18.03 Problem Set 1 Solutions

0. (a) $\frac{dv}{dt} = 9 - v^2$. Separate variables: $\frac{dv}{9-v^2} = dt$. Integrate this using partial fractions:

$$\frac{1}{9-v^2} = \frac{1}{6}\left(\frac{1}{3-v} + \frac{1}{3+v}\right)$$

so $-\ln|3-v| + \ln|3+v| = 6t + c$ or

$$\ln\left|\frac{3+v}{3-v}\right| = 6t + c\,.$$

This is an implicit solution, but we can go ahead and solve for v. Exponentiate and eliminate the absolute values (and return the "missing solution"):

$$\frac{3+v}{3-v} = Ce^{6t} \tag{1}$$

Multiply through by the denominator and collect terms involving v: $(1+Ce^{6t})v = 3(Ce^{6t}-1)$: $v = 3\,\frac{Ce^{6t}-1}{Ce^{6t}+1}$. This is more transparent when written

$$v = 3\,\frac{C - e^{-6t}}{C + e^{-6t}}\,.$$

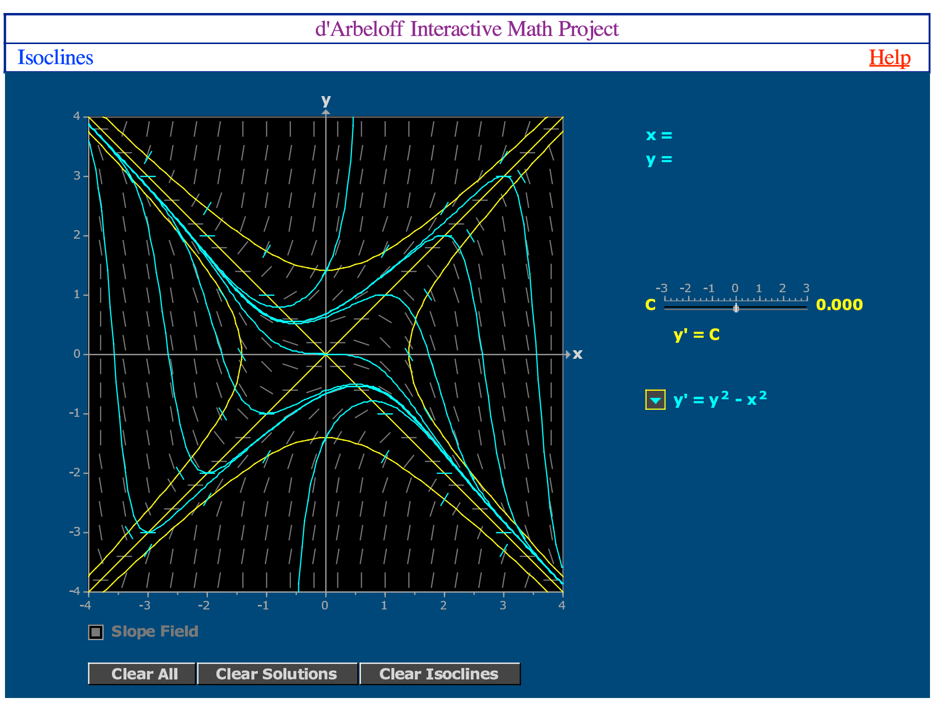
(b) From (1) with $v = 0 = t$, $1 = C$: so the particular solution is

$$v = 3\,\frac{1 - e^{-6t}}{1 + e^{-6t}}\,.$$

(c) Yes: as $t \to \infty$, $v \to 3$, no matter what C is.

(d) When $v < 0$ the equation is asserting that friction is decreasing the velocity even further, i.e., increasing the speed.

1. (a)



(b) By clicking on various points on the y axis and watching the fate of the solutions, $.66 < y_0 < .68$.

(c) Actually, *all* solutions with $y(0) > 0$ always lie above the graph of a certain function $f(x) =$ for $x > 0$. Of course the best function wtih this property is the solution of the differential equation itself satisfying $y(0) = 0$. More concretely but less precisely, the function $f(x) = -x$ will serve: it is on the nullcline, and its slope is negative, so solutions which cross it must cross it "from below." Thus solutions which begin (at $x = 0$) above it must stay above it.

