Dynamic Decisions, Multiple Equilibria and Complexity

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- I. Introduction: Literature, Methodological Remarks
- II. Examples of Models with Multiple Equilibria
- III. Mechanisms leading to Multiple Equilibria
- IV. Numerical Methods to Compute the Global Dynamics
- V. Example: State Dependent Risk Premium…
- VI. Multiple Attractors, Heterogeneity and Empirics
- VII. Policy Implications: Enlarging Domains of Attraction
- VIII. Conclusions

I. Introduction: Literature…

Foundation and Survey:

- W. Brock and Malliaris (1989), “Differential Equations, Stability and Chaos in Dynamic Economics”, Skiba (1978), and see Brock`s Web-site at Madison University…
- M. Sieveking and W. Semmler (1997), “The Present Value of Resources with large Discount Rate”, Appl. Math & Optimization, Renewable Resources Models..
- C. Deisenberg, G. Feichtinger, W. Semmler, F. Wirl (2003), “History Dependence and Global Dynamics in Models with Multiple Equilibria”, Cambridge University Press, ed. , Barnett et al.

I. Introduction: Literature…

Credit and Financial Markets:

- W. Semmler and M. Sieveking (2000): “Critical Debt and Debt Control‟, JEDC
- L. Grüne, W. Semmler and M. Sieveking (2003), “Creditworthiness and Threshold in a Credit Market Model with Multiple Equilibria”, Economic Theory,
- L.Grüne, and W. Semmler (2005), Default Risk, Asset Pricing and Debt Control, Journal of Financial Econometrics
- L. Grüne, W. Semmler and B. Lucas (2007), Firm Value, Diversified Capital Assets and Credit Risk,: Toward a theory of default correlation, Journal of Credit Risk

I. Introduction: Literature…

Other Applications:

- W. Semmler, and A. Greiner (2005), "Economic Growth and Global Warming: A Model of Multiple Equilibria and Thresholds", Journal of Economic Behaviour and Organization,, ed. W. Semmler (see also OUP-book)
- W. Semmler and M. Ofori (2007), "On Poverty Traps, Thresholds and Take-Offs, Journal of Structural Change and Economic Dynamics",
- M. Kato and W. Semmler (2007/8), on Firms Size Dynamics, Ecological Management Problem, Poverty Traps and Inequality, Metroeconomica, forthcoming

I. Introduction: Methodological Remarks; **Economic Agents** (intentional behavior: memory, expectations, bounded rationality, learning…)

Lotka-Volterra Dynamics (see Sieveking and Semmler, 1997)

(1) Infinite Horizon

$(\delta < \infty)$ Price: $P(x_2 u)$

$$V \max \int_0^\infty e^{-\delta t} \overbrace{(p(x_2 u_2) x_2 - c) u_2}^{\text{rent}} dt$$

s.t.

1. $\dot{x}_1 = x_1(a_0 - a_1 x_1 - a_2 x_2) - x_1 u_1(x)$
2. $\dot{x}_2 = x_2(-b_0 + b_1 x_1 - b_2 x_2) - x_2 u_2(x)$

(1) predator/ Prey: $-a_2 x_2, +b_1 x_1$
(2) competitive: $-a_1 x_1, -b_2 x_2$; innerspecific competition: $(-a_1, -a_2, -b_1, -b_2)$
(3) cooperative interaction: $(+a_2, +b_1)$

(2) Zero Horizon $(\delta \Rightarrow \infty)$

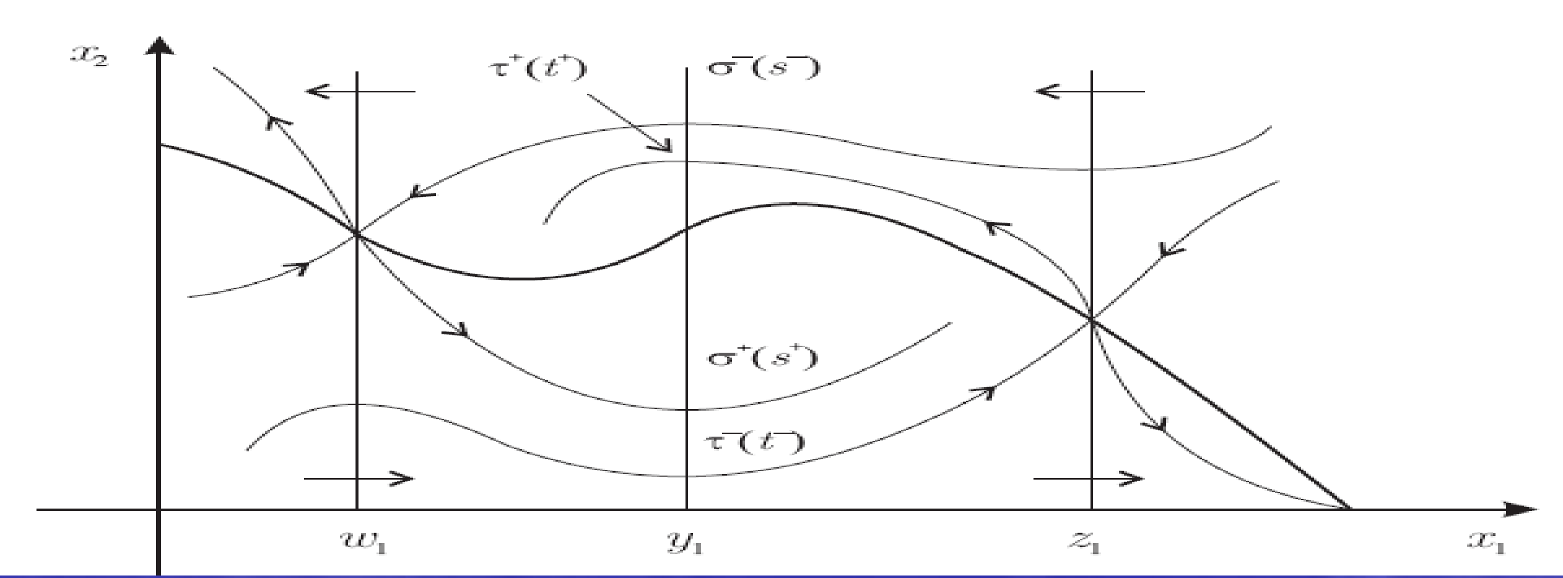
$$\max \overbrace{(p(x_2 u_2) x_2 - c) u_2}^{\text{rent}} > 0$$

