FLUCTATION SCALING: TAYLOR'S LAW AND BEYOND

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OUTLINE

- General observation: Fluctuation Scaling (FS) is ubiquitous in complex systems
- Examples from population dynamics, internet traffic, stock market, etc.
- Categorization: Temporal and ensemble FS
- Interesting effects: Window dependence, multiscaling
- Possible scenarios:
  - Central Limit Theorems
  - Strong driving
  - Random Walk with impact
  - Finite Size Scaling
- Summary

GENERAL OBSERVATION: FLUCTATION SCALING

• 1938 Fairfield Smith: For fixed size *A* of lands the average yield $\overline{f_A}$ and the variance σ_A was measured. Varying *A*, the two quantities show scaling:

$$\sigma_A \propto \overline{f_A}^{\alpha_E} \qquad (1) \qquad \text{with} \qquad \alpha_E \approx 0.62$$

• 1961 L.R. Taylor (1924-2007) in Nature: ”Aggregation, variance and the mean”. Counted # of animals in given areas, and stated that (1) is a ”universal law” (today called Taylor’s law in population dynamics).

•Triggered more than 1000 studies. Widely accepted as one of the few universal laws in ecology.

$$f_i = \sum_{n=1}^{N_i} V_{i,n} \qquad f_i^{\Delta t}(t) = \sum_{n=1}^{N_i^{\Delta t}(t)} V_{i,n}^{\Delta t}(t)$$

constituents → nodes/elements → complex system

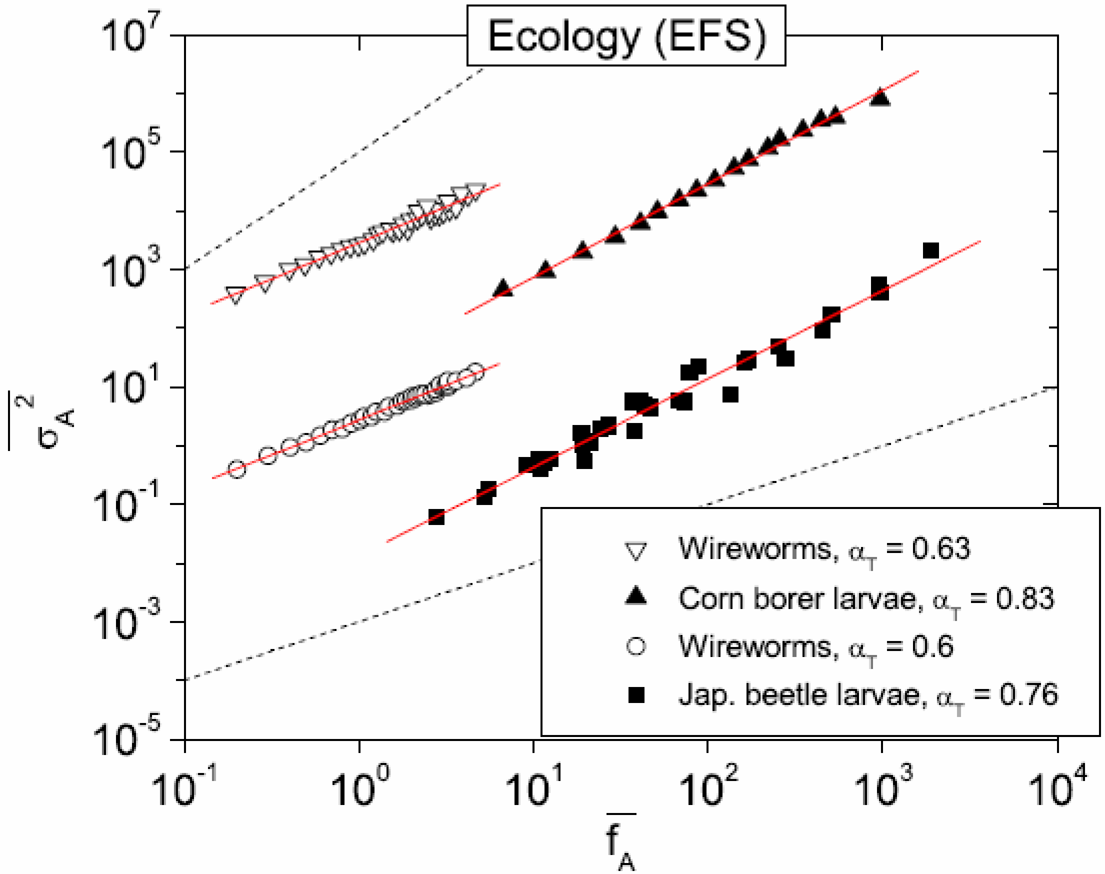
$V_{i,n}, n = 1, \ldots, N_i \rightarrow f_i, i = 1, \ldots, M \rightarrow$ total activity

a group of individuals → a population → a species

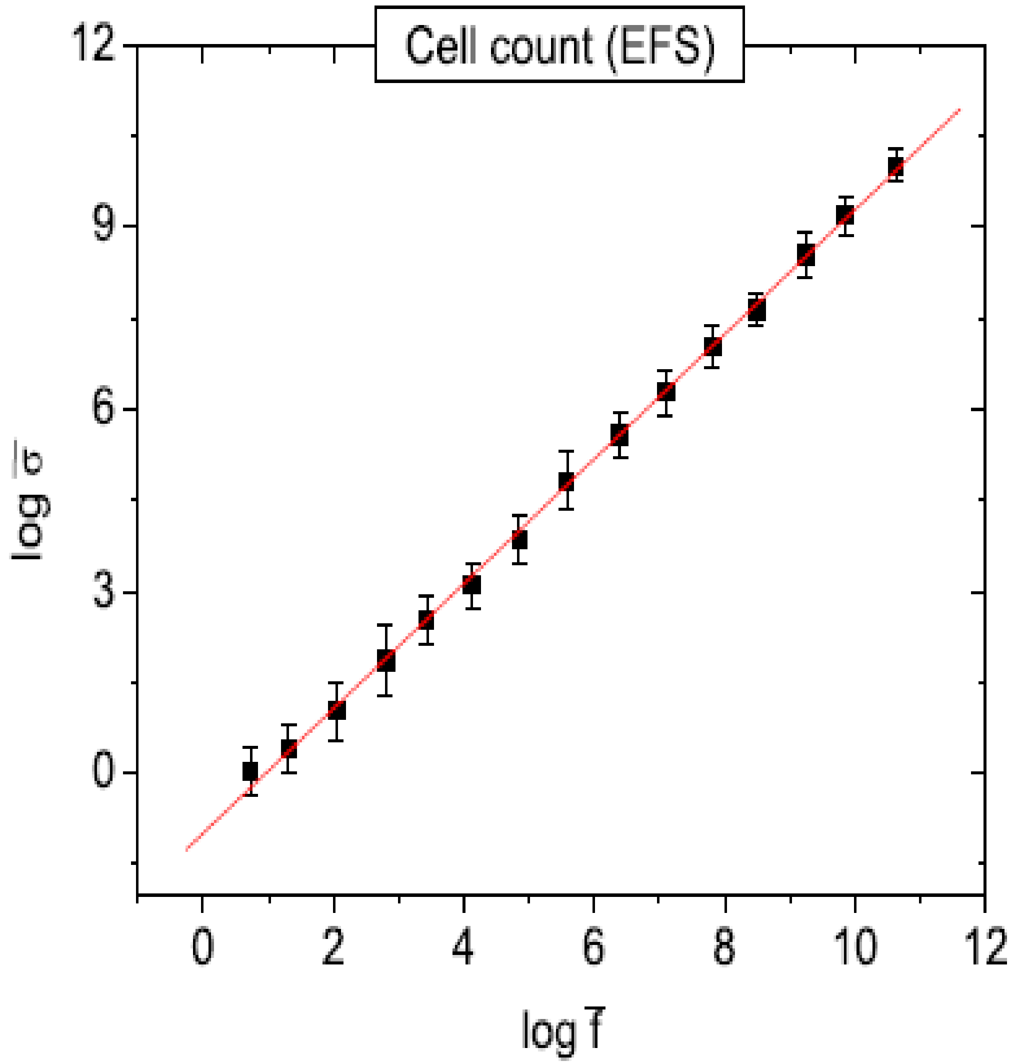
a single tree → a single forest → all forests of a continent

a single data packet → router → Internet

a single car → measurement point → highway system



Fluctuation scaling for ensemble averages of the population of four species. Every point represents the mean $\overline{f_A}$ and variance σ_A over an ensemble of areas of the same size A. The bottom dashed line corresponds to α_E= 1/2, the top one to α_E = 1. Points were shifted both vertically and horizontally for better visibility. Data from Taylor (1961).



