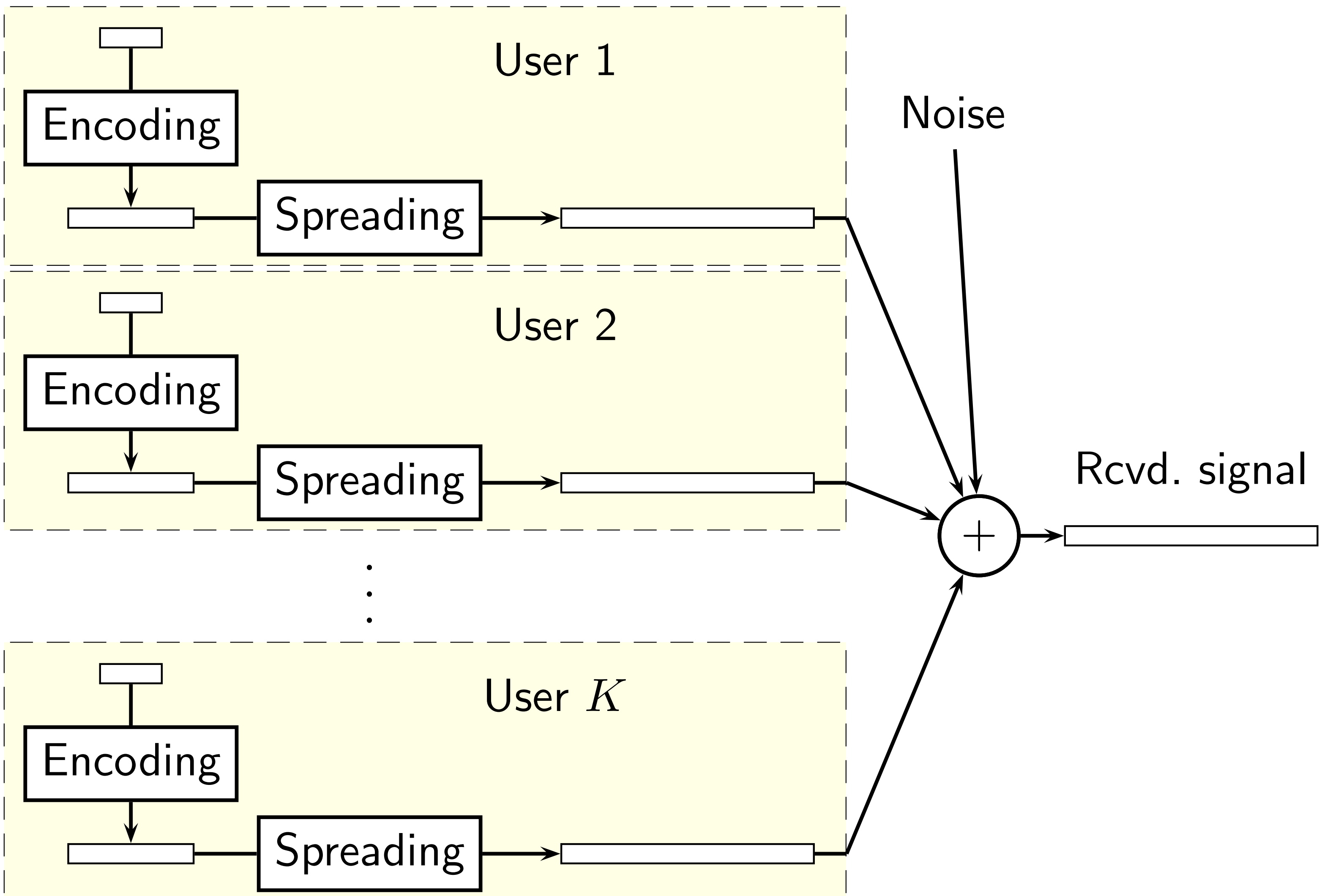
$\mathcal{A} = [\boldsymbol{A} \mid \boldsymbol{B}]$ sparse $(M-N)\times M$ binary matrix

The problem: $\boldsymbol{z} = \mathcal{A}\boldsymbol{\tau} \pmod 2$
carried out by various methods (e.g., BP)

Motivation: LDPC-coded CDMA

- Need for coding: Performance of uncoded CDMA system $<$ minimal QoS
- LDPC codes: High rate /low decoding complexity
- Related work: LDPC-coded CDMA on the basis of stripping (Caire et al 2003)