II. Examples

Example 1: Resource Economics (Lotka Volterra dynamics

Interacting Renewable Resources, Sieveking and Semmler (1997)

$$
\begin{aligned}
\dot{x}_1 &= x_1(a_0 - a_1 x_1 - a_2 x_2 - v_1(x_1)), \\
\dot{x}_2 &= x_2(b_0 - b_1 x_1 - b_2 x_2 - k_1 v_1(x_1)).
\end{aligned}
$$

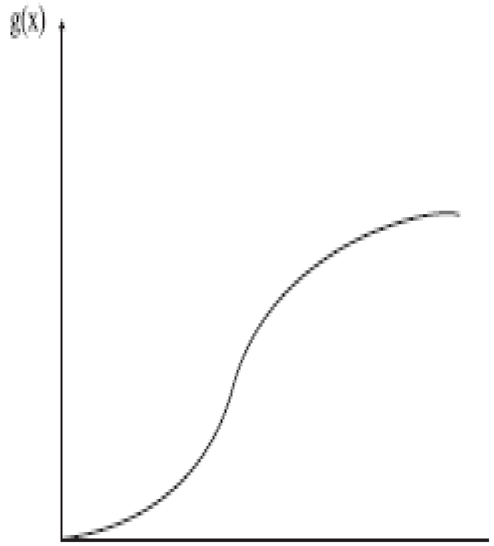


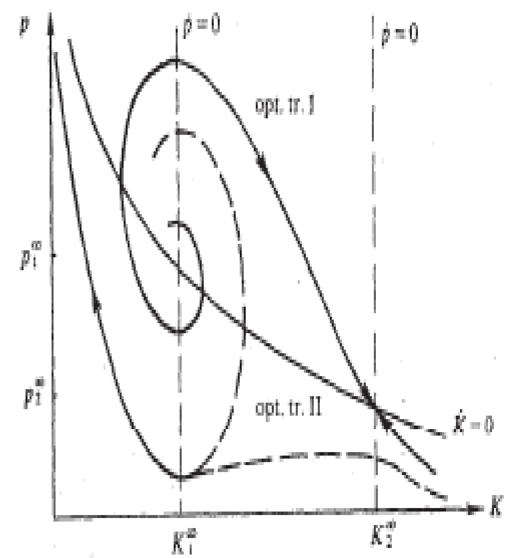
II. Examples…

Example 2: Development Economics

Development Economics

Skiba (1978)



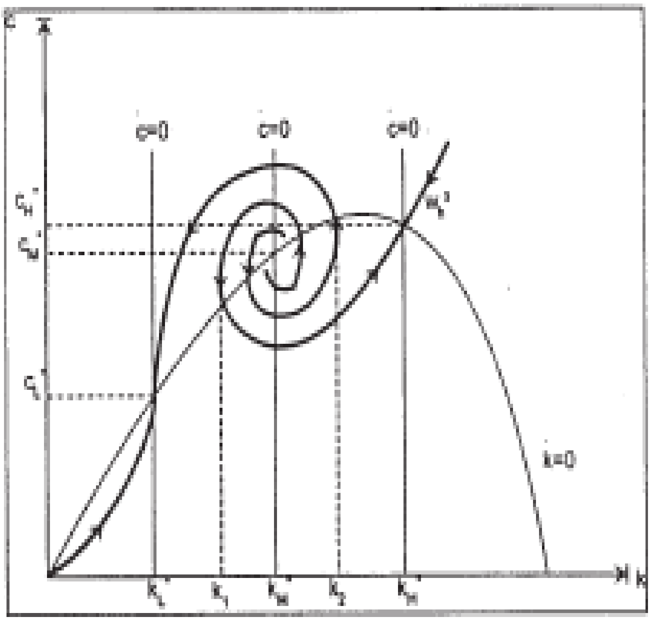


II. Examples..

Example 3: Growth Theory

Growth Theory with Externalities

Santos (1999) / Matsuyama (1991)