Fluctuation scaling for the cell count of species. For every species the average cell count $\overline{f}$ and its variance $\overline{\sigma^2}$ was calculated separately. Then these points were binned logarithmically for better visibility. $\alpha_E = 1$.

Temporal Fluctuation Scaling

Consider population time series in different habitats. The average and the varience of the time series scale like

$$\sigma_i \propto \langle f_i \rangle^{\alpha_T}$$

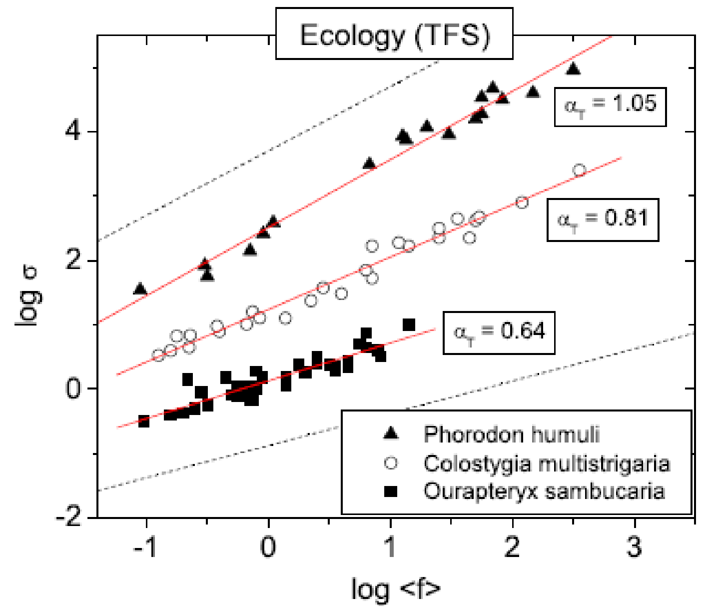


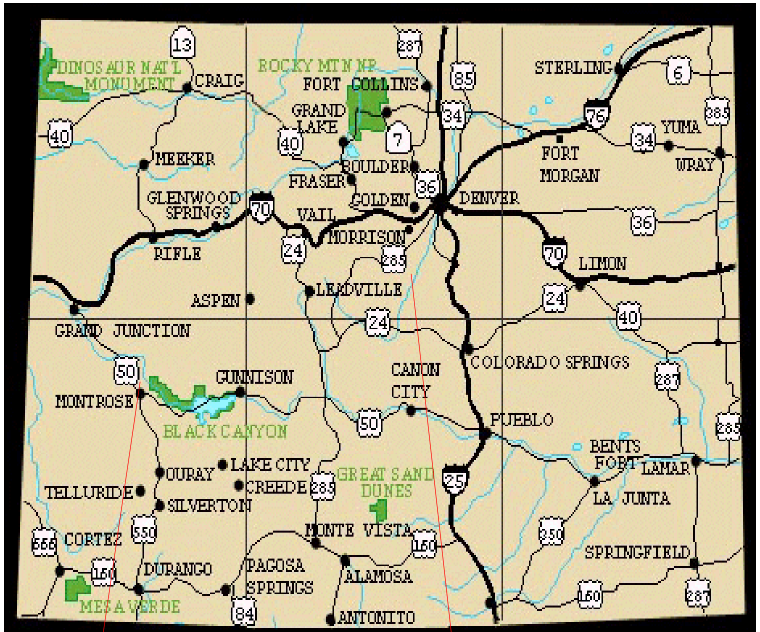
Figure 4. Fluctuation scaling for temporal averages of the population of three species. A point represents the temporal mean $\langle f_i \rangle$ and variance σ_i^2 of a population. The bottom dashed line corresponds to $\alpha_T = 1/2$, the top one to $\alpha_T = 1$. Points were shifted both vertically for better visibility. Data courtesy of Marm Kilpatrick [36, 37].

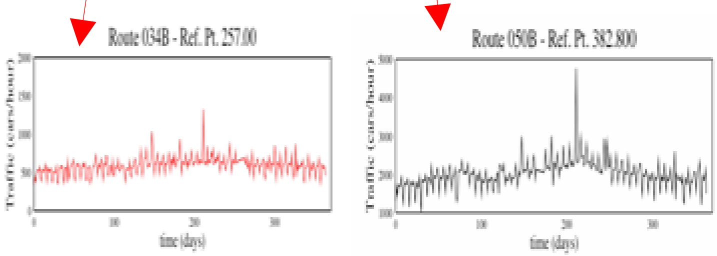
Multichannel observations in complex systems

- Usually processes take place on networks
  - Internet
  - traffic networks
  - stock market
- Coupling to external world/drive present
- Multichanel observation: Activity measured on the *i-th node (or link)* : $f_i(t)$

Highways

•$f_i(t)$ =traffic at a given point of a road *i* at day *t*.

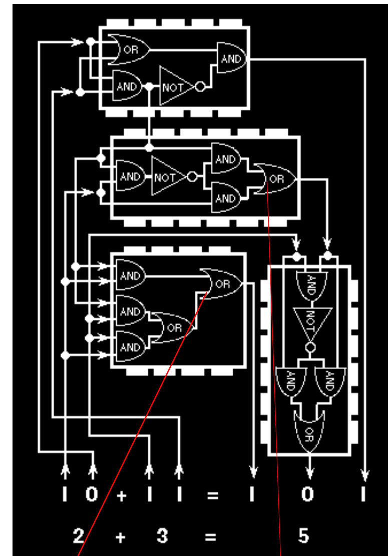


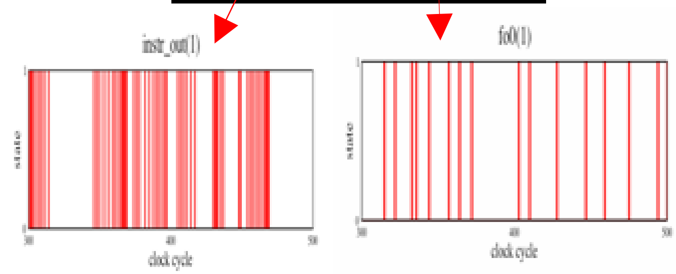


•Daily traffic on 127 Colorado roads from 1998 to 2001.

Computer chip

•$f_i(t)$ =state of a given logic component *i* at clock cycle *t*.





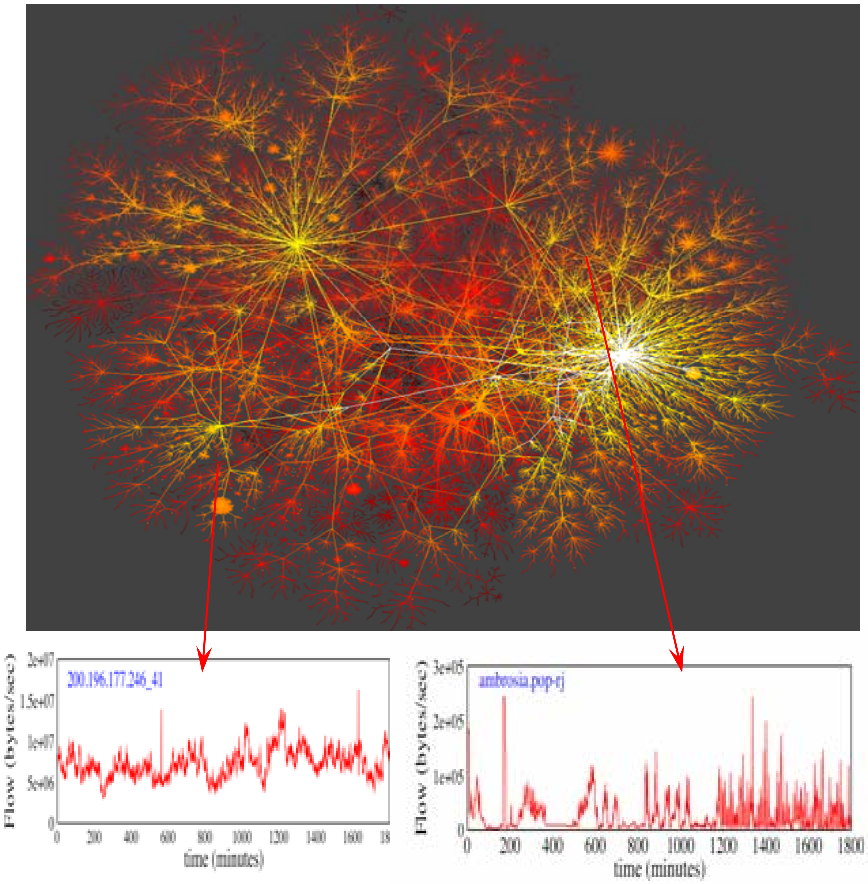
• 462 signal carriers

• 8,862 clock cycles.

