Learning bounded unions of Noetherian closed set systems via characteristic sets

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Key Concepts

- In this talk “learning” is identification in the limit from positive data.
- Noetherian Closed Set Systems are classes of languages with algebraic structure as targets of learning.
- Characteristic sets are used for sufficient conditions for learnability of language classes. They have also another meaning in regarding languages as algebraic objects.

Results

- Previous work:

  It is shown that bounded unions of languages are learnable theoretically[Kobayashi, Kapur et al.] and several learning procedures have been proposed on various viewpoints.

- This talk:

  We give a schema of learning procedure on bounded unions of certain class of languages, called Noetherian closed set system, by using characteristic sets.

Outline

- Inductive Inference and Algebra
- Precise Definitions of Noetherian Closed Set System and Characteristic sets
- Main Result
- Example

The purpose of our research

- Clarifying the properties of learning languages with algebraic structure on both the viewpoint of learning and mathematics.
- Current subject

  Learning bounded unions of languages, which have not attracted mathematicians, but are very popular for learning people.

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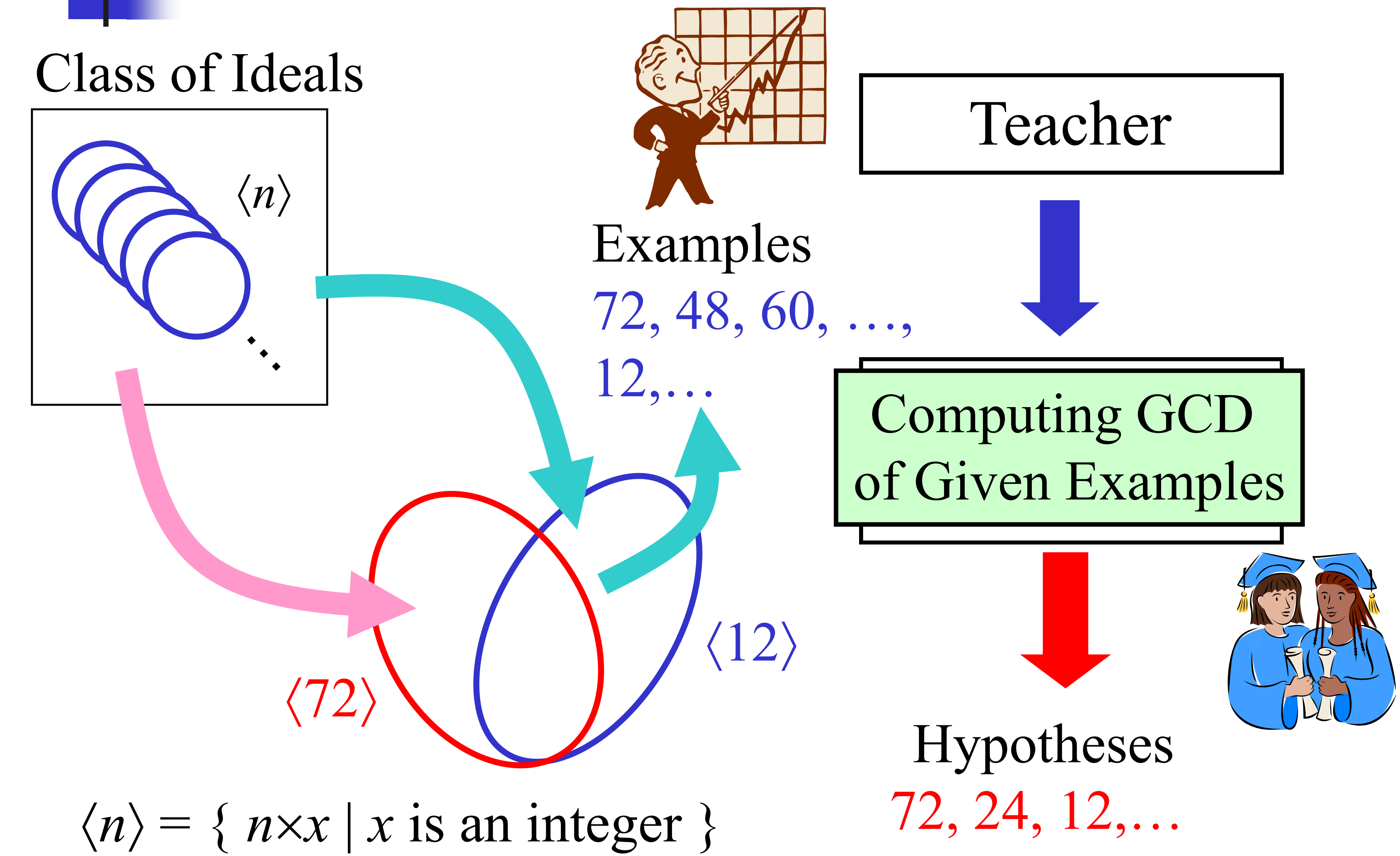
Inductive Inference and Algebra