LDPC-coded CDMA



LDPC-coded CDMA

- Serial concatenation of LDPC code and CDMA channel
- Regular Gallager codes used (C, L: Left/Right degree)
- Performance of optimum joint detection/decoding scheme
- Assume: $N, M \to \infty$ but $R = N/M = O(1)$
- Assume: $K, G \to \infty$ but $\beta = K/G = O(1)$
- Averages taken w.r.t both s and $\mathcal{A}$

Free energy

What we want to calculate is the free energy (mutual information per symbol per user between received and sent symbols)

$$f = \frac{1}{MK}\left[-\left\langle \log P(\boldsymbol{y})\right\rangle_{P_0(\boldsymbol{y})} + \left\langle\left\langle \log P(\boldsymbol{y}|T)\right\rangle_{P_0(\boldsymbol{y}|T)}\right\rangle_{P(T)}\right].$$

where $T \equiv \{\boldsymbol{\tau}_k\}$

We average over A and S

$$\bar{f} = \lim_{M,K\to\infty} E_{S,A}[f(S,A)]$$

LDPC-coded CDMA - Hamiltonian

- The Hamiltonian has components (for each user)

$$\chi\left(0 \;= [\mathcal{A}\,\boldsymbol{\tau}]_{k,\mu}\right)$$

The parity checks $\chi\left(\cdots\right) = \infty$ if parity checks are obeyed and 0 otherwise

- For each codeword bit, spreading chip and user

$$\left[-\frac{1}{2\sigma_0^2}\left(y_t - \frac{1}{\sqrt{G}}\sum_{k=1}^{K}\tau_k s_{kt}\right)^2\right]$$

representing the channel noise

- Use Nishimori's conition = correct prior; RS

LDPC-coded CDMA

RS saddle-point equations:

$$m = \int \tanh\left(\sqrt{F}z + E + \sum_{c=1}^{C} \tanh^{-1} \hat{x}_c\right) Dz \prod_{c=1}^{C} [\hat{\pi}(\hat{x}_c)\, d\hat{x}_c]$$

$$q = \int \tanh^2\left(\sqrt{F}z + E + \sum_{c=1}^{C} \tanh^{-1} \hat{x}_c\right) Dz \prod_{c=1}^{C} [\hat{\pi}(\hat{x}_c)\, d\hat{x}_c]$$

$$E = \frac{1}{\sigma^2 + \beta(1-q)}, \qquad F = \frac{\sigma_0^2 + \beta(1 - 2m + q)}{[\sigma^2 + \beta(1-q)]^2}$$

$$\pi(x) = \int \delta\left[x - \tanh\left(\sqrt{F}z + E + \sum_{c=1}^{C-1} \tanh^{-1} \hat{x}_c\right)\right] Dz \prod_{c=1}^{C-1} [\hat{\pi}(\hat{x}_c)\, d\hat{x}_c]$$

$$\hat{\pi}(\hat{x}) = \int \delta\left(\hat{x} - \prod_{i=1}^{L-1} x_i\right) \prod_{i=1}^{L-1} [\pi(x_i)\, dx_i]$$

LDPC-coded CDMA

Bit error rate:

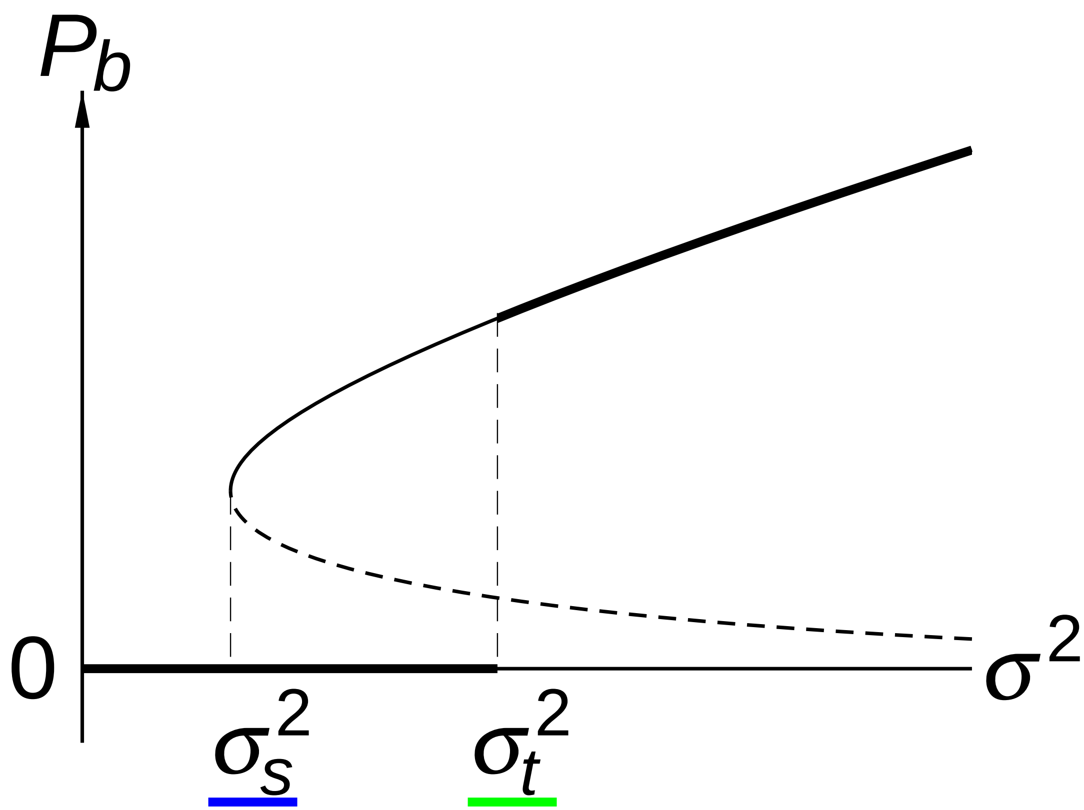
$$P_b = \int_{-\infty}^{0} P(u)\, du$$

$$P(u) = \int \delta\left[u - \mathsf{tanh}\left(\sqrt{F}z + E + \sum_{c=1}^{C} \mathsf{tanh}^{-1} \hat{x}_c\right)\right] Dz \prod_{c=1}^{C} [\hat{\pi}(\hat{x}_c)\, d\hat{x}_c]$$

