Information Evolution of Optimal Learning

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- Bayesian estimation, choice under uncertainty (Wald, 1950; Neumann & Morgenstern, 1944; Savage, 1954; Anscombe & Aumann, 1963).

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- Learning and adaptation?

Optimisation & information in learning

Proposition

There is no need to learn, if there is nothing to optimise

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There is nothing new to learn, if one has full information

Choice and optimisation

Definition

A preference relation is $\lesssim \subseteq A^2$ which is

1. Total: $(a_1 \lesssim a_2) \vee (a_2 \lesssim a_1) = A^2$
2. Transitive: $(a_1 \lesssim a_2) \wedge (a_2 \lesssim a_3) \Longrightarrow (a_1 \lesssim a_3)$

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Lemma

Every set with preference relation is Borel measurable.

Preference morphisms

Lemma

For any $f : A \to B$ *the following are equivalent:*

1. f *is monotone:* $a_1 \lesssim_A a_2 \implies f(a_1) \lesssim_B f(a_2)$;
2. f *is continuous:* $f(a) = b \implies f(O_a) \subset O_b$;
3. f *is measurable:* $f^{-1}(B' \in \mathcal{B}) = A' \in \mathcal{A}$

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Definition

Function $u : A \to \overline{\mathbb{R}}$ is called a utility representation of preference relation $(A, \lesssim)$ if $\forall\, a_1,\, a_2 \in A$

$$a_1 \lesssim a_2 \implies u(a_1) \leq u(a_2)$$

Preliminaries

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$x \in X$ and $y \in Y$ are coupled in H (i.e. $X \subset H \subset Y$) by a bilinear form $(\cdot, \cdot) : X \times Y \to \overline{\mathbb{R}}$, such as the inner product

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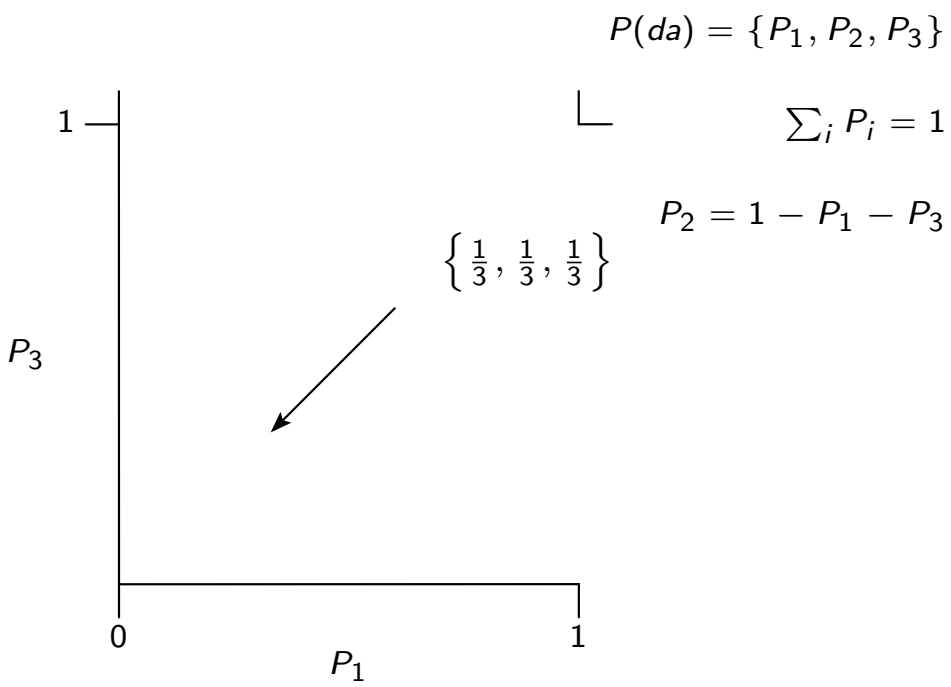
Expected value

$$E\{x\} = \int_A x(a)P(da) = (x, y)$$

where $y(a) \geq 0$, $\int_A y(a)da = P(A) = 1$

Probability triangle (simplex)

$E\{x\}$ is a unique representation of $(\mathcal{P}, \lesssim_{\mathcal{P}})$, where $\mathcal{P}$ is the set of all probability measures in Y (Neumann & Morgenstern, 1944).



Convex functions

Definition

$F : X \to \mathbb{R}_\infty$ is convex if $\forall x \in X$

$$F((1-t)x_1 + t\,x_2) \le (1-t)F(x_1) + t\,F(x_2), \qquad t \in [0,1]$$

F is proper if $\mathrm{dom}(F) \equiv \{x \in X : F(x) < \infty\} \neq \emptyset$

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Information (KL) divergence (relative entropy)

$$I(y, y_0) = \int_A \ln \frac{y}{y_0}\, y(a)da = \int_A \ln \frac{d\mu}{d\nu}\, d\mu(a)$$

Here $y(a) = \mu(da)/da$ is the density function.

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Proposition

If $F : X \to \mathbb{R}_\infty$ is proper, convex and l.s.c. on Banach X, then F is continuous on $\mathrm{Int}(\mathrm{dom}(F))$.(e.g. Showalter, 1997)

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Proposition

Strictly convex F has unique minimum on $\mathrm{dom}(F)$.

Proof.

If not, then $F((1-t)x_1 + tx_2) < (1-t)F(x_1) + tF(x_2) = C$ □

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Definition

Subdifferential $\partial F(x) \equiv \{y \in Y : (y, x' - x) \leq F(x') - F(x)\}$

Support and distance of a convex body

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Definition

K_x and K_y are polar if support of K_x is distance in K_y.

$$\begin{aligned} F(y) &= \sup\{(y,x) : x \in K_x\} = \inf\{C : y \in CK_y\} \\ \tilde{F}(x) &= \sup\{(y,x) : y \in K_y\} = \inf\{D : x \in DK_x\} \end{aligned}$$

Functions of polar sets

- Replace conditions $x \in K_x$, $y \in K_y$ by $\tilde{F}(x) \leq D < \infty$, $F(y) \leq C < \infty$

$$F(y) = \sup\{(y,x) : \tilde{F}(x) \leq D\}, \quad \tilde{F}(x) = \sup\{(y,x) : F(y) \leq C\}$$

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- Dual problems

  $$D(C) = \inf\{\tilde{F}(x) : (y,x) \geq C\}, \quad C(D) = \inf\{F(y) : (x,y) \geq D\}$$

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- Legendre–Fenchel transform

  $$F(y) = \sup\{(y,x) - \tilde{F}(x)\}, \qquad \tilde{F}(x) = \sup\{(x,y) - F(y)\}$$
  $$x \in \partial F(y), \qquad y \in \partial \tilde{F}(x)$$

  In fact, $\partial F = (\partial \tilde{F})^{-1}$

Necessary conditions of extrema

Theorem

The extremum $y^* \in K_y$ *for* $\sup\{(x, y) : F(y) \leq C\} = D$ *satisfies*

$$\beta x \in \partial F(y^*), \quad F(y^*) = C, \quad \beta^{-1}(C) = D'(C)$$

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Proof.