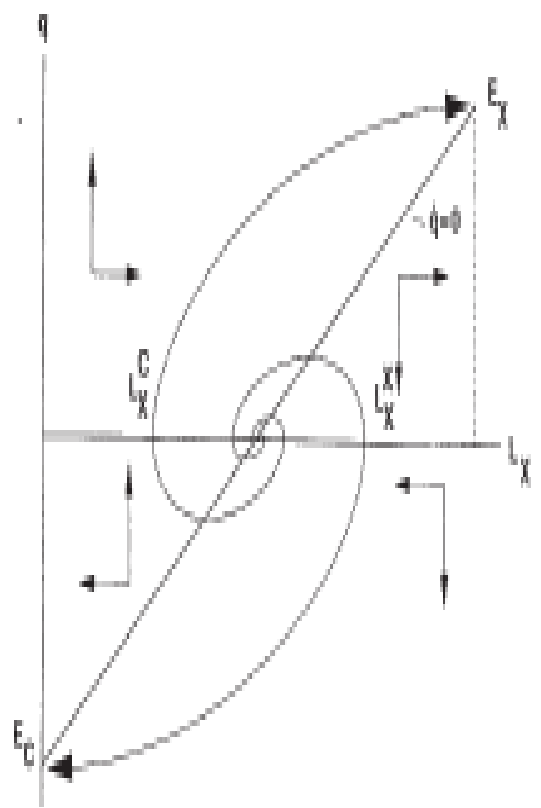


II. Examples

Example 4: Trade and Expectations

Krugman (1991)

Trade Model with Expectations



II. Examples

Example 5: Firms` investment with relative adjustment costs, see Feichtinger et al (2000), Kato, Semmler and Ofori (2006)

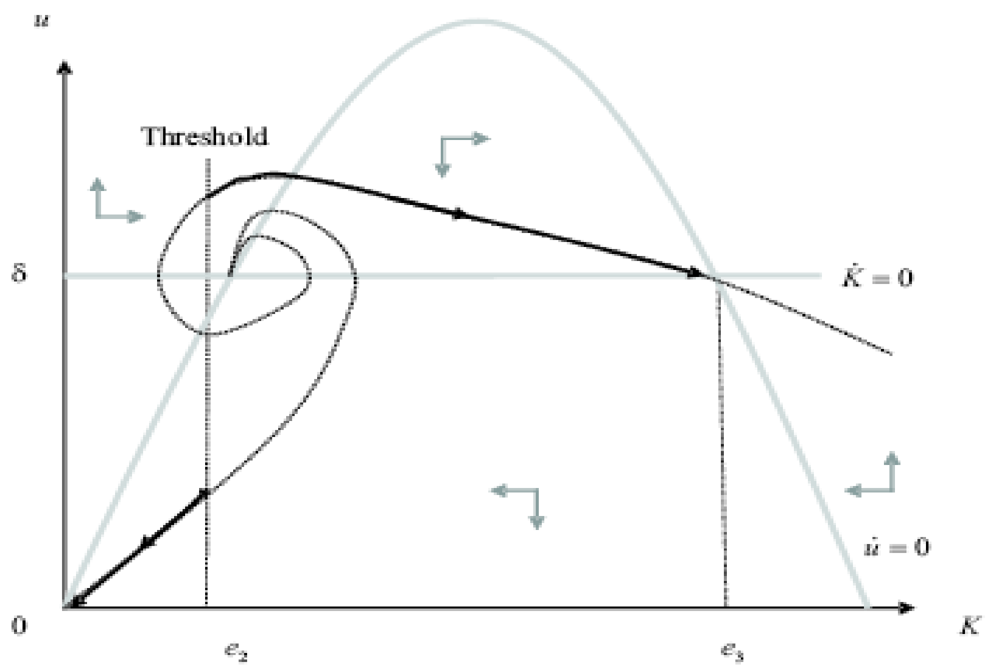
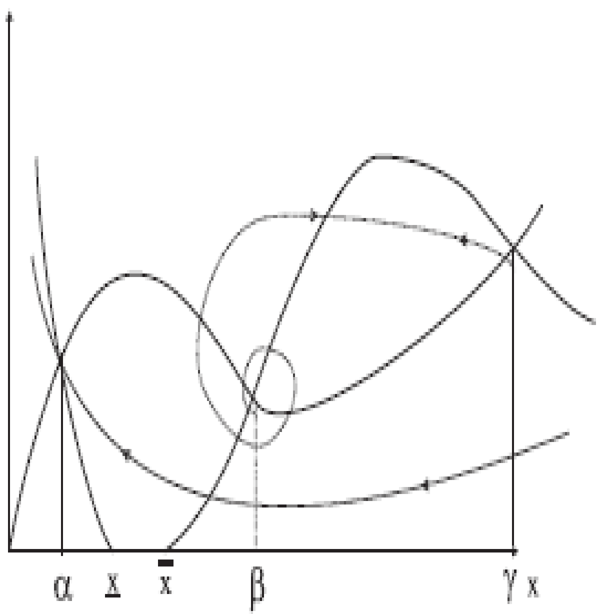