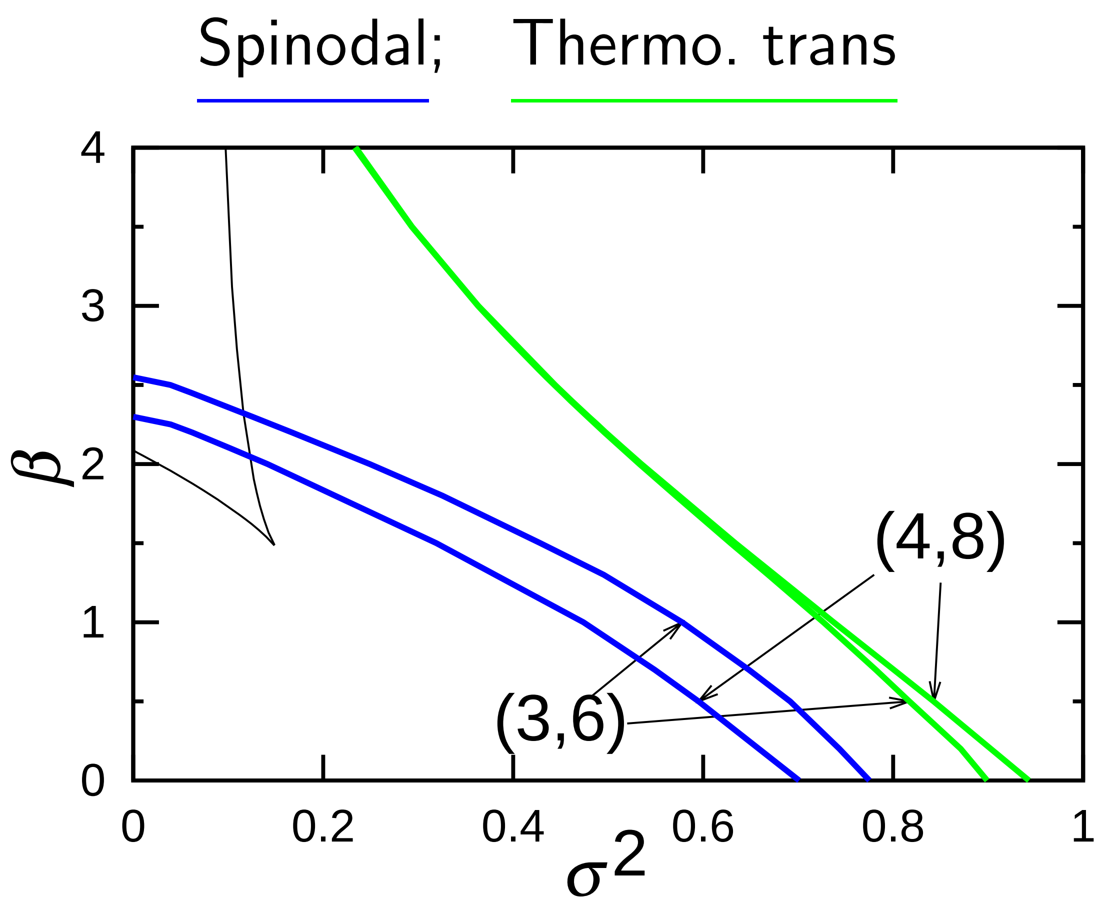
LDPC-coded CDMA

Typical structure of P_b-σ^2 diagram

Spinodal; Thermo. trans



LDPC-coded CDMA

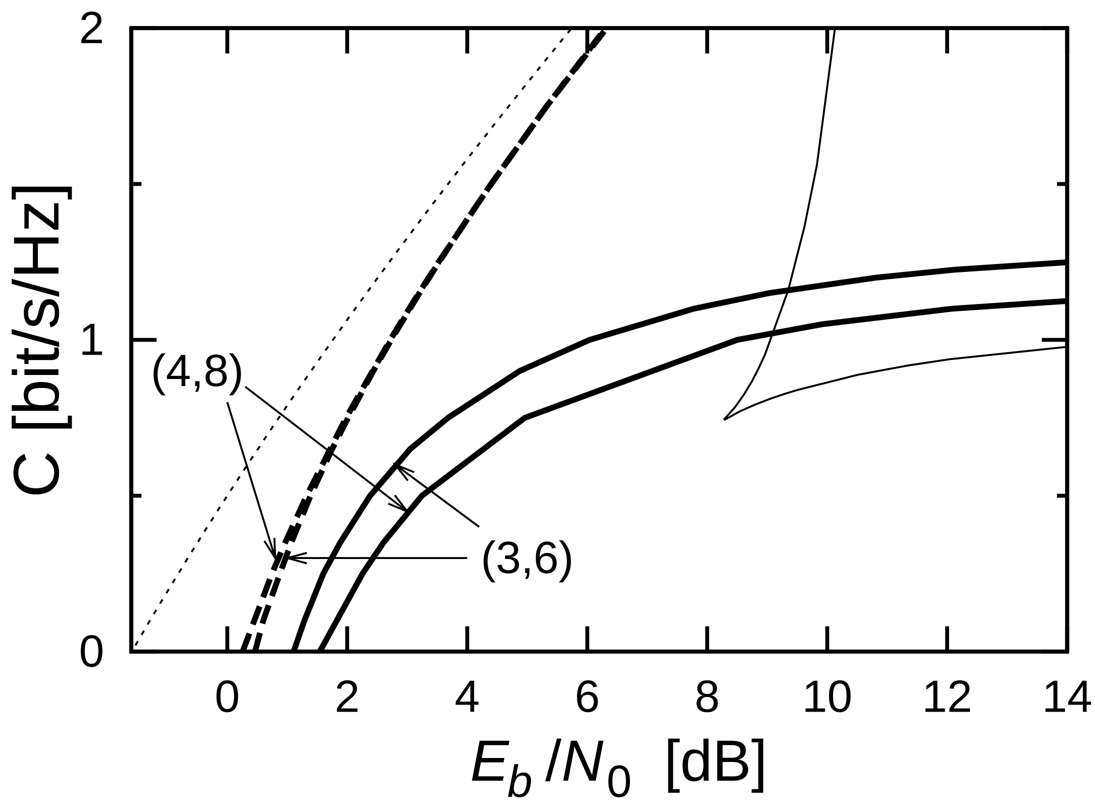


As β increases $\sigma_s \rightarrow 0$; irregular codes?

