18.03 Practice Hour Exam I, February, 2006

1. What is the solution of the ODE $\frac{dy}{dx} = 2xy^2$ for which $y(0) = 1$?

What is x when y reaches infinity?

2. Suppose that the rate of increase of the population of the country Brobdingnag proportional to the current population, with some fixed proportionality constant a (in units of year^{-1}). At a certain point, say at year 0, the Brobdingnagian government decides to foster immigration, which thenceforth occurs at the constant level of b individuals per year (continuously, of course). It is assumed that the immigrants assimilate immediately and do not affect the rate a.

Set up the ODE controlling this population $p(t)$. You do not need to solve it, though.

3. This problem concerns $\dot{x} = tx^2$.

(a) What is the general solution? (Don't forget the "lost" solution(s).)

(b) All but one of these solutions blows up in finite time (that is, they have vertical asymptotes). Which solution does not blow up? At what time does the solution with $x(0) = 1$ blow up?

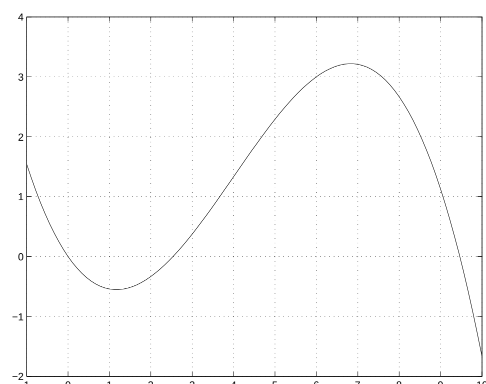
4. What is the general solution to $(t^2 + 1)\dot{x} + 2tx = 1$?

5. Find the general solution to $t\dot{x} = 4t - 3x$.

6. Find the sinusoidal solution of $\dot{x} + 2x = \sin(3t)$.

What is the amplitude of this solution?

7. The population of Whooping Cranes has been found to obey the equation $\dot{x} = g(x)$ (in appropriate units). Here's a graph of this function for x between -1 and 10.



(a) Sketch the corresponding direction field, and the phase line. Identify the equilibria and classify them as stable or unstable.

(b) The Whooping Crane population crashed in the 1930's. The US is attempting to bring this species back to a viable population by releasing captive-raised birds. For simplicity suppose the release happens at a constant rate. What is the minimal rate of release, $r = r_0$, so that any greater rate of release eventually results in a population something like the original healthy population, no matter how small the starting population? (Identify r_0 in terms of the graph above.)

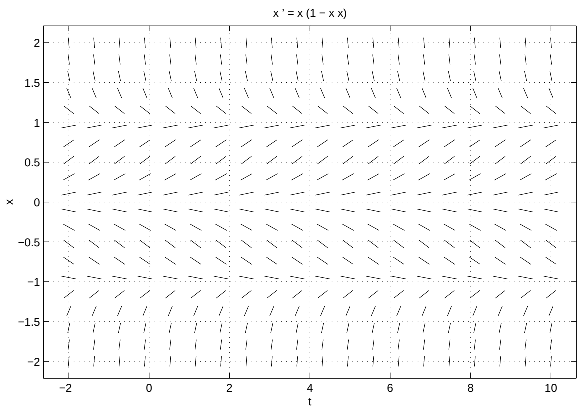
8. (a) Find real numbers r and θ such that $\sqrt{3} - i = re^{i\theta}$.

(b) For a certain complex number $a + bi$, the curve in the complex plane parametrized by $e^{(a+bi)t}$ spirals away from the origin and passes through 1 with a velocity vector making an angle of $-45°$ with the positive real axis, and speed $\sqrt{2}$. What is $a+bi$?

(c) Find a, b, and ω such that $\operatorname{Re} \dfrac{e^{2\pi i t}}{1+i} = a\cos(\omega t) + b\sin(\omega t)$.

9. Use two steps of Euler's method to estimate the value $y(1)$ of the solution of the equation $y' = 1 + (x/y)$ having $y(0) = 1$. Will this estimate be too large or too small?

10. Here is part of the direction field of a certain first order autonomous ODE.
(a) Sketch the phase line. Mark all critical points, and label them as stable or unstable.
(b) Any first order autonomous equation has the form $\dot{x} = g(x)$. Sketch the graph of $g(x)$ in this example.



11. **(a)** Compute the real and imaginary parts of $e^{(-1+i)t}/(1+i)$.

(b) Sketch the curve in the complex plane parametrized by $2e^{(-1+i)t}$.

(c) Express the real and imaginary parts of $e^{2it}/(1+i)$ in the form $A\cos(\omega t - \phi)$, and sketch a graph of each.

12. Suppose $\dot{x} + 3x = 4\cos(2t)$.

(a) Write down a complex valued ODE of which $\dot{x} + 3x = 4\cos(2t)$ is the real part.

(b) What is the is the periodic solution of $\dot{x} + 3x = 4\cos(2t)$? (Write it in the form $a\cos(\omega t) + b\sin(\omega t)$.)

(c) What is the transient in the solution with $x(0) = 0$?