Figure 3: Discontinuous Policy Function in $\{u, K\}$ Space

II. Examples

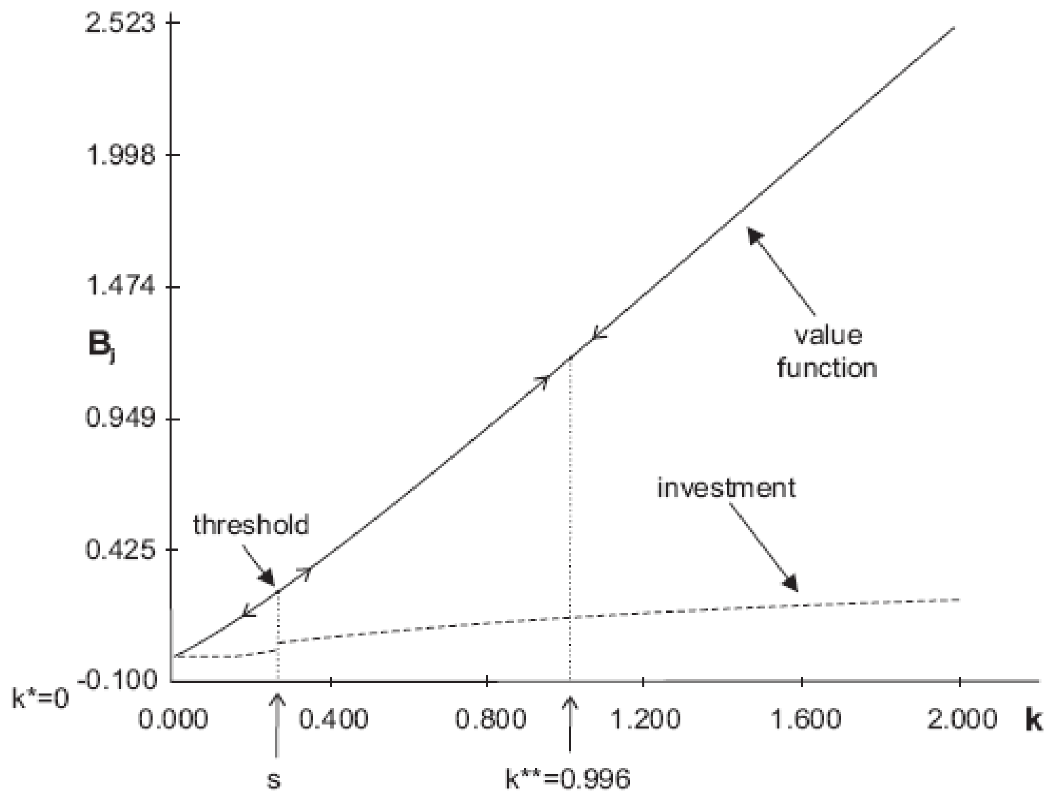
Example 6: Ecological management problem (Brock and Starret, 1999, Grüne, Kato and Semmler, 2005)

Brock and Starret (1999)
Ecological Management
Problem



II. Examples

Example 7: Credit and state dependent risk premium, Grüne, Semmler et al. (2005, 2007),



III: Mechanisms of Multiple Equilibria and Thresholds

- Nonlinear interaction of renewable resources and agents`interventions (example 1)
- Convex-concave production function (example 2)
- Externalities in economic development and growth (example 3)
- Wealths effects on households` utility (Kurz, 1968)
- Expectations formation (example 4, Krugman model)

III: Mechanisms of Multiple Equilibria and Thresholds…

- Nonlinear adjustment costs (Example 5, Feichtinger et al, Kato and Semmler)
- Nonlinear interaction of growth and climate change (Greiner and Semmler, OUP-book)
- Nonlinear absorption capacity of the lake (example 6, Brock and Starret)
- State dependent risk premium in credit markets (example 7, Grüne, Sieveking and Semmler)

IV. Numerical Methods to Compute the Global Dynamics

Step 1: Maximum Principle and Hamiltonian to find the equilibria and local dynamics:

The **first method** uses the maximum principle and Hamiltonian

$$H(u, x, \lambda) = f_0(u, x) + \lambda f(u, x)$$

First order condition for an optimal policy u

$$\begin{aligned} H_u &= 0 \\ \dot{\lambda} &= \delta\lambda - H_x \\ \lim_{t\to\infty} e^{-r\delta t}\lambda(t) &= 0 \end{aligned}$$

We assume $H_{uu} \leq 0$.

IV. Numerical Methods to Compute the Global Dynamics

Remark:

- **Concave** models model: H is jointly concave in state and control (at least over a compact and for the model relevant set), $H_{uu}H_{xx} - H_{ux}^2 \geq 0$,

- **Non-concave** model: $H_{uu}H_{xx} - H_{ux}^2 \leq 0$, **convex** model, $H_{xx} > 0$ (often only local domain of non-concavity and of convexity)

- Associated Jacobian:

$$J = \begin{bmatrix} f_x - \frac{f_u H_{ux}}{H_{uu}} & -\frac{f_u^2}{H_{uu}} \\ \frac{H_{ux}^2}{H_{uu}} - H_{xx} & \delta - f_x + \frac{f_u H_{ux}}{H_{uu}} \end{bmatrix}$$

IV. Numerical Methods to Compute the Global Dynamics

Step 2: The HJB-Equation and Dynamic Pogramming

HJB equation in continuous time:

$$V(x) = \max_{u \in \mathbb{U}} \int_0^\infty e^{-\delta t} g(x(t), u(t)) dt$$

where

$$\frac{d}{dt} x(t) = f(x(t), u(t)), \quad x(0) = x_0 \in \mathbb{R}^n$$

$$(HJB) \qquad \delta V(x) = \underset{u}{Max} \{ f_0(x, u) + V'(x) f(x, u) \}$$

Get $V'(x)$ explicitly as a function of x and $V(x)$ and use

$$V'(x) = G(V(x), x).$$

IV. Numerical Methods to Compute the Global Dynamics

Is solved through discrete time Dynamic Programming (Grüne and Semmler 2004, JEDC)

$$T_h(V_h)(x) = \max_{u \in U} \{hg(x,u) + \beta V_h(x_h(1))\}$$

then V_h can be characterized as the unique solution of the fixed point equation

$$V_h(x) = T_h(V_h)(x) \text{ for all } x \in \mathbb{R}^n.$$

Denoting the nodes of the grid Γ by x^i, $i = 1, \ldots, P$, we are now looking for approximation V_h^Γ satisfying

$$V_h^\Gamma(x^i) = T_h(V_h^\Gamma)(x^i)$$

IV. Numerical Methods to Compute the Global Dynamics

Global dynamics explored with flexible grid size, gridding error measured by

$$\eta_l := \max_{k \in C_l} \left| T_h(V_h^\Gamma)(k) - V_h^\Gamma(k) \right|$$

(0) Pick an initial grid Γ_0 and set $i = 0$. Fix a refinement parameter $\theta \in (0, 1)$ and a tolerance $tol > 0$.