Lagrangian function $L(y, \beta^{-1}) = (x,y) + \beta^{-1}[C - F(y)]$

$$\partial_y L(y, \beta^{-1}) = x - \beta^{-1}\partial F(y) \ni 0, \quad \Rightarrow \quad \beta x \in \partial F(y^*)$$

$$\partial_{\beta^{-1}} L(y, \beta^{-1}) = C - F(y) \ni 0, \quad \Rightarrow \quad F(y^*) = C$$

$$D'(C) = \beta^{-1} \qquad (C'(D) = \beta)$$

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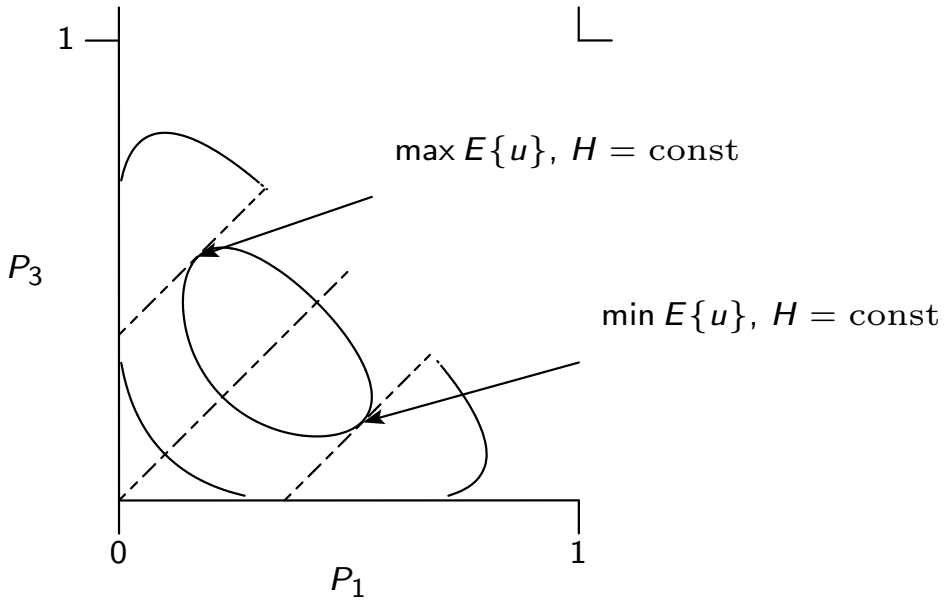
When $y \in Y$ are probabilities, and $F(y)$ is information divergence

$$\beta x - \gamma(\beta) = \ln \frac{y^*}{y_0} \quad \Longrightarrow \quad y^* = y_0 e^{\beta x - \gamma(\beta)}$$

If $x = -|t|^2$, then y^* is Gaussian.

Why 'extrema'?

There is one extremum of $F(y)$ (minimum), but there are two extrema of (x, y): maximum ($\beta \geq 0$) and minimum ($\beta < 0$).



Characterisation of the solution at ∞

Consider $a^* \in da \neq 0$ such that $x(a^*) = \pm\infty$

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Theorem

Linear fnl. $(x, y) \neq \pm\infty$ *iff* $|(x, y)| < \infty$ *for* $|x(a)|$, $|y(a)| < \infty$ *and*

$$x(a^*) = \pm\infty \implies y(a^*) = 0 \qquad x(a^*) = 0 \impliedby y(a^*) = \pm\infty$$

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Proof.

($\Rightarrow$) By convention $\pm\infty \cdot 0 = 0$ we have

$$(x, y) = (x, y)|_{x,y\neq\pm\infty} + (x, 0)|_{x=\pm\infty} + (0, y)|_{y=\pm\infty} = D + 0 = D \neq \infty$$

($\Leftarrow$) By convention $\infty - \infty = -\infty$

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Characterisation of the solution at 0

Theorem

Let $\tilde{F}(x)$, $F(y)$ *be dual convex fnls, and* $\beta x = \partial F(y^*)$. *Then* $\forall x \in X$ *s.t.* $(x, y) \neq \text{const}$, $\partial \tilde{F}(x_0) \ni 0$

$$y^*(a) = 0 \implies \beta x(a) = x_0(a)$$

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Proof.

$$\partial F(y^*) \ni \beta x \;\Rightarrow\; y^*(a) = 0 = \partial \tilde{F}(\beta x(a)) \in \partial \tilde{F}(x_0(a))$$ □

Characterisation of the solution at 0

Theorem

Let $\tilde{F}(x)$, $F(y)$ be dual convex fnls, and $\beta x = \partial F(y^*)$. Then $\forall x \in X$ s.t. $(x, y) \neq \text{const}$, $\partial \tilde{F}(x_0) \ni 0$

$$y^*(a) = 0 \implies \beta x(a) = x_0(a)$$

If $F(y)$ is information divergence, then its dual is

$$\tilde{F}(\beta x) = \ln \int_A y_0 e^{\beta x(a)} da$$

which achieves minimum at $x_0 = -\infty$.
Thus, $y^*(a_0) = 0$ implies $x(a_0) = -\infty$

Generalised characteristic potentials

Definition

Given $x \in X$ and $y^* = \partial\tilde{F}(\beta x)$ for constraints $(x, y^*) = D$ (or $F(y^*) = C$), let

$$\Psi(\beta) \equiv \sup[\beta D - C(D)] \qquad \Phi(\beta^{-1}) \equiv \inf[\beta^{-1} C - D(C)]$$

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$$\begin{aligned} C(D) &= \sup[\beta D - \Psi(\beta)], & D(\beta) &= \Psi'(\beta) \\ D(C) &= \inf[\beta^{-1} C - \Phi(\beta^{-1})], & C(\beta^{-1}) &= \Phi'(\beta^{-1}) \end{aligned}$$

Examples

Let $F(y) = I(y, da) = \int_A \ln \frac{y}{da} y(a) da$, where da is Lebesgue measure. Thus, $F(y)$ is minus entropy, and

$$\Psi(\beta) = \ln \int_A e^{\beta x(a)} da = \ln Z(\beta) = \Gamma(\beta), \quad \Phi(\beta^{-1}) = -\beta^{-1}\Gamma(\beta)$$

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Example (Finite set)

Let $A = \{a_1, a_2\}$, $da = 1$, and $x : A \to \{c - d, c + d\}$.

$$\begin{aligned} Z(\beta) &= e^{\beta(c-d)} + e^{\beta(c+d)} = 2e^{\beta c} \cosh(\beta d) \\ \Gamma(\beta) &= \ln 2 + \beta c + \ln \cosh(\beta d) \\ D(\beta) &= \Gamma'(\beta) = c + d \tanh(\beta d) \end{aligned}$$