- Some learning people have investigated learning languages with algebraic structure:
  - ideals of Z [Angluin]
  - pattern languages [Angluin]
  - ideals in commutative rings [Stephen & Ventsov]

Identification of Ideals of Z from positive data



Algebra and Inductive Inference

- Recently some researchers of History of Mathematics have found that the original version of Hilbert’s basis theorem can be regarded as learning. This means Mathematician used "learning" in algebra in late 19th century.
- In fact, Hilbert’s basis theorem can be regarded as learning.

Hilbert's original paper

16. Über die Theorie der algebraischen Formen[1].

[Mathem. Annalen Bd. 36, S. 473—534 (1890).]

I. Die Endlichkeit der Formen in einem beliebigen Formensysteme.

Unter einer algebraischen Form verstehen wir in üblicher Weise eine ganze rationale *homogene* Funktion von gewissen Veränderlichen, und die Koeffizienten der Form denken wir uns als Zahlen eines bestimmten Rationalitätsbereiches. Ist dann durch irgend ein Gesetz ein System von unbegrenzt vielen Formen von beliebigen Ordnungen in den Veränderlichen vorgelegt, so entsteht die Frage, ob es stets möglich ist, aus diesem Formensysteme eine endliche Zahl von Formen derart auszuwählen, daß jede andere Form des Systems durch lineare Kombination jener ausgewählten Formen erhalten werden kann, d. h. ob eine jede Form des Systems sich in die Gestalt

$$F = A_1 F_1 + A_2 F_2 + \cdots + A_m F_m$$

bringen läßt, wo $F_1, F_2, \ldots, F_m$ bestimmt ausgewählte Formen des gegebenen Systems und $A_1, A_2, \ldots, A_m$ irgendwelche, dem nämlichen Rationalitätsbereiche angehörige Formen der Veränderlichen sind. Um diese Frage zu entscheiden, beweisen wir zunächst das folgende für unsere weiteren Unter-

Hilbert's original paper

suchungen grundlegende Theorem:

Theorem I. *Ist irgend eine nicht abbrechende Reihe von Formen der n Veränderlichen $x_1, x_2, \ldots, x_n$ vorgelegt, etwa $F_1, F_2, F_3, \ldots$, so gibt es stets eine Zahl m von der Art, daß eine jede Form jener Reihe sich in die Gestalt*

$$F = A_1 F_1 + A_2 F_2 + \cdots + A_m F_m$$

bringen läßt, wo $A_1, A_2, \ldots, A_m$ geeignete Formen der nämlichen n Veränderlichen sind.

Die Ordnungen der einzelnen Formen der vorgelegten Reihe sowie ihre Koeffizienten unterliegen keinerlei Beschränkungen. Denken wir uns die letzteren als Zahlen eines bestimmten Rationalitätsbereiches, so dürfen wir annehmen, daß die Koeffizienten der Formen $A_1, A_2, \ldots, A_m$ dem nämlichen

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[1] Vgl. die vorläufigen Mitteilungen des Verfassers: „Zur Theorie der algebraischen Gebilde", Nachr. Ges. Wiss. Göttingen 1888 (erste Note) und 1889 (zweite und dritte Note). Dieser Band Abh. 13 bis 15.

Why Set-Union of Ideals?

- In Mathematics :

  The set-union of two ideals has not been interested because it is difficult to give its “meaning”.

- In Learning Theory :

  Finite elasticity of a class of languages ensures identifiability of set union of two languages in it, but does not effective procedure to compute its finite tell-tale.