Tokyo Metropolitan University

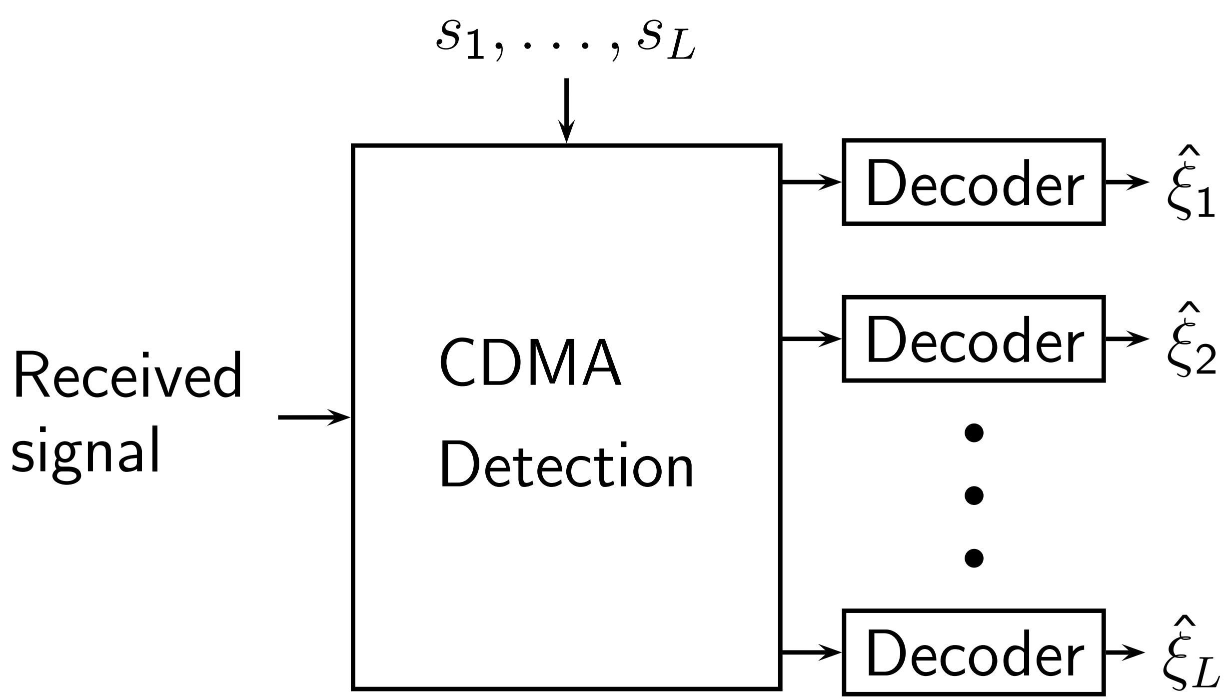
Aston University

LDPC-coded CDMA



As the load β increases, theoretical thresholds approach single -user channel capacity