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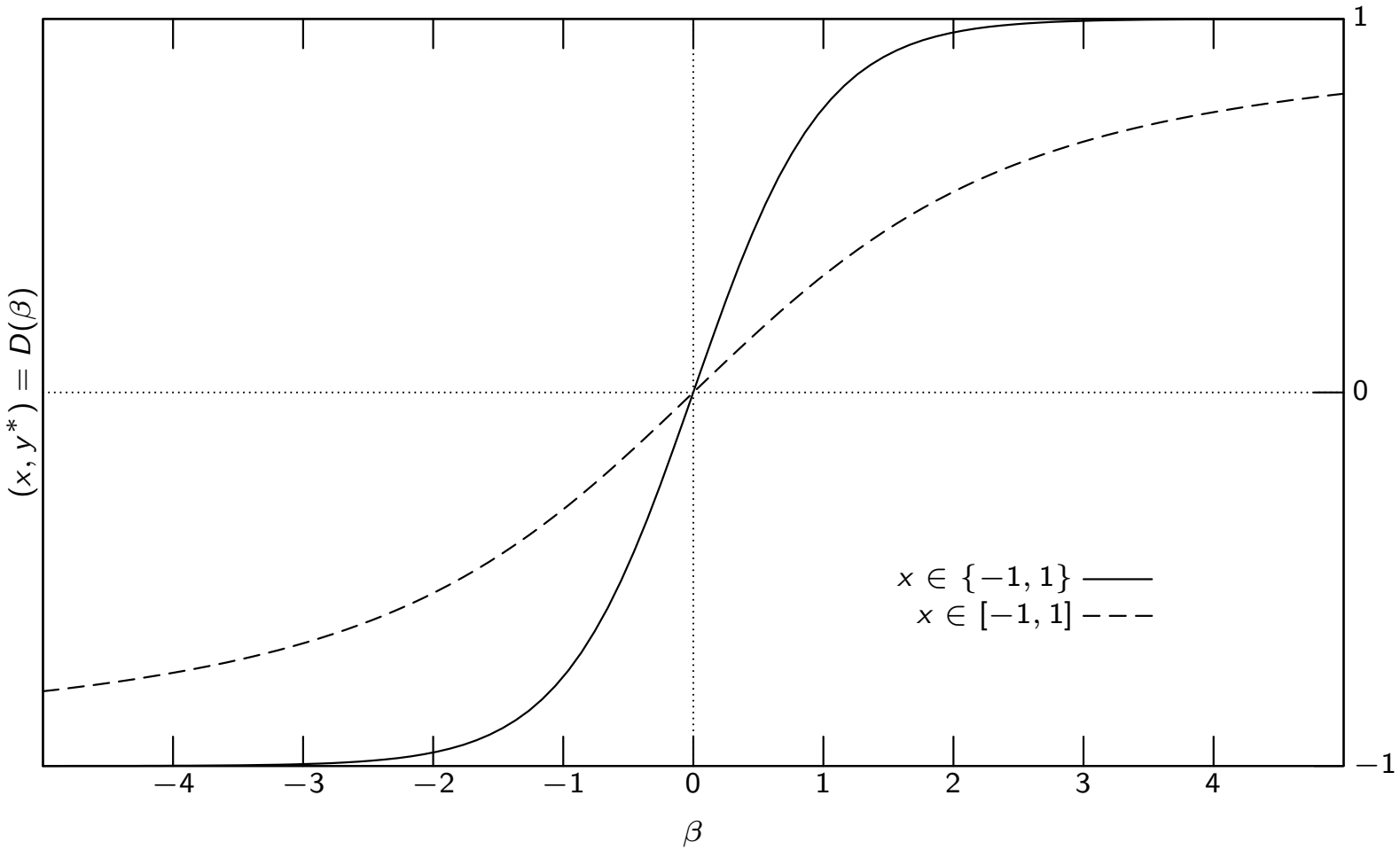
Example (Uncountable set)

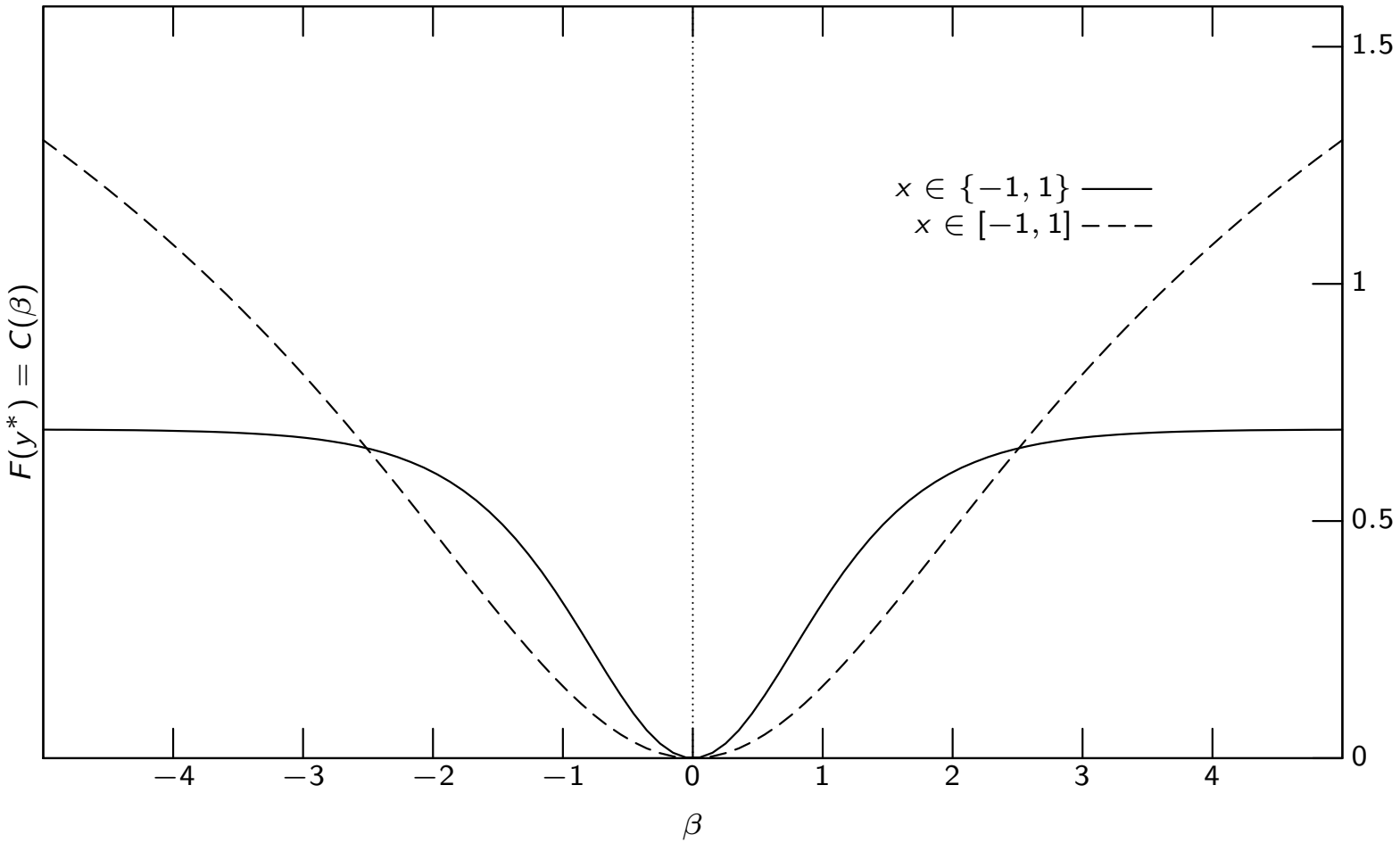
Let A be a continuum, $da = \mathrm{const}$, $x : A \to [c-d, c+d] \subseteq \overline{\mathbb{R}}$.