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Ideals in Commutative Ring

- A subset I of a ring R is an ideal if the followings are satisfied:

$$0 \in I$$

$$\text{If } f \in I \text{ and } g \in I,\ f + g \in I$$

$$\text{If } f \in I \text{ and } h \in R,\ h f \in I$$

Examples

- $\langle n \rangle = \{\, x \times n \mid x \in Z \,\}$ for every $n \geq 0$ in Z
- $\langle f, g \rangle = \{\, h f + kg \mid h, k \in \mathrm{Q}[X_1, \ldots, X_n]\}$

  for every $f, g \in \mathrm{Q}[X_1, \ldots, X_n]$

Closed Set Systems(CSS)

- A mapping $C: 2^U \rightarrow 2^U$ is a closure operator if it satisfies:

  $$X \subset C(X)$$

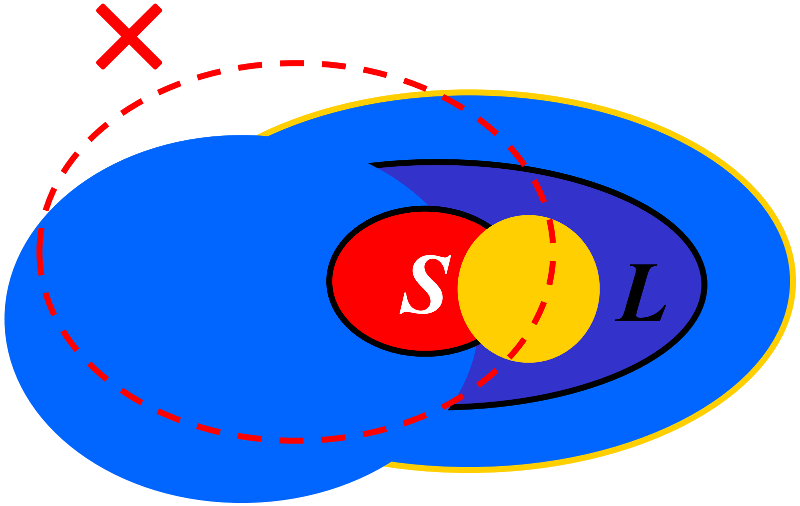
  $$C(C(X)) = C(X)$$

  $$X \subset Y \Rightarrow C(X) \subset C(Y) \quad (\forall X, Y \subset U).$$

- $X \subset U$ is closed if $C(X) = X$.
- A closed set system is a class of closed sets.