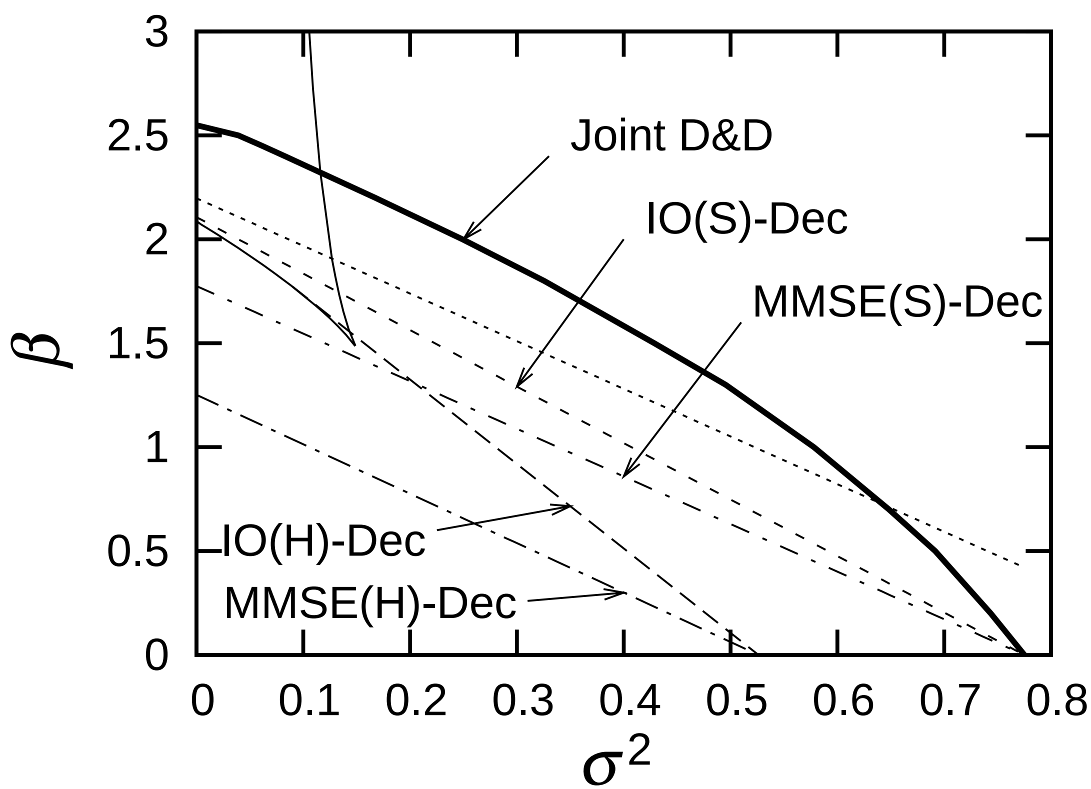
Individual optimum decoding



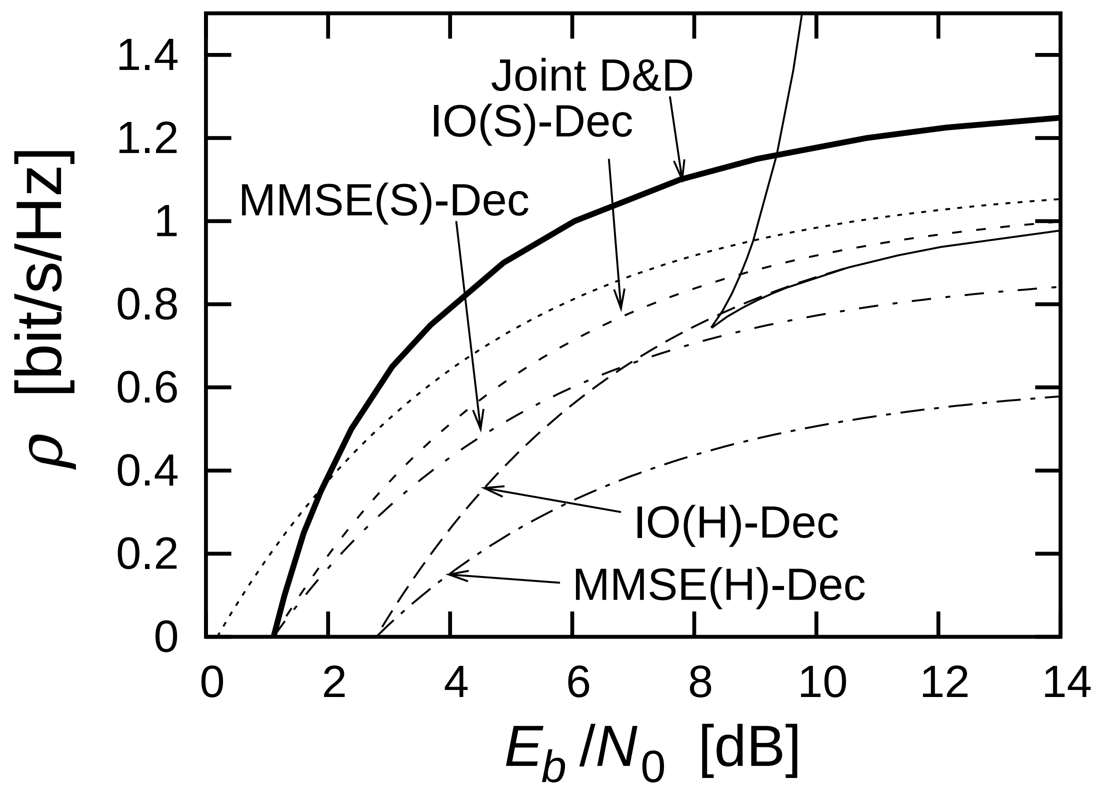
Detection and decoding

- Joint detection and decoding
- Individual optimum detection (H/S) and decoding
- Minimum MSE multi-user detection (H/S), per user decoding

Individual optimum decoding I



Individual optimum decoding II



Summary and future directions

- Introduction of CDMA multiuser detecction problem
- Statistical-mechanics analysis using replica method
- Coding prior to modulation has great potential
- Current problem - dynamical transition point
- Future directions - irregular constructions, joint modulation and coding

http://www.ncrg.aston.ac.uk

Supported by FP6 IP - EVERGROW