$$\begin{aligned} Z(\beta) &= \int_{c-d}^{c+d} e^{\beta x(a)} da = 2e^{\beta c} \beta^{-1} \sinh(\beta d) \\ \Gamma(\beta) &= \beta c - \ln|\beta| d + \ln|\sinh(\beta d)| \\ D(\beta) &= c - \beta^{-1} + d \coth(\beta d) \end{aligned}$$

Expected utility



Information



Dynamics

Theorem

Let $C = C(t)$, $D = D(t)$ be monotone functions describing constraints $F(y) \leq C$ or $(x, y) \geq D$ on interval $t \in [t_1, t_2]$.

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$$\begin{aligned} \int_{t_1}^{t_2} (x, y(t))\, dt &\leq \Psi(\beta_2) - \Psi(\beta_1) \\ \int_{t_1}^{t_2} F(y(t))\, dt &\geq \Phi(\beta_1^{-1}) - \Phi(\beta_2^{-1}) \end{aligned}$$

where β_1, β_2 are determined from

$$\beta^{-1} = (\Phi')^{-1}(C), \qquad \beta = (\Psi')^{-1}(D)$$

Dynamics

Proof.

An integral is maxi-(mini)-mised if each integrand is maxi-(mini)-mised:

$$\int_{t_1}^{t_2} (x, y(t))\, dt \quad \leq \quad \int_{\beta_1}^{\beta_2} D(\beta) d\beta = \int_{\beta_1}^{\beta_2} \Psi'(\beta) d\beta = \Psi|_{\beta_1}^{\beta_2}$$

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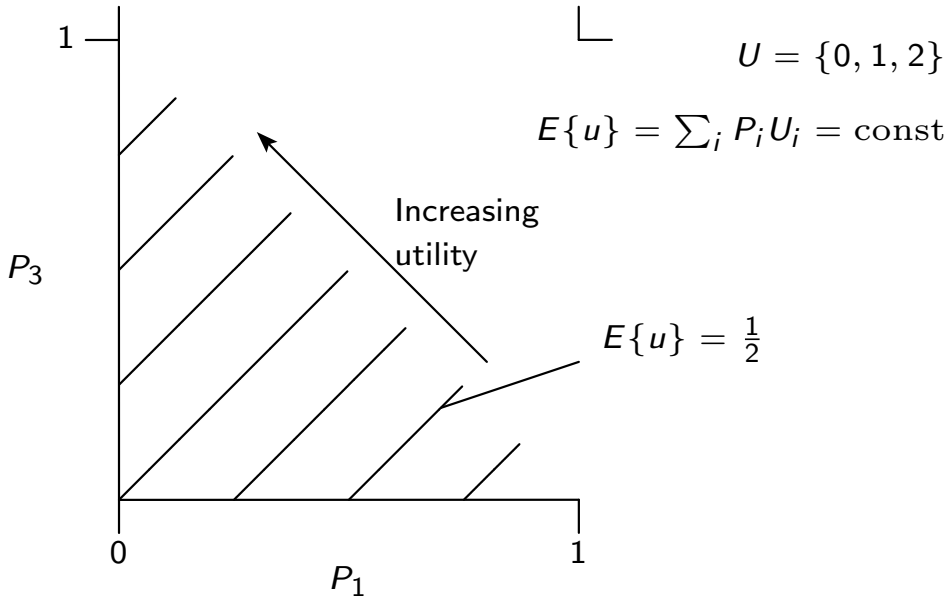
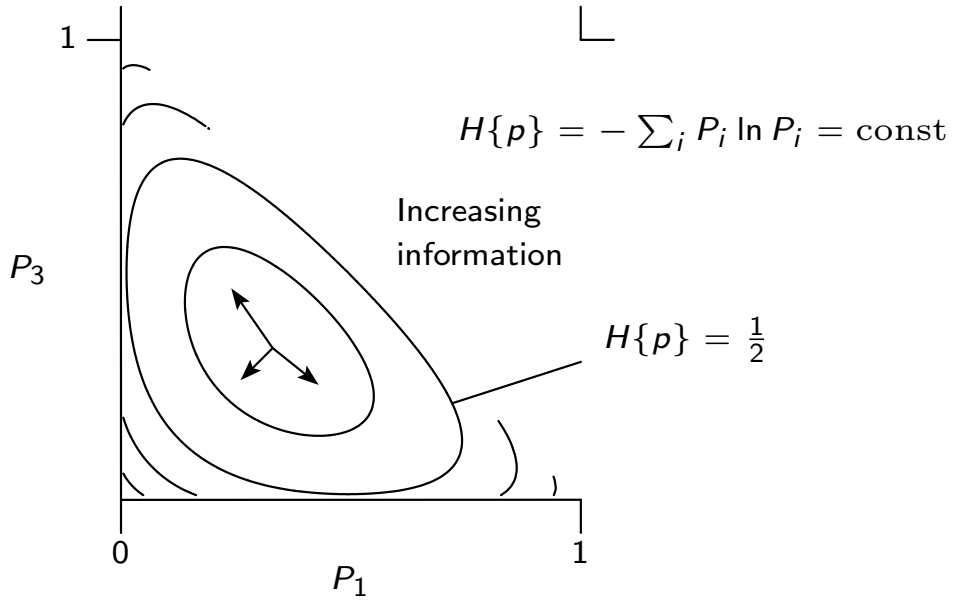
$$\int_{t_1}^{t_2} F(y(t))\,dt \geq \int_{\beta_2^{-1}}^{\beta_1^{-1}} C(\beta^{-1})d\beta^{-1} = \Phi(\beta^{-1})\Big|_{\beta_2^{-1}}^{\beta_1^{-1}}$$