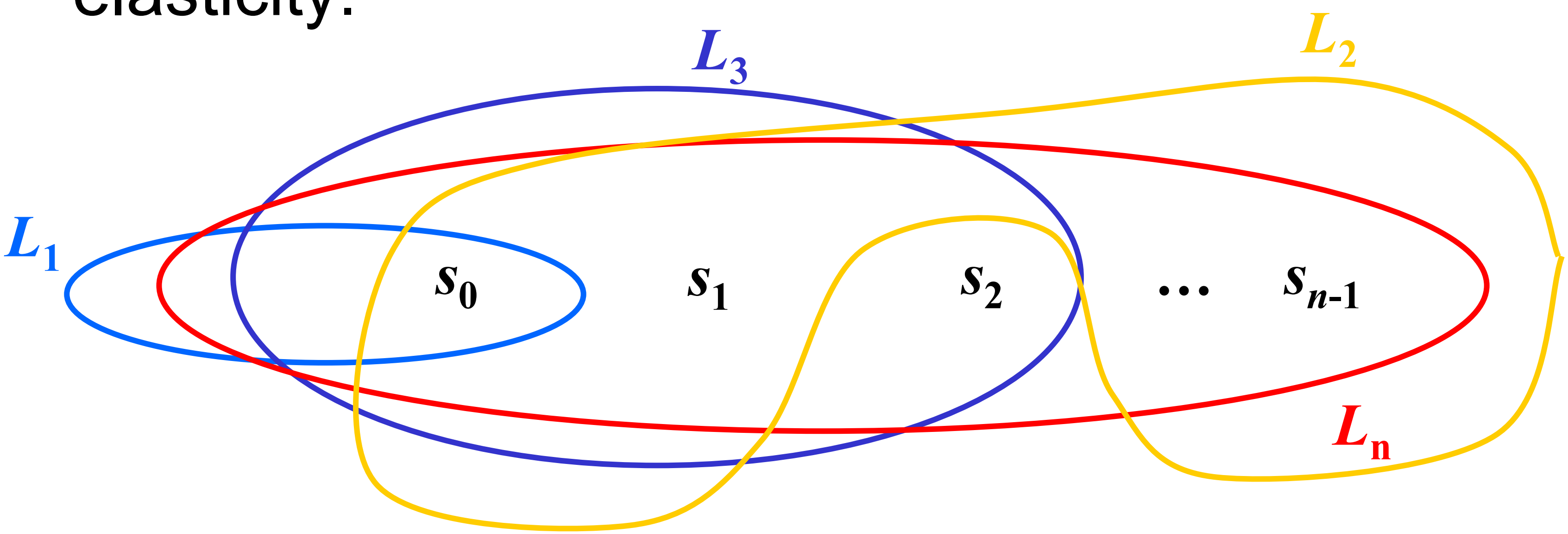
Characteristic Sets

- A finite subset $S \subset L \in \mathcal{L}$ is a characteristic set of L in $\mathcal{L} \Leftrightarrow \forall L' \in \mathcal{L}, S \subset L' \Rightarrow L \subset L'$.



Finite Elasticity(F.E.)

- $\mathcal{L}$ has infinite elasticity if there exist an infinite sequence of elements $s_0, s_1, \ldots$ and languages $L_1, L_2, \ldots$ such that $\{s_0, s_1, \ldots, s_{n-1}\} \subset L_n$ and $s_n \notin L_n$.
- $\mathcal{L}$ has finite elasticity $\Leftrightarrow$ $\mathcal{L}$ does not have infinite elasticity.



Closed Set System and F.E.

- If $\mathcal{L}$ is CSS, then

Theorem[de Brecht et al.]

$\mathcal{L}$ has F.E. $\Leftrightarrow$ $\mathcal{L}$ satisfies the ascending chain condition, i.e. $\mathcal{L}$ has no infinite chain of languages

$$L_1 \subsetneq L_2 \subsetneq \dots \subsetneq L_n \subsetneq \dots.$$

- A Noetherian closed set system(NCSS) is a CSS that has F.E..

Bounded Unions of Languages

- $\mathcal{L}$: a class of language.

$\cup^{\leq k}\mathcal{L} := \{ L_1 \cup \ldots \cup L_m \mid \forall L_i \in \mathcal{L},\ m \leq k \}$.

- Theorem[Motoki-Shinohara-Wright] If $\mathcal{L}$ has finite elasticity, then $\cup^{\leq k}\mathcal{L}$ also has.

Bounded Unions of NCSS

$\mathcal{L}$: Noetherian closed set system.

$$\Downarrow$$

1. $\mathrm{U}^{\leq k}\mathcal{L}$ has finite elasticity.
2. Every element of $\mathrm{U}^{\leq k}\mathcal{L}$ has a characteristic set.
3. $\mathrm{U}^{\leq k}\mathcal{L}$ is identifiable from positive data.

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Main Result

- Suppose that $\mathcal{L}$ is NCSS and $\cup^{\leq k}\mathcal{L}$ is compact. We give an algorithm schema for learning $\cup^{\leq k}\mathcal{L}$ under the condition:

  for $\forall L \in \mathcal{L}$, a char. set of L in $\cup^{\leq k}\mathcal{L}$ is computable from char. set of L in $\mathcal{L}$.

- Compactness

  $\cup^{\leq k}\mathcal{L}$ is compact iff $L \subset L_1 \cup \ldots \cup L_m \Rightarrow \exists i$ s.t. $L \subset L_i$ $(\forall L \in \mathcal{L}, L_1 \cup \ldots \cup L_m \in \cup^{\leq k}\mathcal{L})$.

Learning Schema

Target: $L_1 \cup \ldots \cup L_m \in \cup^{\leq k} \mathcal{L}$

Positive data: $f_1, f_2, \ldots, f_n, \ldots$

Step n: By using hypergraph, construct a hypothesis $H \in \cup^{\leq k} \mathcal{L}$ s.t.