M. de Menezes and A.-L. Barabási,

Internet

• $f_i(t)$ = number of bytes passing through router i at time t.

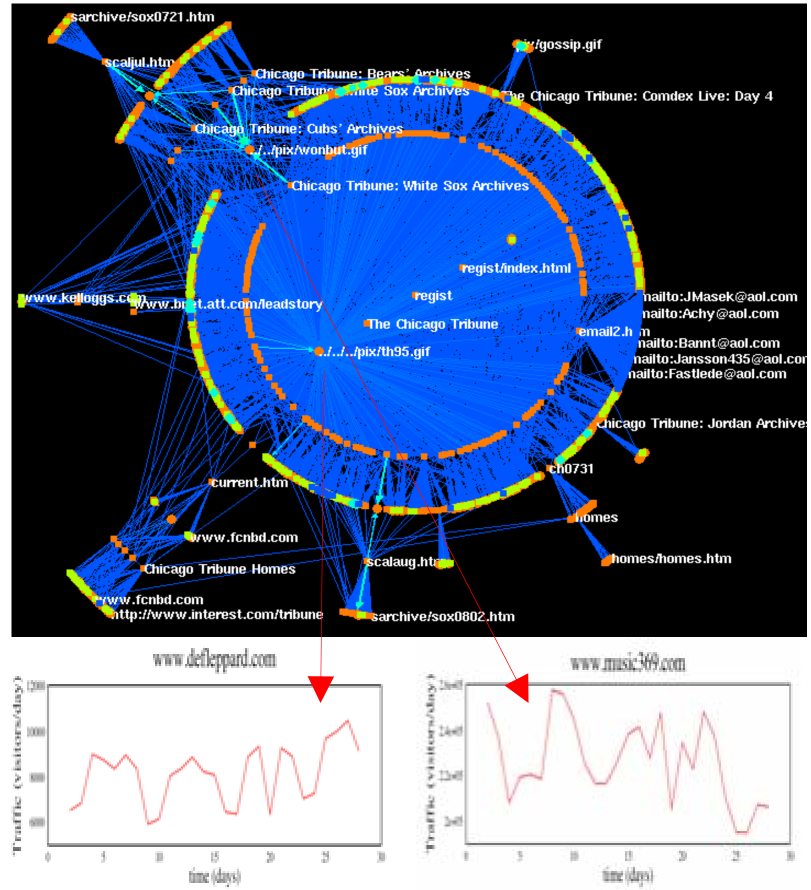


• 347 routers

• t_{max}=2 days (5 min. resolution)

World Wide Web

• $f_i(t)$ = number of visits to website i at day t

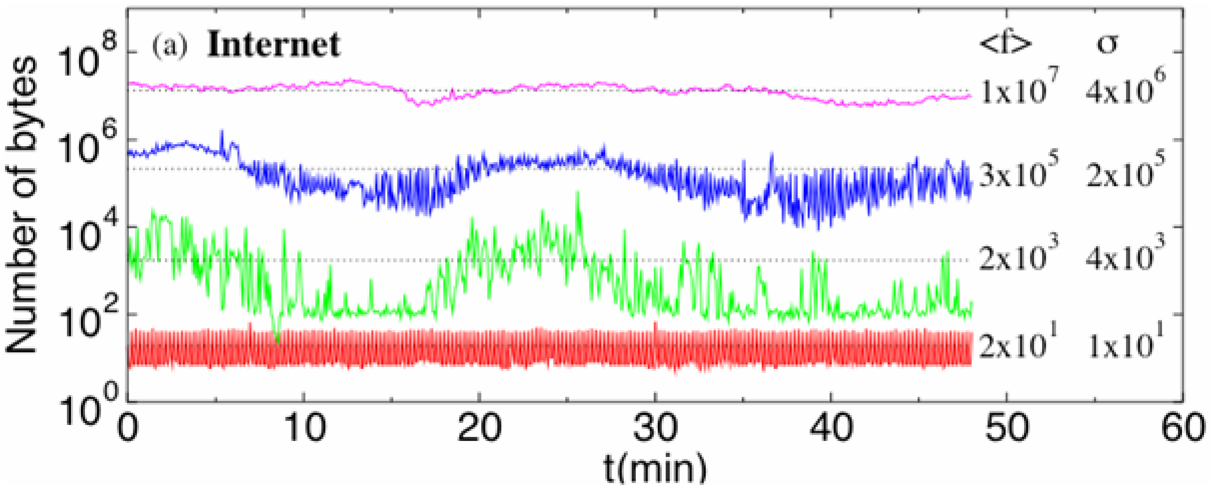


• 3000 web sites.

• Daily visitation for a 30 day period

M. de Menezes and A.-L. Barabási

What can we learn from this?

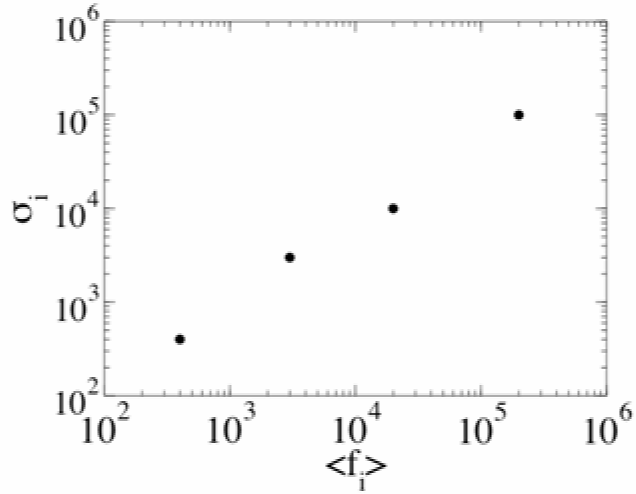


1) For all *i* nodes:

$$\langle f_i \rangle = \frac{1}{T} \sum_{t=0}^{T} f_i(t)$$

$$\sigma_i = \sqrt{\langle f_i(t)^2 \rangle - \langle f_i(t) \rangle^2}$$

2) Plotted results:

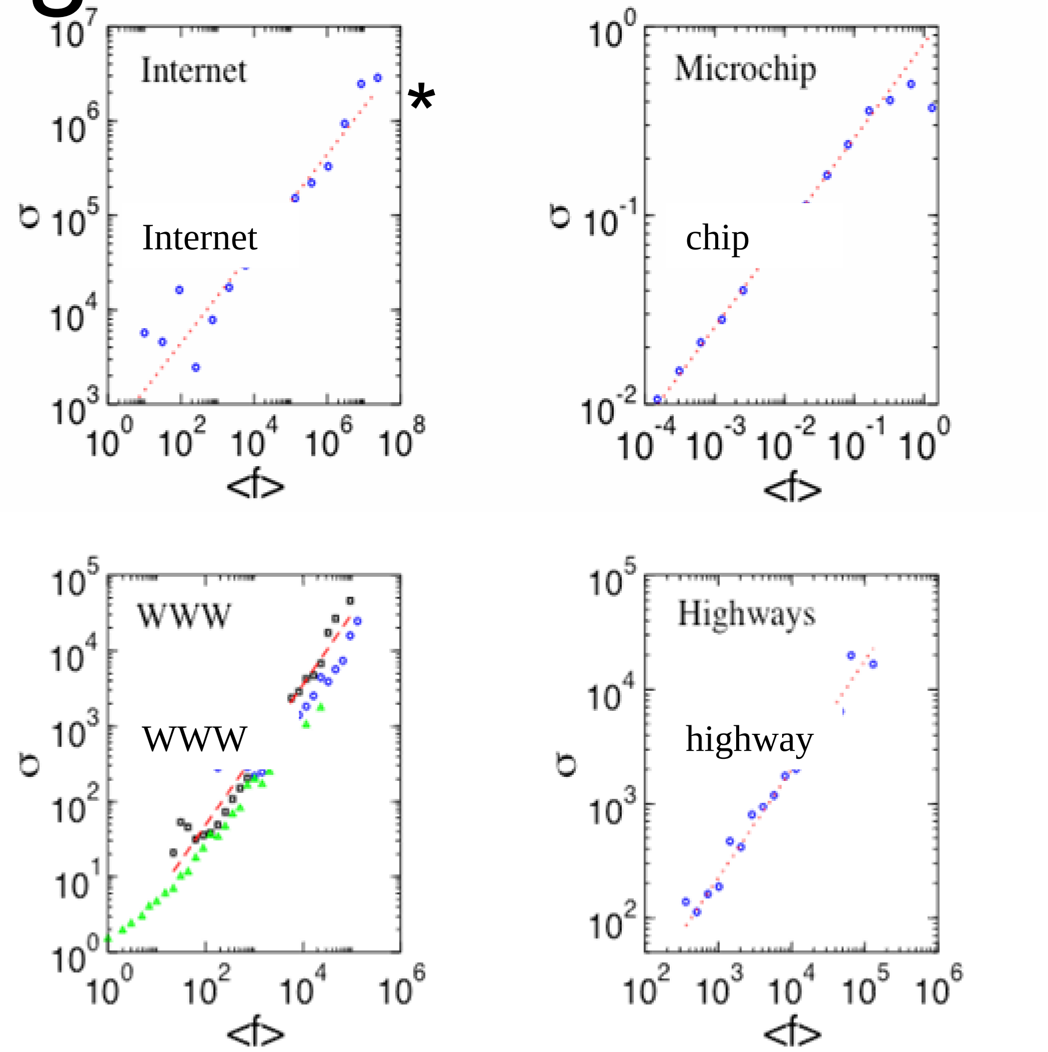


The scaling of fluctuations

$\alpha = 1/2$

$$\sigma_i \sim \langle f_i(t) \rangle^{\alpha}$$

$\alpha = 1$



M. de Menezes and A.-L. Barabási

Two universality classes?

Simple random walk model:
Network with
N random wa
average num
Activity at no
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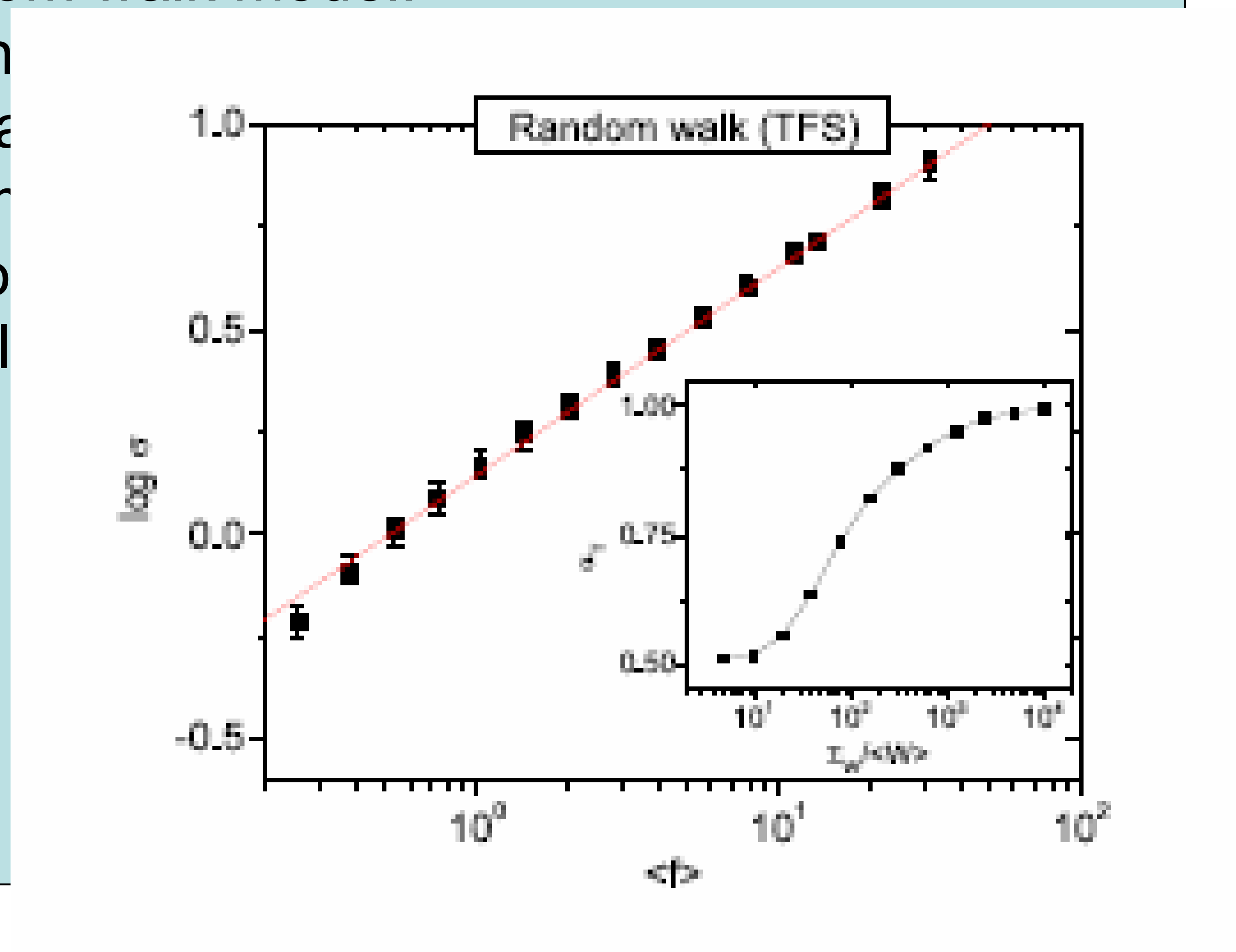
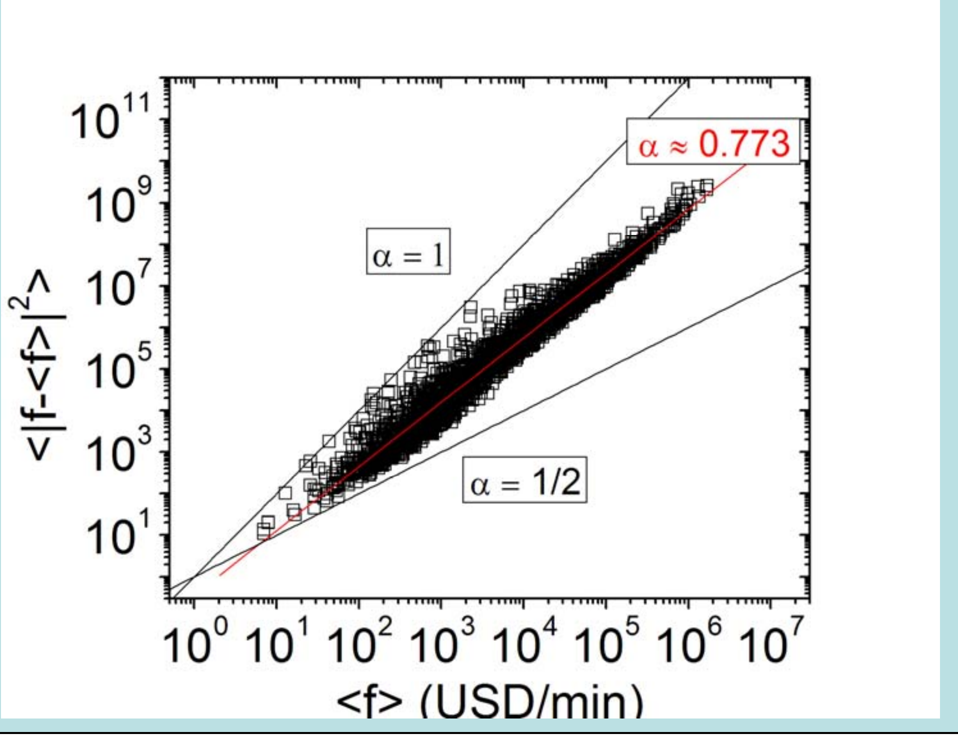


Two universality classes?