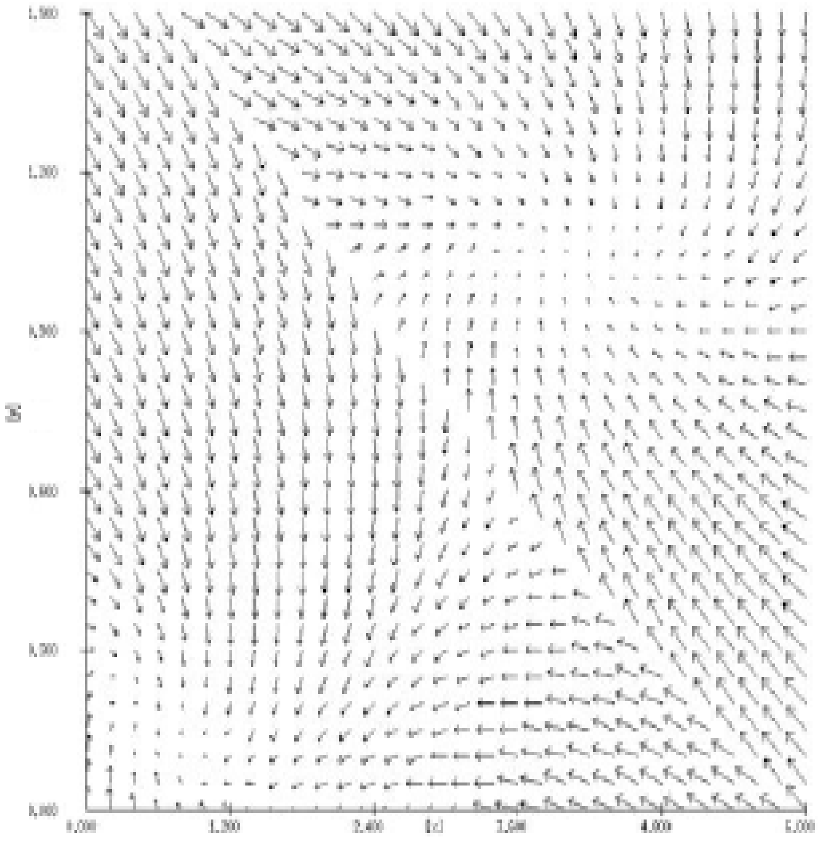
(1) Compute the solution $V_h^{\Gamma_i}$ on Γ_i

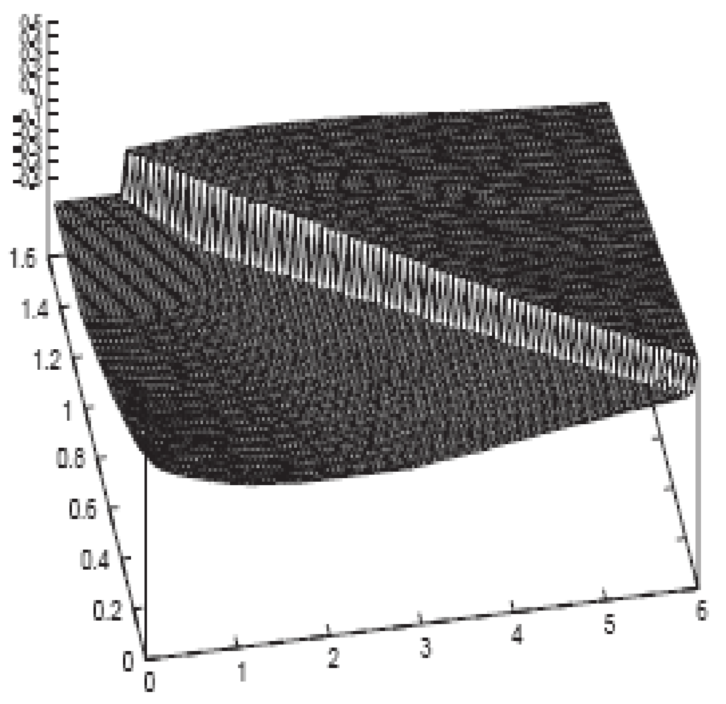
(2) Evaluate the error estimates η_l. If $\eta_l < tol$ for all l then stop

(3) Refine all cells C_j with $\eta_j \geq \theta \max_l \eta_l$, set $i = i + 1$ and go to (1).