- $H \subset L_1 \cup \ldots \cup L_m$,
- H contains elements of $\{f_1, \ldots, f_n\}$ as many as possible

from $\{f_1, \ldots, f_n\}$.

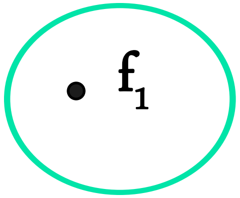
Construction of Hypergraph

- Step 1

  Set of examples: $\{f_1\}$

  Set of vertices $\mathcal{V}_1$: $\{f_1\}$

  Set of hyperedges $\mathcal{HE}_1$: $\{\{f_1\}\}$



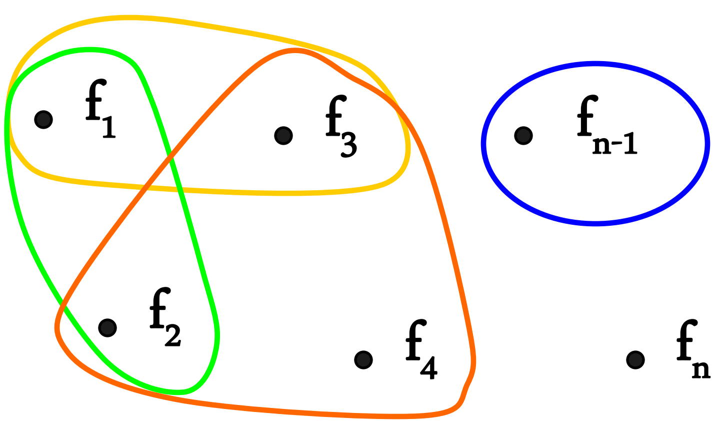
Construction of Hypergraph

- Step $_n$

  Set of examples: $\{f_1,\ldots,f_n\}$

  Set of vertices $\mathcal{V}_n$: $\{f_1,\ldots,f_n\} = \mathcal{V}_{n-1} \cup \{f_n\}$

  1. Set $\mathcal{HE}_n = \mathcal{HE}_{n-1}$.



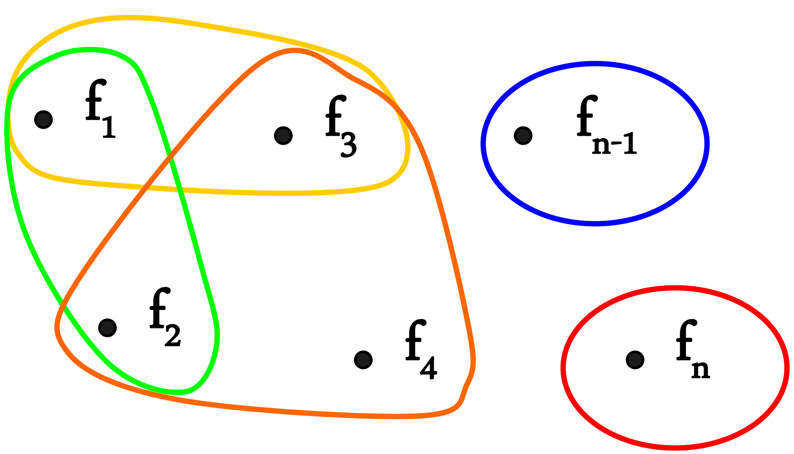
Construction of Hypergraph

2. For $\forall s \subset \mathcal{V}_n$ do

   If a char. set of $c(s)$ in $\cup^{\leq k}\mathcal{L}$ is contained by $\mathcal{V}_n$, then

   Add s to $\mathcal{HE}_n$, and remove all hyperedges contained by s.

3. If no hyperedge has f_n, then add $\{f_n\}$.



Learning Algorithm of $U^{\leq k}\mathcal{L}$

Repeat

1. Put $n=1$.
2. Construct a hypergraph $\mathcal{G}_n$ from $\{f_1,\ldots,f_n\}$.
3. Choose at most k maximal hyperedges of $\mathcal{G}_n$ w.r.t. some ordering.
4. Output (at-most) k-tuple in 3.
5. Add 1 to n.

forever.