We have seen many $\alpha_E \neq 1 \text{ or } 1/2$ for the ensemble averages. What about the time averages?

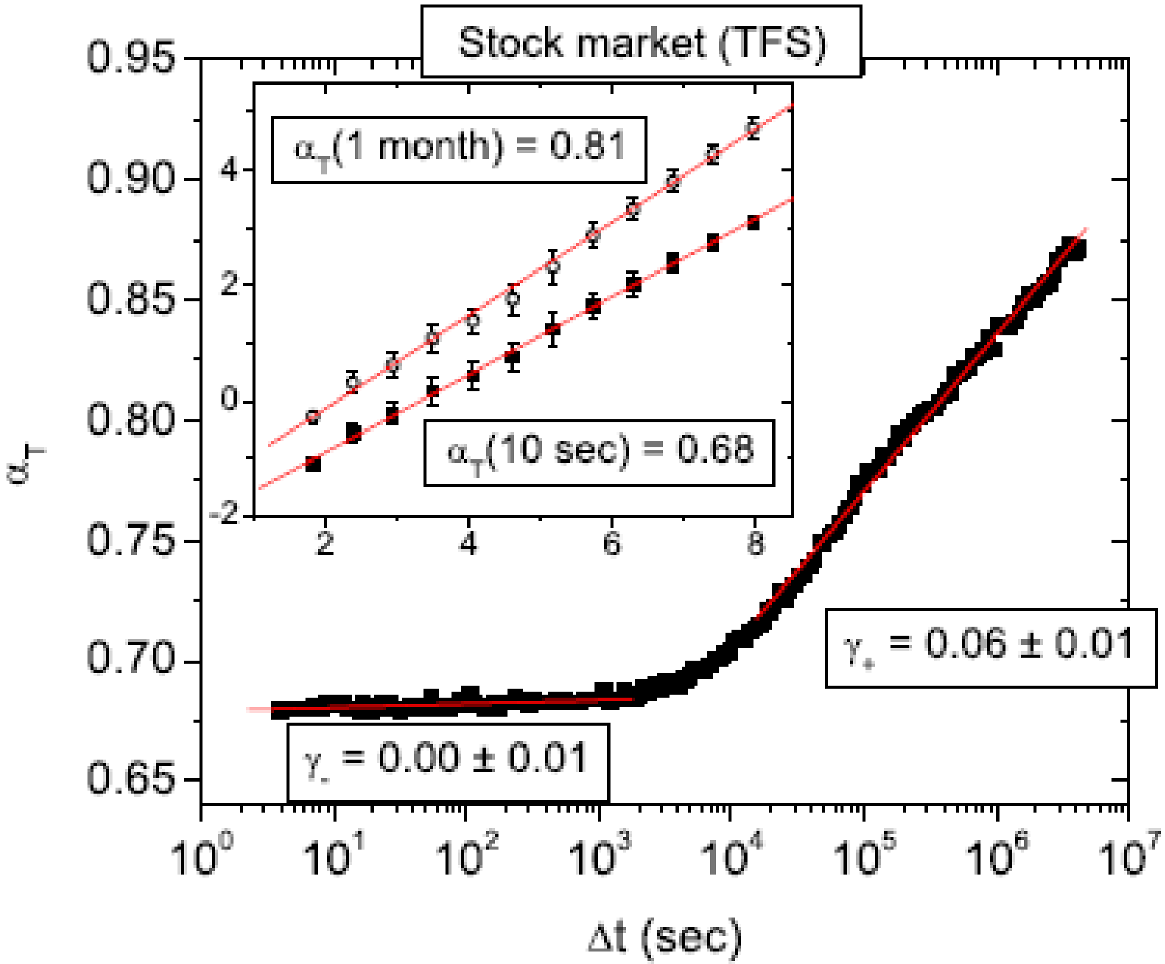
Stock market data: Take a window of size Δt=10min, and consider the volume of a stock i traded during this time activity



Non-universal scaling over 6 orders of magnitude

Eisler et al. 2005

Dependence on Δt



* Note:

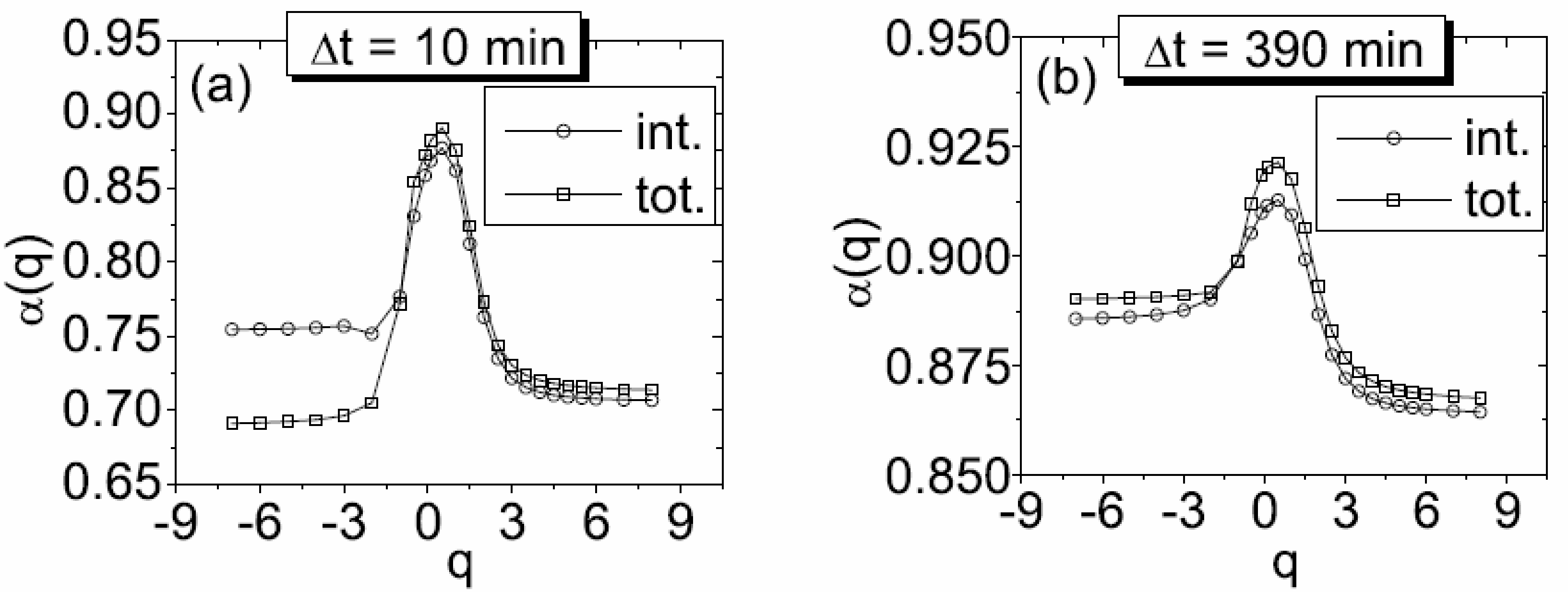
Duch and Arenas Phys. Rev. Lett. 96, 218702 (2006) got α = 0.75 for Inernet

Multiscaling

If in the scaling form

$$\langle |f_i - \langle f_i \rangle|^q \rangle = C_F^q(\Delta t, q)\, \langle f_i \rangle^{q\alpha(q)}$$