□

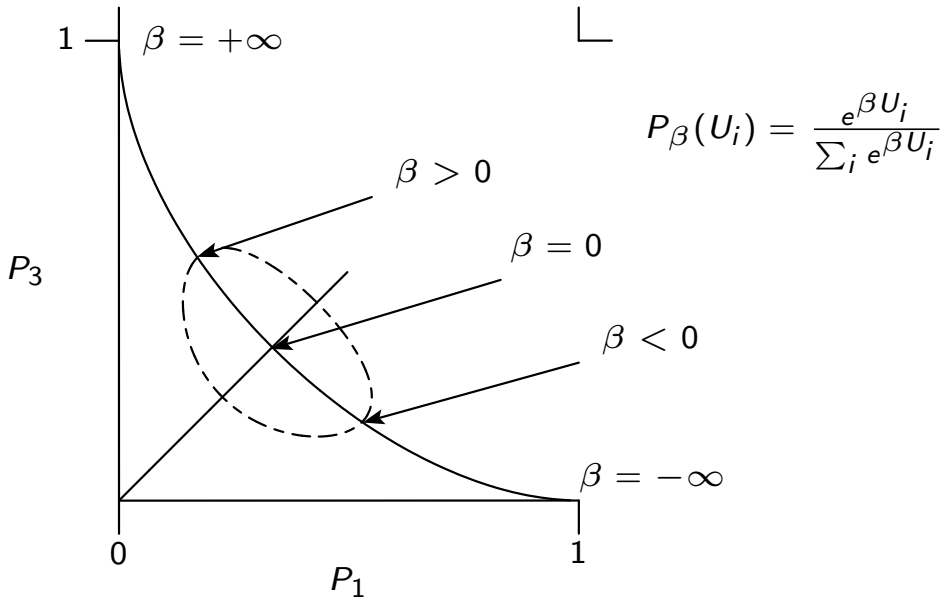
Expected utility



Information divergence



Information brachistochrone



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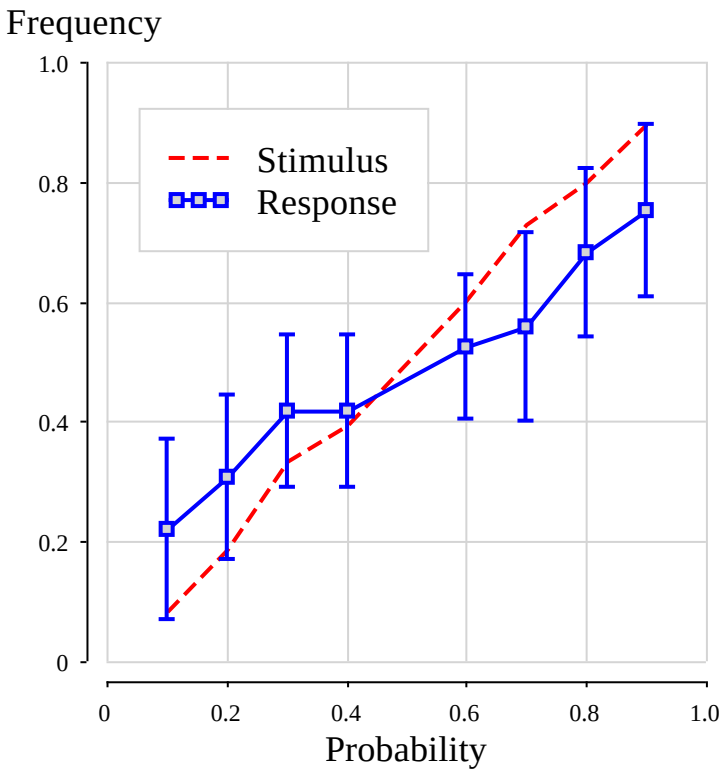
- The exponential and logarithmic functions are isomorphisms between $\mathbb{R}_{\max}$ and $\mathbb{R}_{+}$ is $x \mapsto e^x$ $(y \mapsto \ln y)$.

Probability matching



(Friedman et al., 1964)

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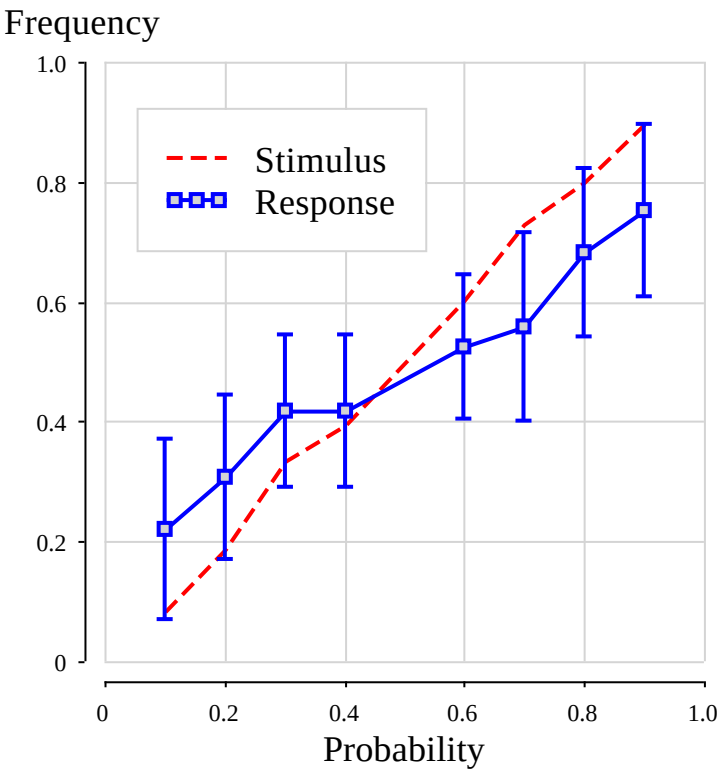