The ordering $\leq$ at 3 can be taken freely provided that

$$c(s) \subset c(s') \Rightarrow s \leq s'.$$

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Example (Polynomial ideal)

Let $\mathcal{I}$ be the class of all ideals of polynomial ring $\mathrm{Q}[x,y]$.

- Target: $\langle x^2,y^3\rangle \cup \langle x^3,y^2\rangle \in \cup^{\leq 2}\mathcal{I}$
- Positive data:

  $x^2,\ y^3,\ y^2,\ x^2+y^3,\ x^3,\ x^3+y^2, \ldots$
- For $\forall f,g \in \mathrm{Q}[x,y]$, $\{f,\ g,\ f+g\}$ is a characteristic set of $\langle f,g\rangle$ in $\cup^{\leq 2}\mathcal{I}$.

Example (Polynomial ideal)

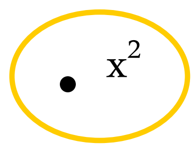
Target: $\langle x^2, y^3 \rangle \cup \langle x^3, y^2 \rangle \in \cup^{\leq 2} I$

Positive data: $x^2,\ y^3,\ y^2,\ x^2+y^3,\ x^3,\ x^3+y^2, \ldots$

- Step 1

The set of examples: $\{x^2\}$

Hypergraph $\mathcal{G}_1$:



Output: $\langle x^2 \rangle$

Example (Polynomial ideal)

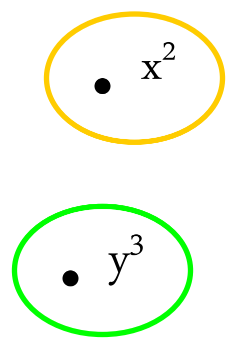
Target: $\langle x^2, y^3 \rangle \cup \langle x^3, y^2 \rangle \in \cup^{\leq 2} I$

Positive data: x^2, y^3, y^2, x^2+y^3, x^3, x^3+y^2,…

- Step 2

The set of examples: $\{x^2, y^3\}$

Hypergraph $\mathcal{G}_2$:



Output: $\langle x^2 \rangle \cup \langle y^3 \rangle$

Example (Polynomial ideal)

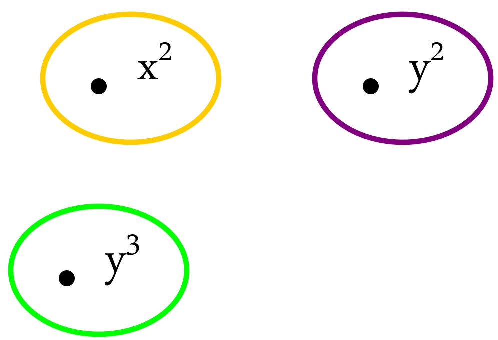
Target: $\langle x^2, y^3 \rangle \cup \langle x^3, y^2 \rangle \in \cup^{\leq 2} I$

Positive data: $x^2,\ y^3,\ y^2,\ x^2+y^3,\ x^3,\ x^3+y^2, \ldots$

- Step 3

The set of examples: $\{x^2,\ y^3,\ y^2\}$

Hypergraph $\mathcal{G}_3$:



Output: $\langle x^2 \rangle \cup \langle y^2 \rangle$

$c(S) \subset c(S') \Rightarrow S \leq S'.$

($\langle y^3 \rangle$ is not maximal: $\langle y^3 \rangle \leq \langle y^2 \rangle$ since $\langle y^3 \rangle \subset \langle y^2 \rangle$)

Example (Polynomial ideal)

Target: $\langle x^2,y^3\rangle \cup \langle x^3,y^2\rangle \in \cup^{\leq 2} I$

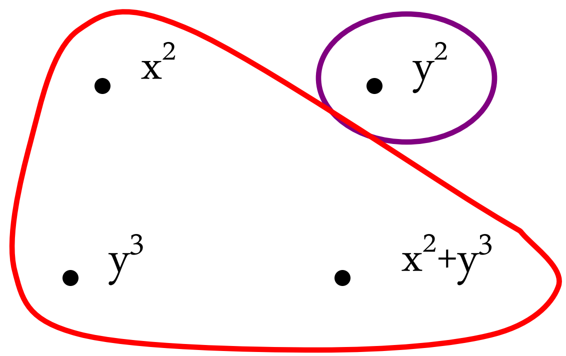
Positive data: $x^2,\ y^3,\ y^2,\ x^2+y^3,\ x^3,\ x^3+y^2,\ldots$

- Step 4

$\{x^2,y^3,x^2+y^3\}$ is a char. set of $\langle x^2,y^3\rangle$ in $\cup^{\leq 2} I$.