α depends on q, we have multiscaling (stock market):



T: time average
E: ensemble average

Subj.	System	T/E	Refs.
Networks	Random walk	T	[7, 31, 33]
	Network models	T	[34, 35]
	Highway network	T	[7, 31]
	World Wide Web	T	[7, 31]
	Internet	T	[7, 31, 32]
Phy.	Heavy ion collisions	E	[26–28]
	Cosmic rays	E	[29, 30]
Soc./econ.	Stock market	T	[8, 55, 56, 59]
	Stock market	E	this review
	Business firm growth rates	E	[60, 61]
	Email traffic	T	this review
	Printing activity	T	this review
Cl.	River flow	T	[62, 63]
	Precipitation	T	[64]
Ecology/pop. dyn.	Forest reproductive rates	T	[45, 46]
	Satake-Iwasa forest model	T	[44]
	Crop yield	T	[6]
	Animal populations	T, E	[5, 10, 15, 16]
	Diffusion Limited population	E	[17]
	Population growth	T	[65, 66]
	Exponential dispersion models	E	[18, 21, 67]
	Interacting population model	T	[36]
Life sciences	Cell numbers	E	[20]
	Protein expression	T	[54]
	Gene expression	T	[68, 69]
	Individual health	E	[70]
	Tumor cells	E	[21]
	Human genome	E	[22, 23]
	Blood flow	E	[67]
	Oncology	E	[21]
	Epidemiology	T	[52, 53]

Many questions:

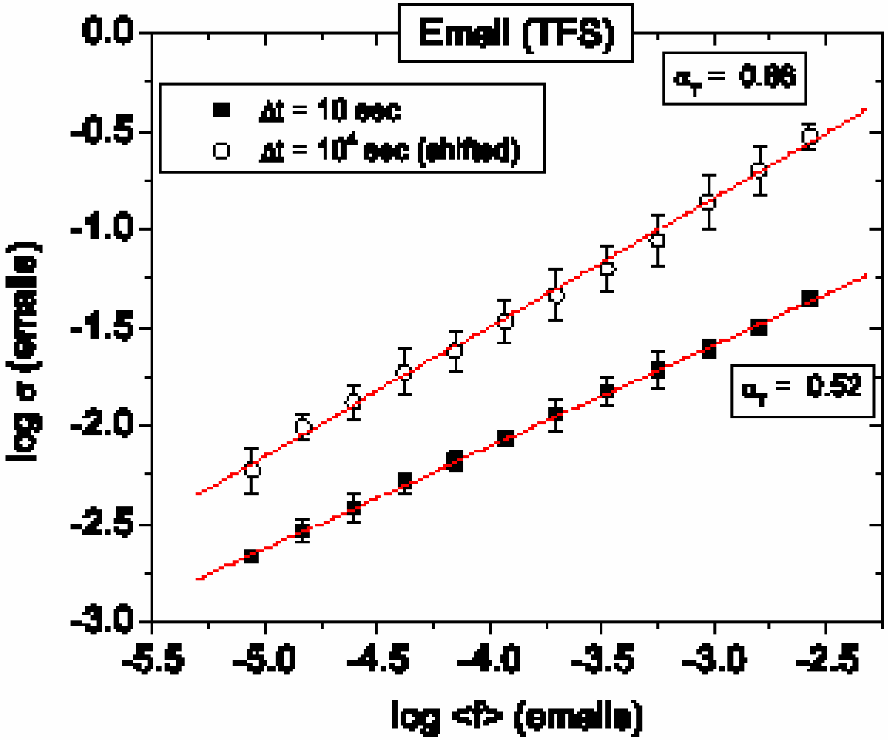
- What is special about α = 1/2 and 1?
- Are there universality classes?
- How to relate Δt dependence to other types of scaling?
- How to relate time and ensemble averages?
- What are the possible scenarios?
- When is multiscaling expected?
- What is its origin?
- What is the ”physical” meaning of the crossovers?
- Corrections to scaling?
- …

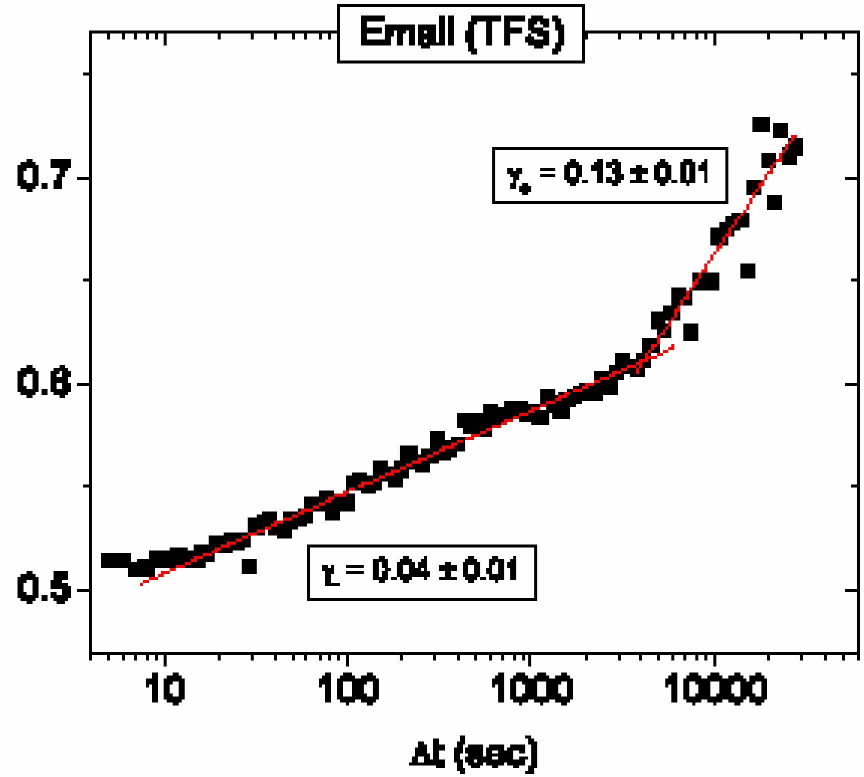
α = 1/2

Simple case: Central Limit Theorem
variance $\propto$ sqrt(mean)

Examples: Stat. phys. fluctuations, random walkers on a network with broad degree distribution and many more.

Another route to α = 1/2:
If the signal is 1 or 0 (# emails) and Δt is short enough such that no multiple events happen, then $f_i(t) = f_i^2(t)$. E.g., for independent events $\sigma^2 = p(1-p)$
Scaling possible only if <*f*> spans many orders of magnitudes → *p* is small → $\sigma^2 \sim p = < f >$ hence α = 1/2