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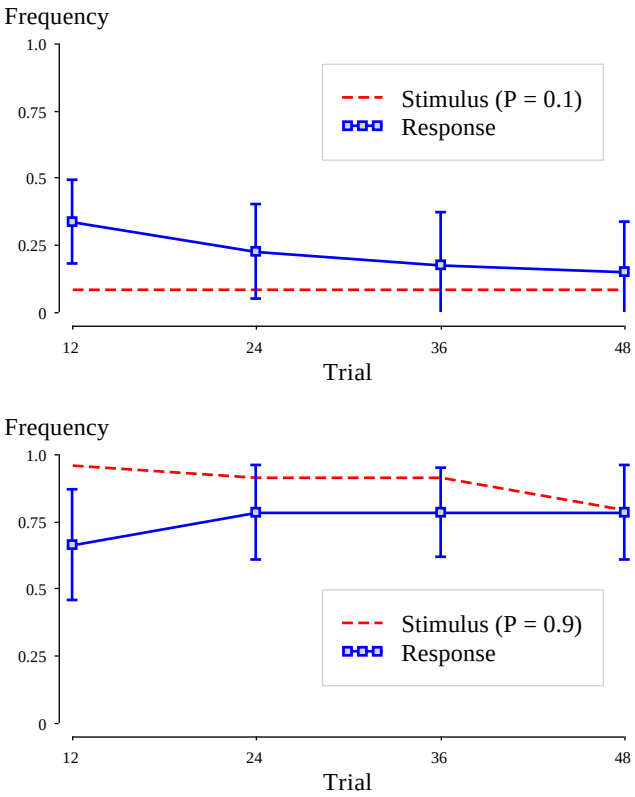
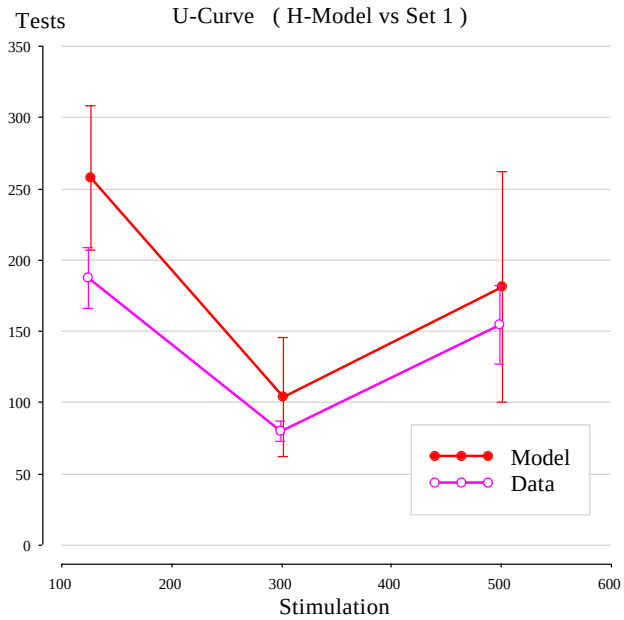
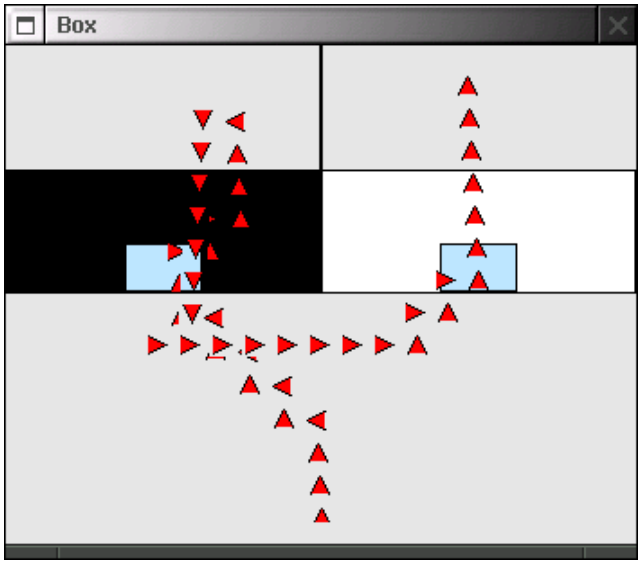


(Belavkin, 2001)

The inverted–U effect



(Yerkes & Dodson, 1908)



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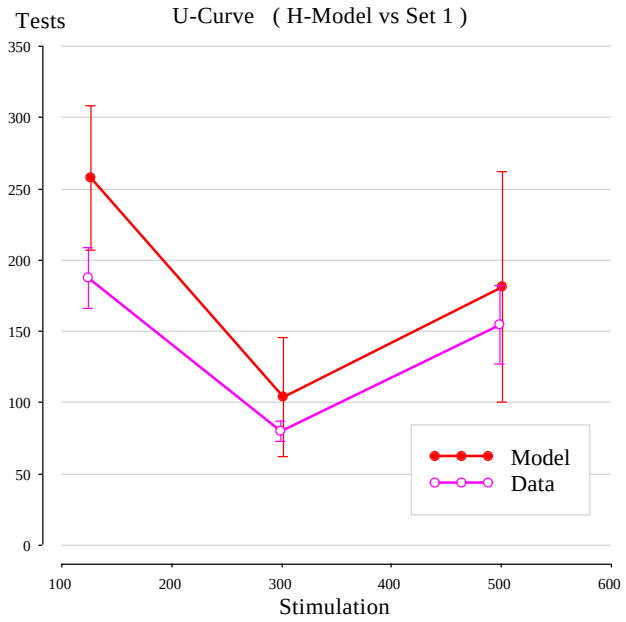

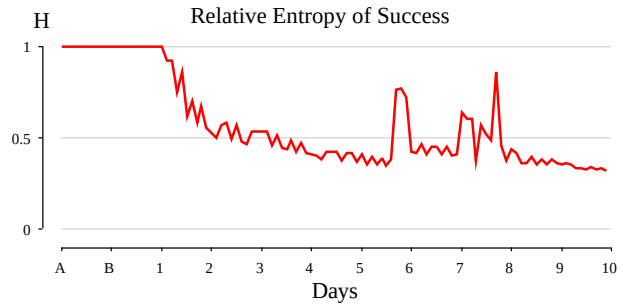

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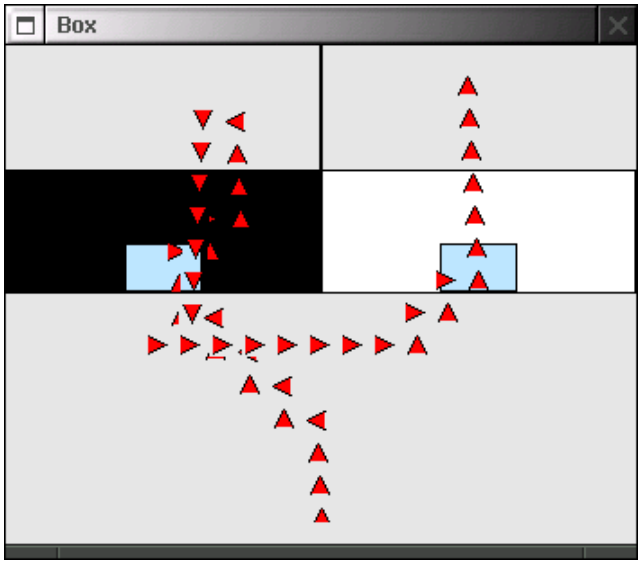

