Approximate graph products

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Motivation for the investigation of approximate graph products

A workable definition of k-approximate graph products

A proof that they can be recognized in polynomial time

The problem to decide when they are unique

Partial answers, that is, answers for similar problems

The need of fast algorithms and heuristics

Motivation for the investigation of approximate graph products – graph drawing.

Furthermore – theoretical biology and computational engineering:

In theoretical biology graph products arise in two rather different context

1. Sequence spaces, which are a convenient framework to discuss the evolution of genetic sequences, are Hamming graphs, that is, Cartesian products of complete graphs (Eigen, Dress, Rumschitzki, Stadler).

2. Topological theory of the relationships between genotypes and phenotypes.

If recombination and sexual inheritance are disregarded, this framework reduces to strong products of graphs (Stadler).

In both cases the graphs in question have to be obtained from computer simulations or need to be estimated from biological data.

They are known only approximately.

In computational engineering the objects that one wishes to investigate are modeled by grids.

This may yield complicated graphs that determine the systems of linear equations whose solutions have to be found repeatedly, fast, and accurately.

If the graphs are products or product-like one can build efficient equation solvers (Clemens Brand).

We thus wish to decompose large graphs into products or product-like graphs.

A framework is needed that allows us to deal with graphs that are only approximate products.

Similar problem: Feigenbaum and Haddad have studied the problems of minimal Cartesian product extensions and maximal Cartesian product subgraphs of arbitrary graphs.

Such problems arise in the design of computer networks and multiprocessing machines.

Both problems were shown to be NP–complete.

We need a different approach.

The distance $d(G, H)$ between two graphs G and H is at most k if they have two representations G',H' such that

$$|E(G') \triangle E(H')| \leq k.$$

A graph G is a k-approximate graph product if there is a product H such that

$$d(G, H) \leq k.$$

Trivi *Let G be a graph on n vertices. Then the number of graphs of distance $\leq k$ is $O(n^{2k})$.*

Proof Let H be a graph of distance $\leq k$ from G. Since isolated vertices are unimportant for the distance we can assume that $|V(H)| = |V(G)| + 2k$.

We estimate in how many ways a graph H of distance $\leq k$ from G can be constructed from G. We begin with $H = G$. Then we add $2k$ isolated vertices to $V(G)$. There are $(n+k)(n+k-1)/2 = O(n^2)$ ways to select a pair of vertices in in $V(H)$. If this pair is an edge in G we delete it from H, otherwise we add it to H. We do this k-times.

Clearly we get, up to isomorphisms, all graphs of distance $\leq k$ from G, and we can do this in $O(n^{2k})$ ways.

Trivi *k-approximate graph products (with respect to the Cartesian and the strong product) can be recognized in polynomial time.*

Proof This only requires a polynomial algorithm for graph factorization.

True for the graph products in question.

Problem *Under which conditions are these k-approximate graph products unique?*

Partial answers (Zerovnik and Imrich) in the case of weak reconstruction of graphs:

Theorem *Finite or infinite connected Cartesian products* G *can be recovered from any of their one vertex deleted subgraphs* $G \setminus v$*.*

Another uniqueness result (Imrich, Zerovnik, Zmazek) extends a conjecture by MacAvaney:

Theorem *Let* G *be a Cartesian product on at least* $k+1$ *prime factors on at least* $k+1$ *vertices each.*

Remove k *vertices – obtain* G'*.*

Then G *can be uniquely recovered from* G'*.*

How fast can we effect this reconstruction?

Zerovnik has an $O(mn(\Delta^2 + m \log n))$ algorithm that reconstructs nontrivial Cartesian products from single vertex deleted subgraphs.

n – number of vertices, m – number of edges, Δ – maximal degree

Currently we are looking for good algorithms for the recognition of k-approximate products.