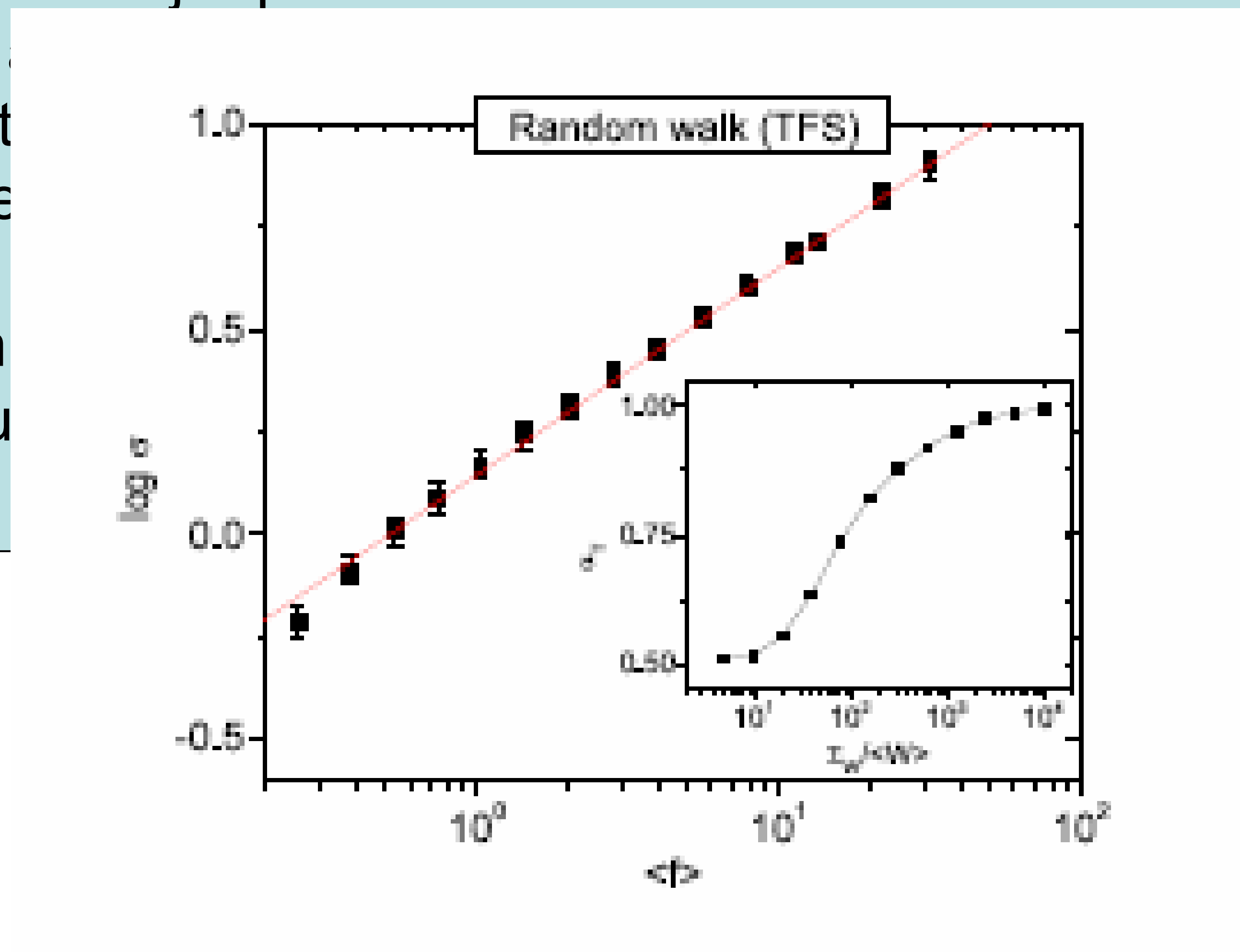


FS from the Enron email database. α_T depends on the time window *Δt* and approaches 1/2 when *Δt* goes to 0.

α = 1

If there is synchronization in the system, mostly due to a strong external drive, the internal fluctuations become irrelevant and the major part of the fluctuations come from the drive itself,
is proportional t
triggered by the

Simple random
system is fluctu
takes place.



General α

α = 1/2 and α = 1 are not “universality classes in the stat. phys. sense. They are trivial extremes.

Generally:

$1/2 \leq \alpha \leq 1$

How to obtain non-trivial α-s?

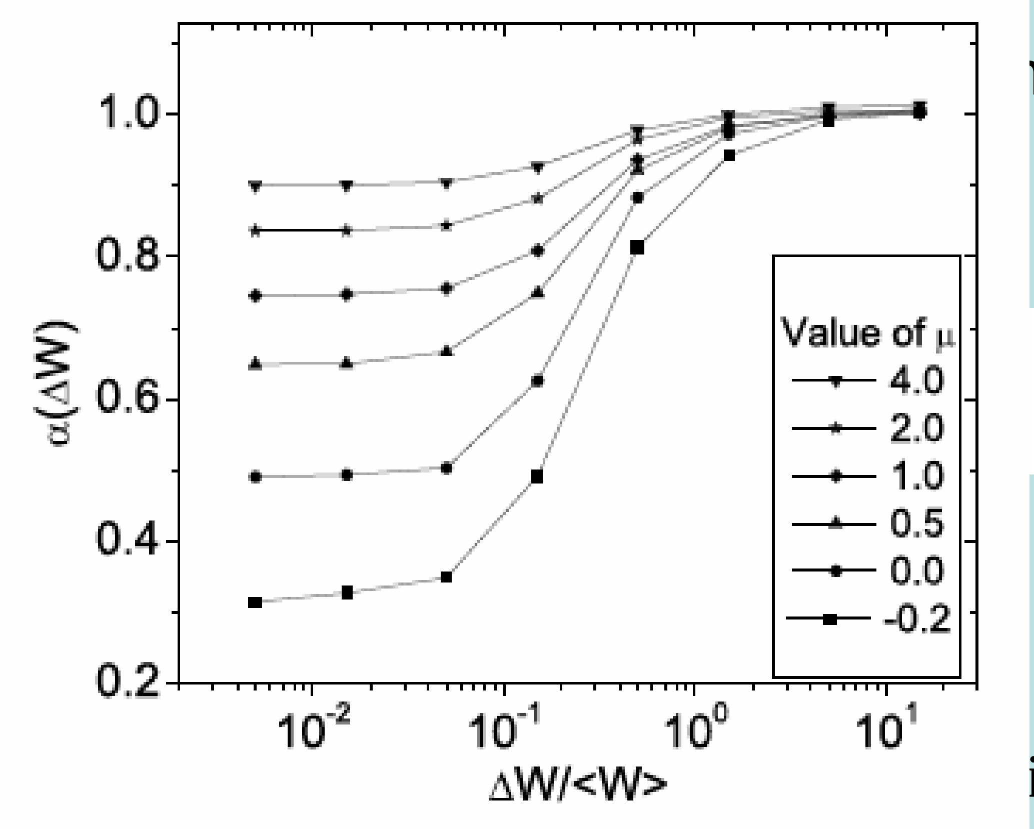
- Impact inhomogeneity
- Finite Size Scaling

Impact inhomogeneity

Imagine that the random walkers make in impact on the node they visit, which is proportional to the (degree)$^{\beta}$ of the node. The ac… …me window.

There are two flu…

$\sigma^2 = \Sigma_V^2 \langle N \rangle$ - … $k^{2\alpha(\beta+1)}$



From which foll…

$$\alpha = \frac{1}{2}\left(1 + \frac{\beta}{\beta+1}\right)$$ … if $\beta = \infty$

Finite Size Scaling (FSS)

Linear size of the system: L
Number of "spins": L^d
Order parameter: M
Susceptibility: $\chi \sim \sigma^2$

FSS at criticality

$$M \propto L^{d-\beta/\nu}$$

$$\chi \propto L^{d+\gamma/\nu}$$

$$\longrightarrow \quad \sigma = \sqrt{\chi} \propto M^{\frac{d+\gamma/\nu}{2(d-\beta/\nu)}} = M$$

Hyperscaling: $\alpha=1$

What if not $M = N_{up} - N_{down}$ but just N_{up} is looked at?
$N_{up} = L^d$ is extensive but with fluctuations like in χ