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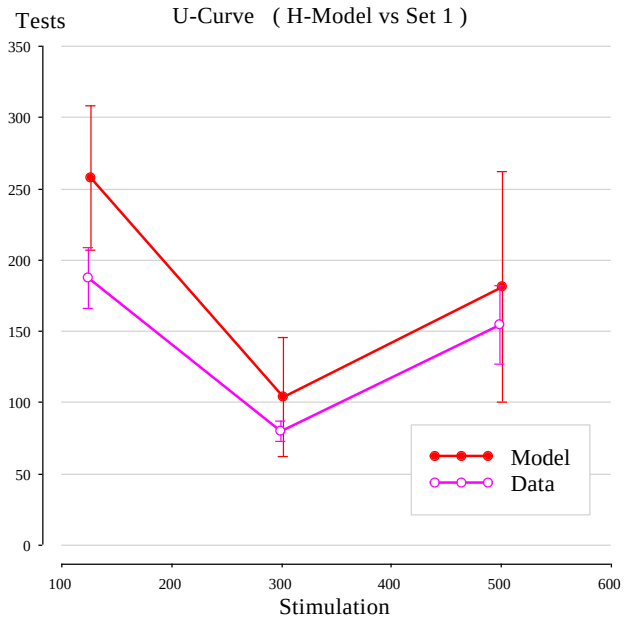

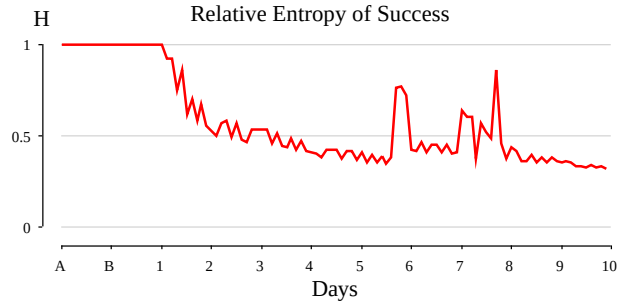

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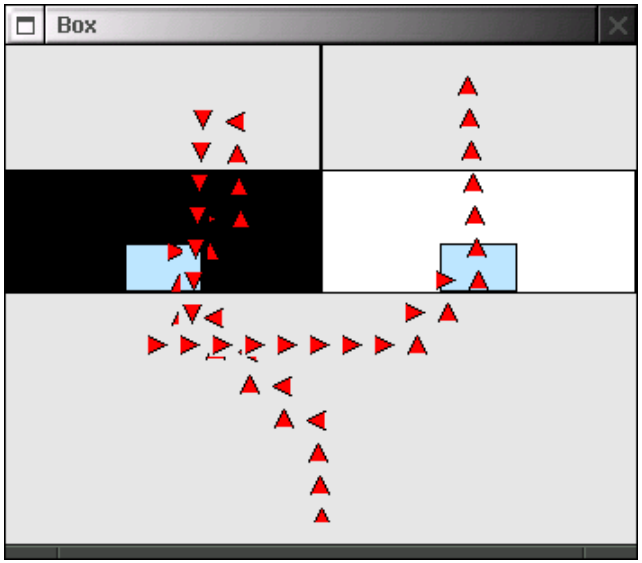

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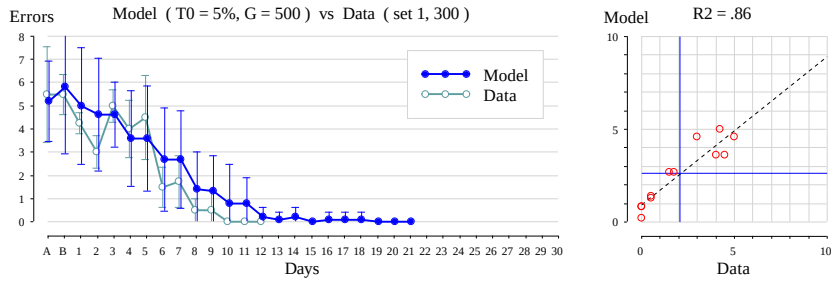
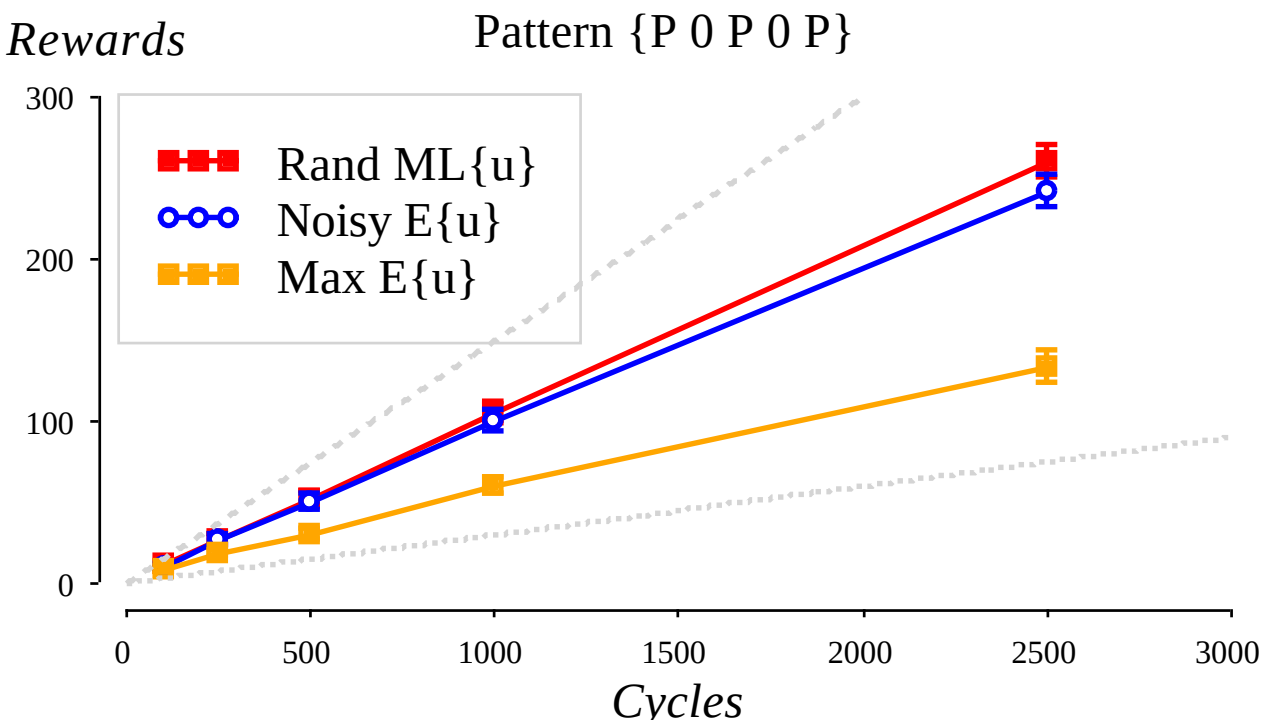


Optimal action selection

(Belavkin, 2005)