IV. Numerical Methods to Compute the Global Dynamics

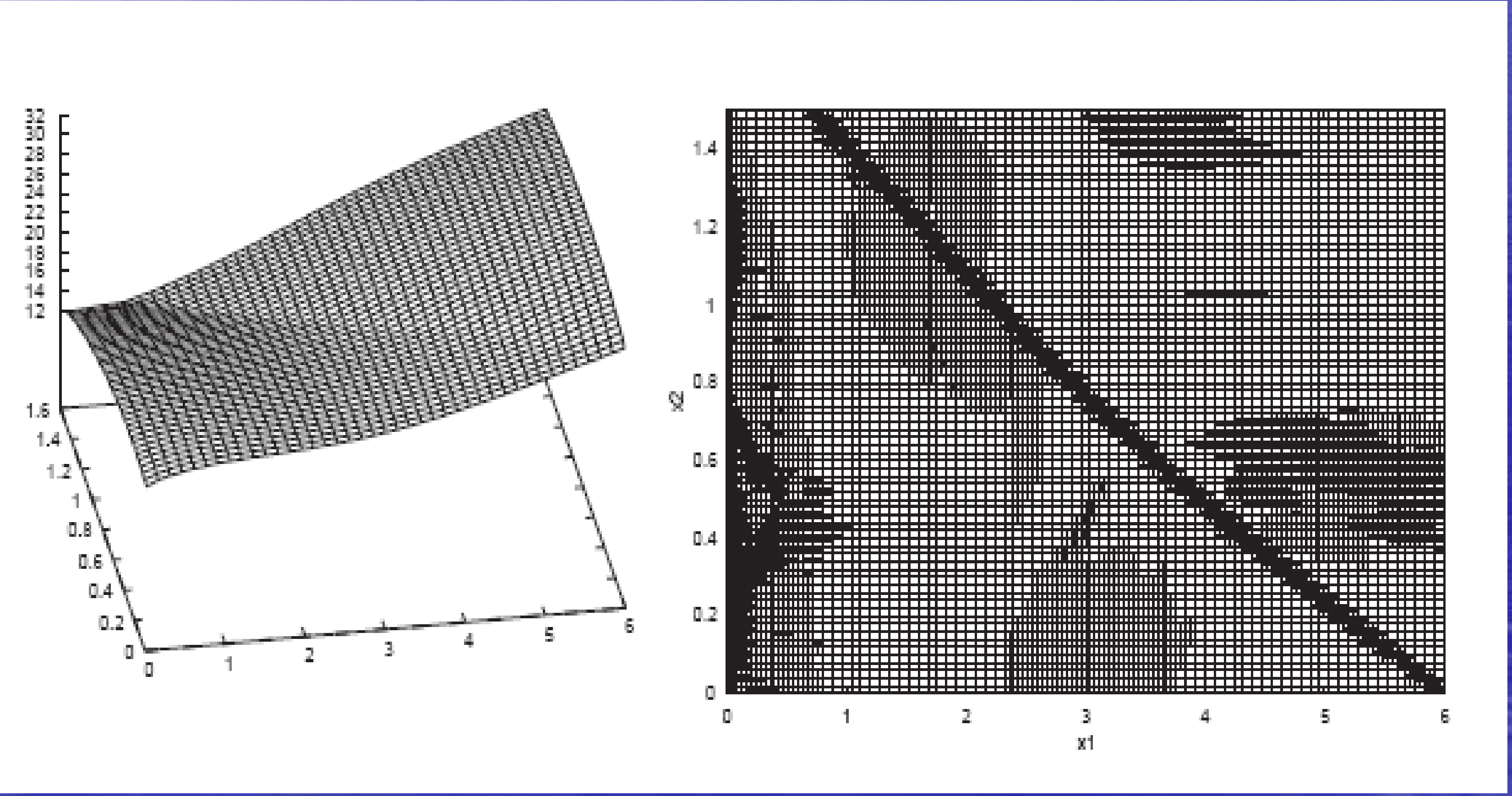
Advantage of adaptive grid size:





IV. Numerical Methods to Compute the Global Dynamics

Value function and Skiba line through adaptive grid



V. Example: State Dependent Risk Premium and Credit Derivatives
(Grüne et al. JFE, 2005, JCR, 2007)

Standard model of credit derivatives (Merton (1974), uses Brownian motions)

- According to Merton (1974), the debt payment at maturity date T is $F(V,T) = \min(V, \bar{B})$ with V

$$dV = (\alpha_V V - C_V)dt + \sigma_V V dz.$$

$$\text{Distance-to-Default} = \frac{\text{[Market Value of Assets]-[Default Point]}}{\text{[Asset Volatility]}}$$