$$\alpha = \frac{1}{2} + \frac{\gamma/\nu}{2d}$$

with $1/2 \leq \alpha \leq 1$, e.g., $\alpha_{MF} = 3/4$

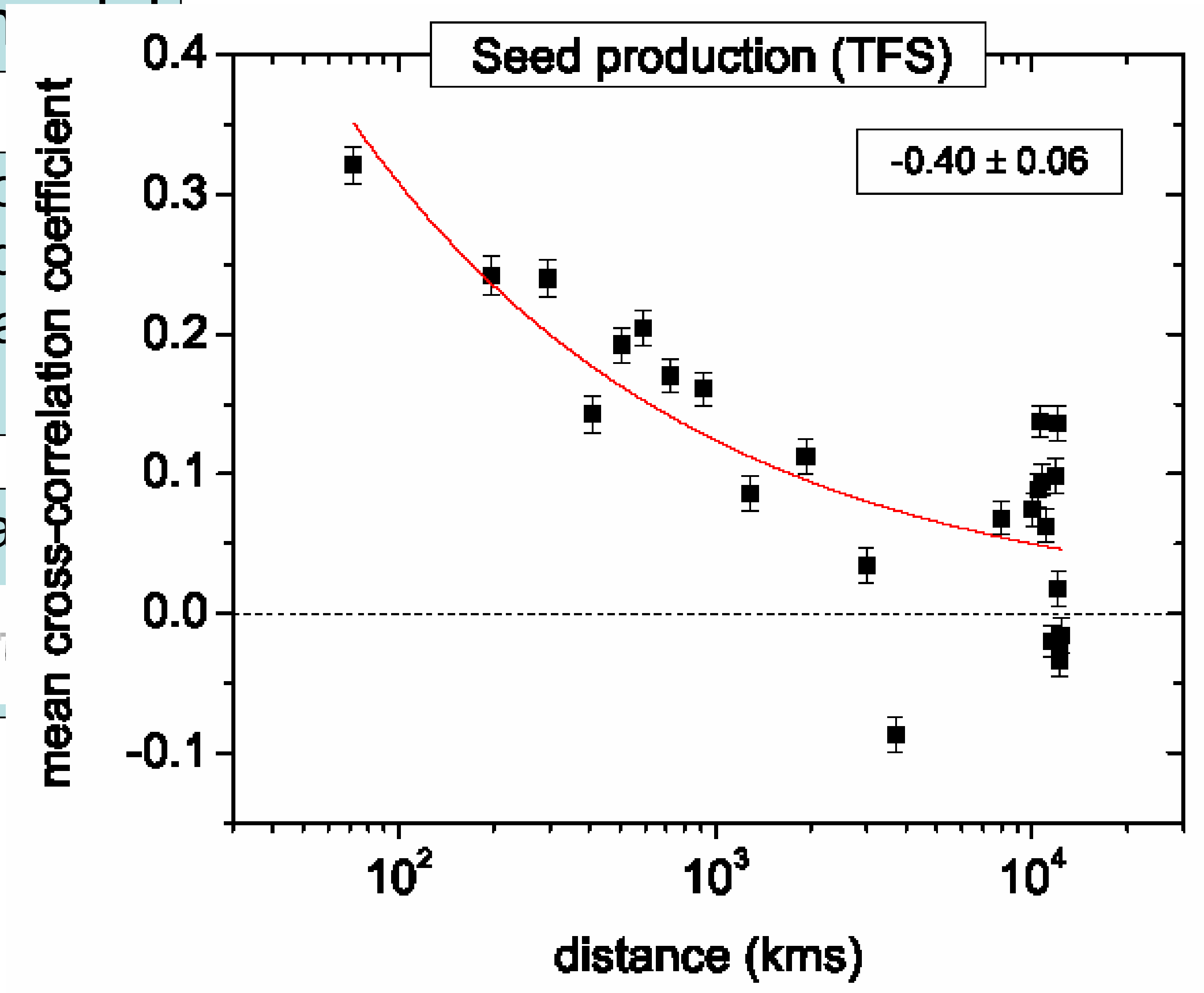
SOC?

Binary forest m

Consider a forest c
In year *t* the reproc
For simplicity we ta
Due to the relative

The reproductive a

$C(\Delta n$



From data fit:
$C(\Delta n) \sim (\Delta n)^{-0.4}$

The total reproductivity is $f_N = \sum_{n=1}^{N} V_n$

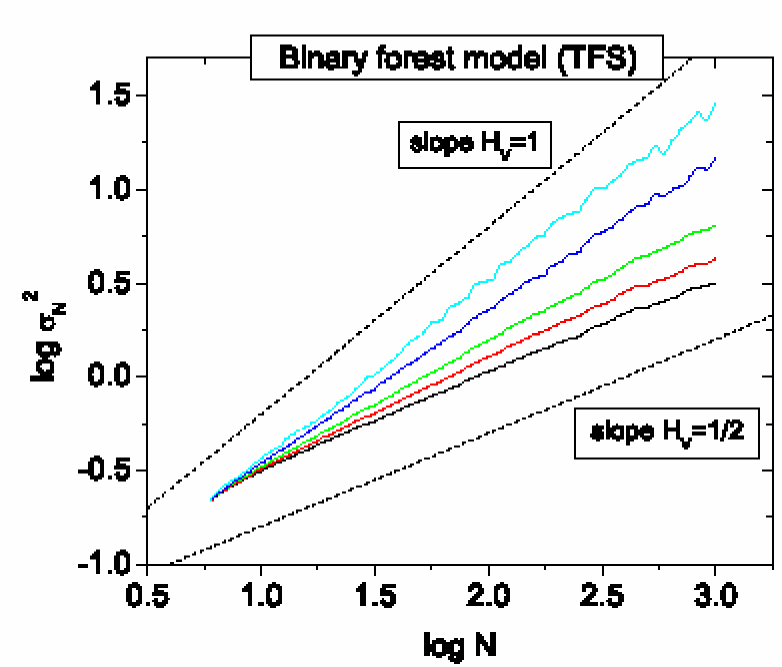
From the correlations (for 1d)

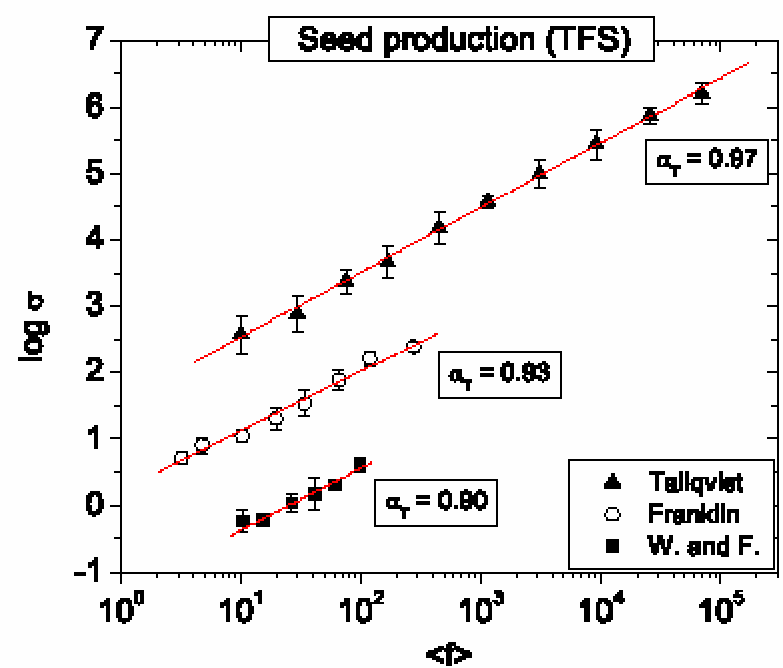
$$\sigma_N = \sqrt{\langle f_N^2 \rangle - \langle f_N \rangle^2} \propto N^{H_V}$$

As the reproductivity is extensive

$\langle f_N \rangle = pN$ → $\alpha_T = H_V$

Correlations increase synchronization





Hurst exponent vs. FS

$$\sigma_i \propto \langle f_i \rangle^{\alpha_T}$$

If f_i comes from time series, σ_i may scale with the length of the series as

$$\sigma_i(\Delta t) = \left\langle \left[f_i^{\Delta t}(t) - \langle f_i^{\Delta t}(t) \rangle \right]^2 \right\rangle^{1/2} \propto \Delta t^{H(i)}$$

The dependence of α on Δt implies a dependence of H on i:

$$\frac{dH(i)}{d(\log \langle f_i \rangle)} \sim \frac{d\alpha(\Delta t)}{d(\log \Delta t)} \sim \gamma$$

which is governed by the same γ.
No universality! (E.g., dependence of *H* on capitalization.)

Limit theorems

FS is always related to sums of random variables.
We have seen that α = 1/2 comes from plain CLT

In the language of limit theorems FS means

$$\frac{\sum_{n=1}^{N} V_n - \langle f \rangle}{K \langle f \rangle^{\alpha}} \to X$$

Possible reasons to get nontrivial α:

- iid, but Levy stable distributions
- dependence of the variables (see, e.g. FSS)

Summary

FS: $\sigma \propto \langle f \rangle^{\alpha}$

general observation over many disciplines and systems

FS: ensemble/temporal

$1/2 \leq \alpha \leq 1$

Trivial limiting cases (no universality classes)

Scenarios to non-trivial α-s

Limit theorems

Review by Z. Eisler, I. Bartos and J. Kertesz:
Adv. Phys. 57, 89-142 (2008), arXiv:0708.2053