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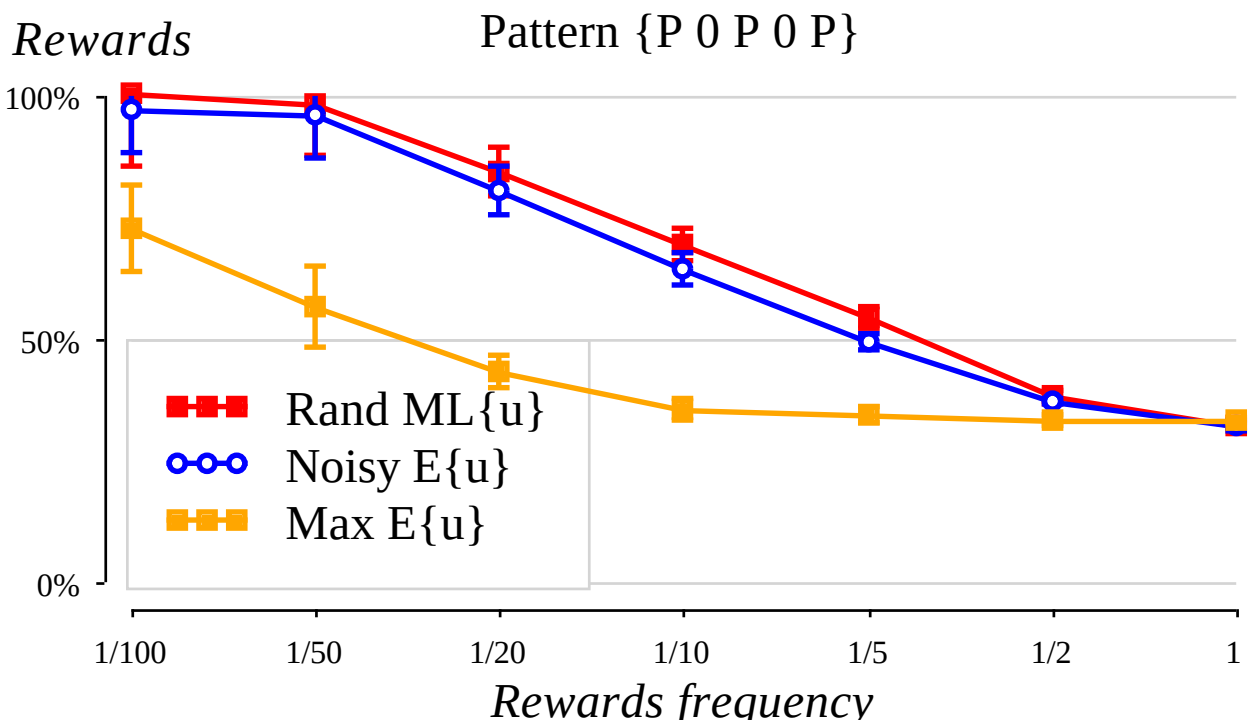

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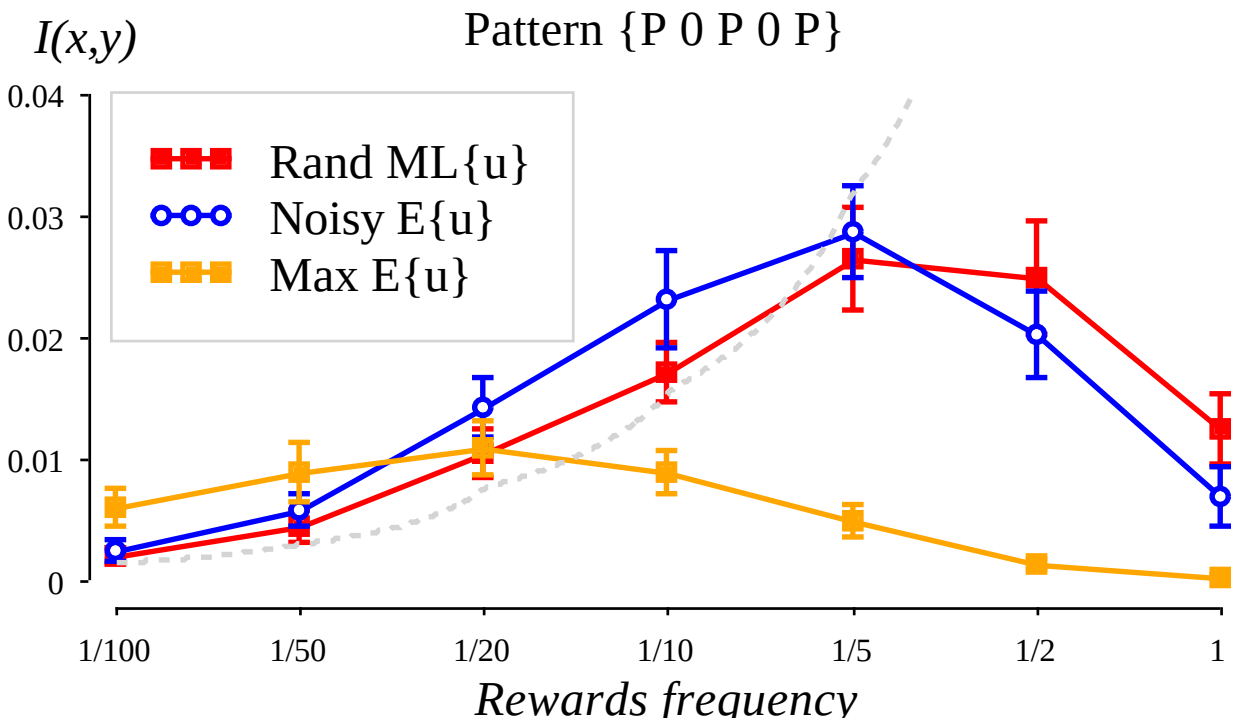
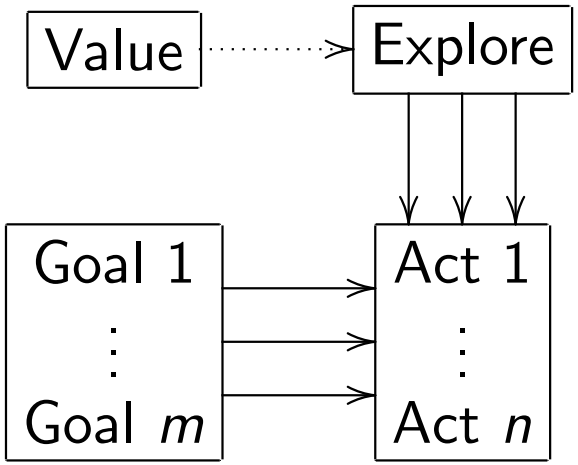


Learning in neural cell assemblies

(Belavkin & Huyck, 2008)



Conclusions

Questions?

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