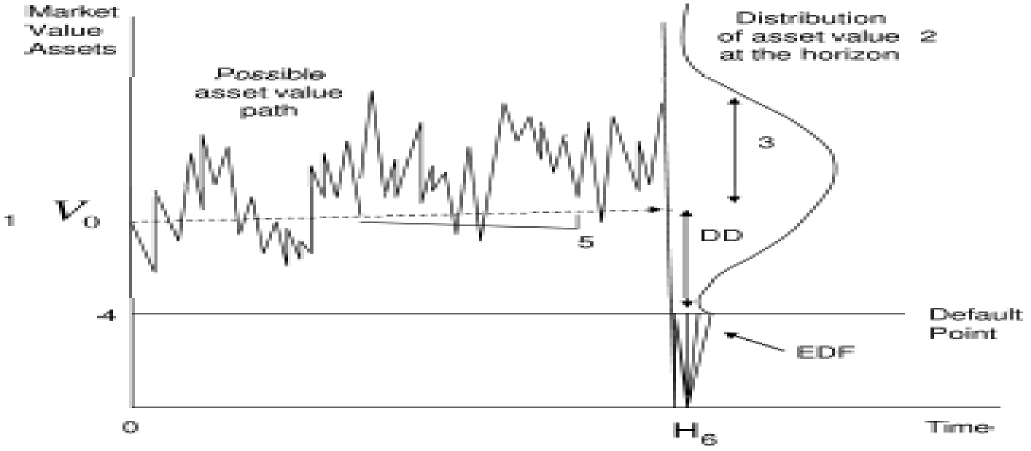
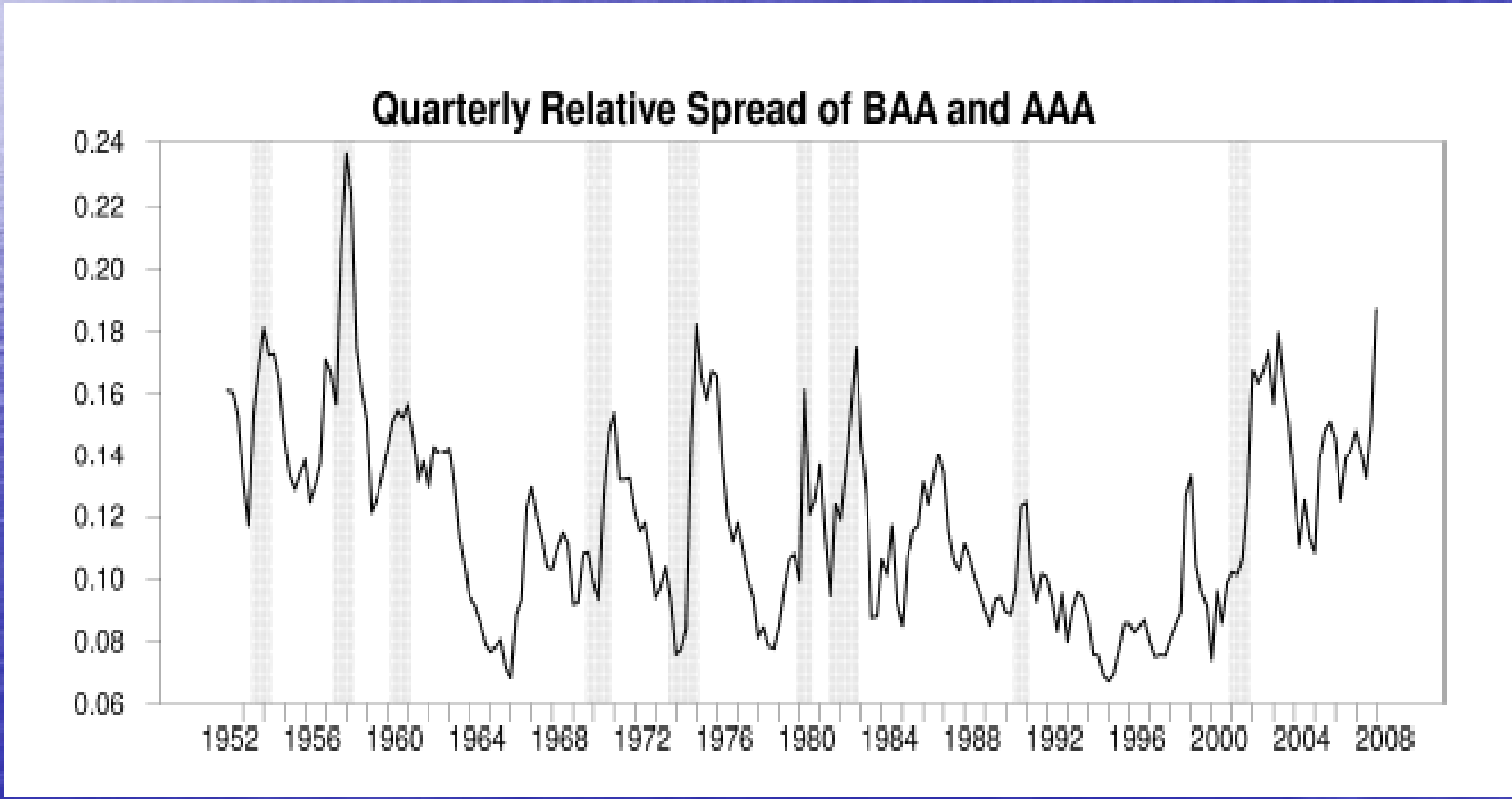


Figure 6: Distance-to default

V. Example: State Dependent Risk Premium and Credit Derivatives

Historical Bond Premia



We construct the value of the underlying asset from a firm value model (discounted expected cash flows: V>B?)

Model with Risk Free Interest Rate

- with $\theta = \frac{H(k,B)}{B}$ a constant, and we have the benchmark case, then $V(k)$ is in fact the present value of k

$$V(k) = \underset{j}{Max} \int_0^\infty e^{-\theta t} f(k(t), j(t))\, dt \tag{12}$$