(d) It is a general fact that critical points of solutions lie on the null-cline: the solution has zero derivative at a critical point, and the derivative equals the slope of the direction field. The null-cline of this equation is $y = \pm x$. From the applet it is clear that the critical points are maxima for $x > 0$ and minima for $x < 0$. This can be checked analytically too: The second derivative of a solution can be read off by differentiating the equation $y'' = 2yy' - 2x$. This can be re-expressed entirely in terms of x and y, to find an equation for the locus of points of inflections of solutions, but we are interested in the sign of the second derivative along the null-cline. There, $y' = 0$, so $y'' = -2x$. This is negative just when $x > 0$, so that is where you get maxima.

2. (F 10 Feb) (a) We compute y_k for a few small values of k, for n fixed but arbitrary. Here's the table.

k	x_k	y_k	$A_k = y_k$	hA_k
0	0	1	1	h
1	h	$1+h$	$1+h$	$h(1+h)$
2	$2h$	$(1+h)+h(1+h) = (1+h)^2$	$(1+h)^2$	$h(1+h)^2$
3	$3h$	$(1+h)^2+h(1+h)^2 = (1+h)^3$	$(1+h)^3$	$h(1+h)^3$
.	$\cdots$	$\cdots$	$\cdots$	$\cdots$

It's fair to conclude that $y_k = (1+h)^k$ for all k, so in particular $y_n = (1+h)^n$. Since $h = 1/n$, this is $(1+(1/n))^n$. [A standard calculus exercise shows that $\lim_{n\to\infty}(1+(1/n))^n = e$.]

(b)

h	$y_{est}(1)$	Error	Error$/h$	Error$/h^2$
1	1.00	.58	.58	.58
.5	1.24	.34	.68	1.36
.25	1.39	.19	.76	3.04
.125	1.48	.10	.80	6.40
Actual	1.58			

The quotients Error$/h$ are not exactly constant, but they do seem to be convering (to .8??) and they are much closer to constant than the quotients by h^2. You can't expect the error term to behave too regularly, after all—if it did, you could use that regularity to compute the actual answer exactly!

(c) the actual is larger than the estimates. The direction field is increasing in this range.

(d) Separate variables: $dy/y = \sin t\, dt$ so $\ln|y| = -\cos t + c$ and $y = Ce^{-\cos t}$. $y(0) = 1$ forces $C = e$ so $y = e \cdot e^{-\cos t}$, or $y = e^{1-\cos t}$. When $t = 1$ a calculator gives 1.5835952.

3. (M 13 Feb) (a) Write x for the principle or loan balance. Over a short period of time, we have to add the interest on that money and subtract the payment we have made:

$$x(t + \Delta t) \simeq x(t) + Ix(t)\Delta t - q\Delta t$$

Check units here: I is measured in years^{-1} and Δt in years, so the second term on the right has units of dollars. So does the third term.

Now put $x(t)$ on the left, divide by Δt, and take the limit:

$$\frac{dx}{dt} = Ix - q\,.$$

(b) Separate: $\dfrac{dx}{Ix - q} = dt$, $I^{-1} \ln |Ix - q| = t + c$, $Ix - q = Ce^{It}$, $x = I^{-1}(q + Ce^{It})$.

(c) $x(T) = I^{-1}(q + Ce^{IT})$ so $C = -qe^{-IT}$ and $x = (q/I)(1 - e^{I(t-T)})$. Finally, $M = x(0) = (q/I)(1 - e^{-IT})$ implies that $q = \dfrac{MI}{1 - e^{-IT}}$.

(d) A typical mortgage has $T = 30$ and $I = 0.05$. Estimate the *monthly* payments if $M = 10^5$.

With $T = 30$ and $I = 0.05$ (not 0.5—that would be usury!), we compute $I/(1 - e^{-IT}) \simeq 0.06430846$. Multiply by $M = 10^5$ to get annual payments of $q = \$6436.0846$, or monthly payments of $q/12 = \$536.34$.

Over the life of the loan you have to pay the bank $IT/(1 - e^{-IT})$ for each dollar you borrow. With $I = 0.05$ and $T = 30$ this comes to 1.9308254.