Solutions

1. $dy/y^2 = 2x\,dx$ integrates to $-y^{-1} = x^2 + c$ or $y = 1/(c - x^2)$. $1 = y(0) = 1/c$ implies $c = 1$, so $y = 1/(1 - x^2)$ is an equation satisfied by the solution.

It reaches infinity when $x = 1$.

2. $\dot{p} = ap + b$.

3. **(a)** $x^{-2}\,dx = t\,dt$ so $-x^{-1} = t^2/2 + c$ or $x = 2/(c - t^2)$ (for a different c). There is also the solution $x = 0$, which was lost when I divided by x^2.

(b) The constant solution $x = 0$ is the only one that does not reach infinity in finite time. The solution with $x(0) = 1$ has $c = 2$, so its denominator becomes zero when $t = \sqrt{2}$.

4. The left hand side is the derivative of $(t^2+1)x$, so $(t^2+1)x = \int dt = t + c$ or $x = (t+c)/(t^2+1)$.

5. $t\dot{x} + 3x = 4t$. By inspection (or by rewriting this in standard form and discovering the integrating factor t^2, or by using variation of parameter), $(d/dt)(t^3 x) = t^3\dot{x} + 3t^2 x = 4t^3$, so $t^3 x = t^4 + c$ or $x = t + ct^{-3}$.

6. Here is one of the many ways to do this. Replace it with the complex ODE $\dot{z}+2z = e^{3it}$, of which it is the imaginary part. The Exponential Response Formula gives a solution $z_p = e^{3it}/(3i+2)$. The imaginary part of this will be the sinusoidal solution of the original equation. Since $1/(2+3i) = (2-3i)/13$, this is $x_p = (1/13)(2\sin(3t) - 3\cos(3t))$.

The amplitude is $1/\sqrt{13}$.

7. (a) The phase line shows stable equilibria at $x = 0$ and $x = 9.5$ and an unstable equilibrium at about $x = 2.5$. Direction arrows point up towards 0, down between 0 and 2.5, up between 2.5 and 9.5, and down above 9.5.

(b) r_0 is the negative of the value of $g(x)$ at its minimum, so r_0 is about 0.5.

8. (a) $r = 2$ and $\theta = -\pi/6$ or $11\pi/6$ or $-\pi/6$ plus any other integer multiple of 2π; or $r = -2$ and $\theta = 5\pi/6$ plus any integer multiple of 2π.

(b) The velocity vector is $1 - i$, which must be $a + bi$.

(c) $((1-i)/2)(\cos(2\pi t) + i\sin(2\pi t))$ has real part $(1/2)(\cos(2\pi t) + \sin(2\pi t))$. One could also compute (for partial credit) $1+i = \sqrt{2}e^{\pi i/4}$, $1/(1+i) = (1/\sqrt{2})e^{-\pi i/4}$, so $e^{2\pi it}/(1+i) = (1/\sqrt{2})e^{(2\pi t-\pi/4)i}$ has real part $(1/\sqrt{2})\cos(2\pi t - \pi/4)$.

9. $h = .5$.

k	x_k	y_k	$A_k = 1 + (x_k/y_k)$	hA_k
0	0	1	1	1/2
1	1/2	3/2	4/3	2/3
2	1	13/6		

The direction field is increasing along the first segment, as you can see: at the start the value is $A_0 = 1$, at the end of that segment it is $A_1 = 4/3$. So the estimate is too low.

10. (a) $>> * << * >> * <<$ where the equilibria are at $-1, 0, +1$.

(b) $g(x)$ is positive for $x < -1$, negative for $-1 < x < 0$, positive for $0 < x < 1$, and negative again for $x > 1$.

11. (a) $e^{(-1+i)t}/(1+i) = ((1-i)/2)e^{-t}(\cos t + i\sin t)$ has real part $(1/2)e^{-t}(\cos t + \sin t)$ and imaginary part $(1/2)e^{-t}(-\cos t + \sin t)$.

(b) This is a spiral which revolves counterclockwise and decays to zero as t increases. When $t = 0$ it passes through the complex number 2.

(c) This can be done by expanding out into real and imaginary parts and using the a, b, A, ϕ triangle. Another way is to begin by expressing $1/(1+i)$ as $Ae^{-\phi i}$. The magnitude of $(1+i)$ is $\sqrt{2}$ and the argument is $\pi/4$, so $1/(i+i) = (1/\sqrt{2})e^{-\pi i/4}$. Thus $e^{2it}/(1+i) = (1/\sqrt{2})e^{(2t-(\pi/4))i}$. The real part of this is $(1/\sqrt{2})\cos(2t - \pi/4)$, and the imaginary part is $(1/\sqrt{2})\sin(2t - \pi/4)$.

12. (a) $\dot{z} + 3z = 4e^{2it}$.

(b) The exponential solution to the complex equation is $z_p = (4/(2i+3))e^{2it}$, by substituting to find the constant or by the Exponential Response Formula. Expanding, $z_p = (4/13)(3-2i)(\cos(2t) + i\sin(2t))$, which has real part $x_p = (4/13)(3\cos(2t) + 2\sin(2t))$.

(c) $x_p(0) = 12/13$, so the transient is $-(12/13)e^{-3t}$.