The set of examples: $\{x^2,\ y^3,\ y^2,\ x^2+y^3\}$

Hypergraph $\mathcal{G}_4$:



Output: $\langle x^2,y^3\rangle \cup \langle y^2\rangle$

Example (Polynomial ideal)

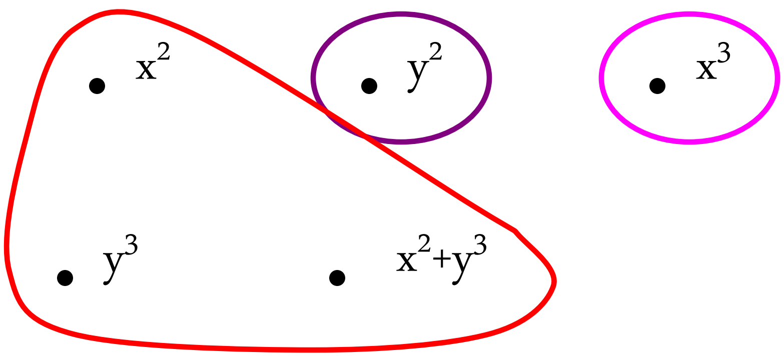
Target: $\langle x^2,y^3\rangle \cup \langle x^3,y^2\rangle \in \cup^{\leq 2} I$

Positive data: $x^2,\ y^3,\ y^2,\ x^2+y^3,\ x^3,\ x^3+y^2, \ldots$

- Step 5

The set of examples: $\{x^2,\ y^3,\ y^2,\ x^2+y^3,\ x^3\}$

Hypergraph $\mathcal{G}_5$:



Output: $\langle x^2,y^3\rangle \cup \langle y^2\rangle$

($\langle x^3\rangle$ is not maximal: $\langle x^3\rangle \leq \langle x^2,y^3\rangle$ since $\langle x^3\rangle \subset \langle x^2,y^3\rangle$)

Example (Polynomial ideal)

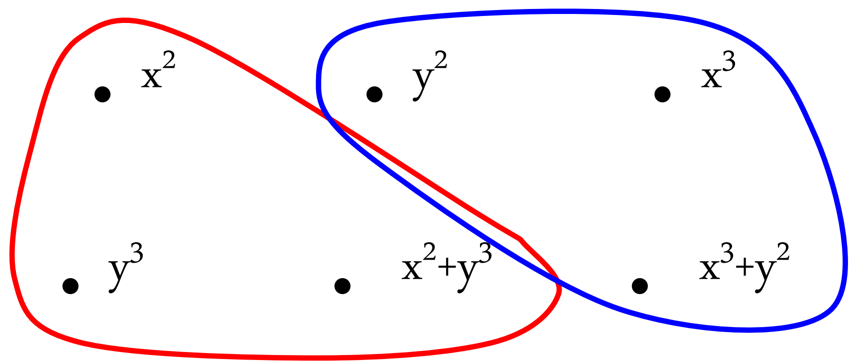
Target: $\langle x^2,y^3\rangle \cup \langle x^3,y^2\rangle \in \cup^{\leq 2} I$

Positive data: $x^2,\ y^3,\ y^2,\ x^2+y^3,\ x^3,\ x^3+y^2, \ldots$

- Step 6 $\{y^2,x^3,x^3+y^2\}$ is a char. set of $\langle x^3,y^2\rangle$ in $\cup^{\leq 2} I$.

The set of examples: $\{x^2,\ y^3,\ y^2,\ x^2+y^3,\ x^3,\ x^3+y^2\}$

Hypergraph $\mathcal{G}_6$:



Output: $\langle x^2,y^3\rangle \cup \langle x^3,y^2\rangle$

↑ Target language

Application

- The algorithm schema can be applied to learning bounded unions of Tree Pattern Languages.

Conclusion

We proposed a learning algorithm schema by combining hypergraph and characteristic sets for bounded unions of Noetherian closed set systems.