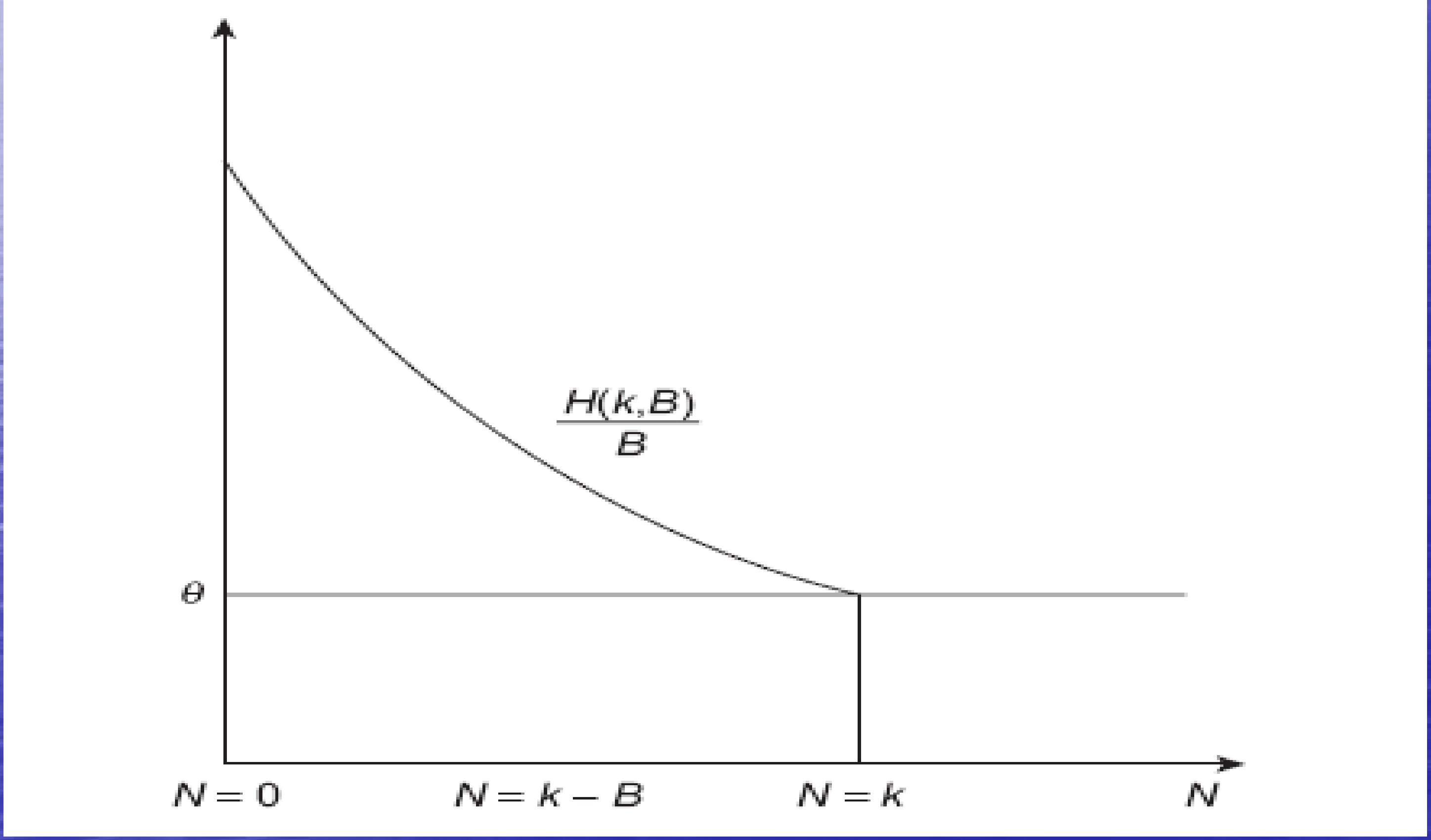
$$\dot{k}(t) = j(t) - \sigma k(t), \quad k(0) = k. \tag{13}$$

$$\dot{B}(t) = \theta B - f(k(t), j(t)), \; B(0) = B_0 \tag{14}$$

with $B^* = V$. If it holds that $V - B > 0$ then the residual remains as equity of the firm.

Motivated by the empirical evidence, we suggest model with state dependent risk premium:

risk premium (or finance premium, Bernanke et al)



With state depending risk premium we compute asset value and debt capacity (and test if B>V)

- With default risk and risk premium, we have for the firm

$$V(k) = \underset{j}{Max} \int_0^\infty e^{-\theta t} f\left(k(t), j(t)\right) dt \tag{8}$$

$$\dot{k}(t) = j(t) - \sigma k(t), \quad k(0) = k. \tag{9}$$

.

$$\dot{B}(t) = H\left(k(t), B(t)\right) - f\left(k(t), j(t)\right), \; B(0) = B_0 \tag{10}$$

The firm's net income

$$f(k, j) = ak^\alpha - j - j^\beta k^{-\gamma} \tag{11}$$

Solution through the HJB Equation

- Example of the appropriate HJB equation:

$$H(k, B^*(k)) = \max_j \left[f(k,j) + \frac{dB^*(k)}{dk}(j - \sigma k) \right]$$

$$B^*(k) = \max_j H^{-1} \left[f(k,j) + \frac{dB^*(k)}{dk}(j - \sigma k) \right]$$

whith $H(k, B) = B^\kappa \theta$ where, with $\kappa > 1$. Thus,

$$B^*(k) = \max_j \left[f(k,j) + \frac{dB^*}{dk}(j - \sigma k) \right]^{\frac{1}{\kappa}} \theta^{-\frac{1}{\kappa}}$$

State Dependent Risk Premium: Asset Value and Debt Capacity

$$\dot{B} = H(k, B) - f(k, j).$$

Multiplying by $e^{-\theta t}$ and using partial integration we find

$$\int_0^T e^{-\theta t} f(k, j) dt = B(0) - e^{-\theta T} B(T) + \int_0^T e^{-\theta t}(H - \theta B) dt.$$

The present value of the external finance premium - with initial value (k, B) - is

$$\int_0^\infty e^{-\theta t}(H - \theta B) dt = V_H(k, B).$$

where we use the optimal investment rate j. Then for $T \to \infty$ we find

$$V(k) = B(0) - lim\ e^{-\theta T} B(T) + V_H(k, B)$$

The term $V_H(k, B)$ is equal to zero for $H(k, B) = \theta B$.
In particular, if $B(0) = B^*(k)$ we have

$$V(k) = B^*(k) + V_H(k, B^*(k))$$

Numerical Solution: Risk Free Interest Rate

- Borrowing at the Risk-free Rate,parameters: $\sigma = 0.15$, $A = 0.29$, $\alpha = 0.7$, $\beta = 2$, $\gamma = 0.3$ and $\theta = 0$, $H(k, B) = \theta B$.

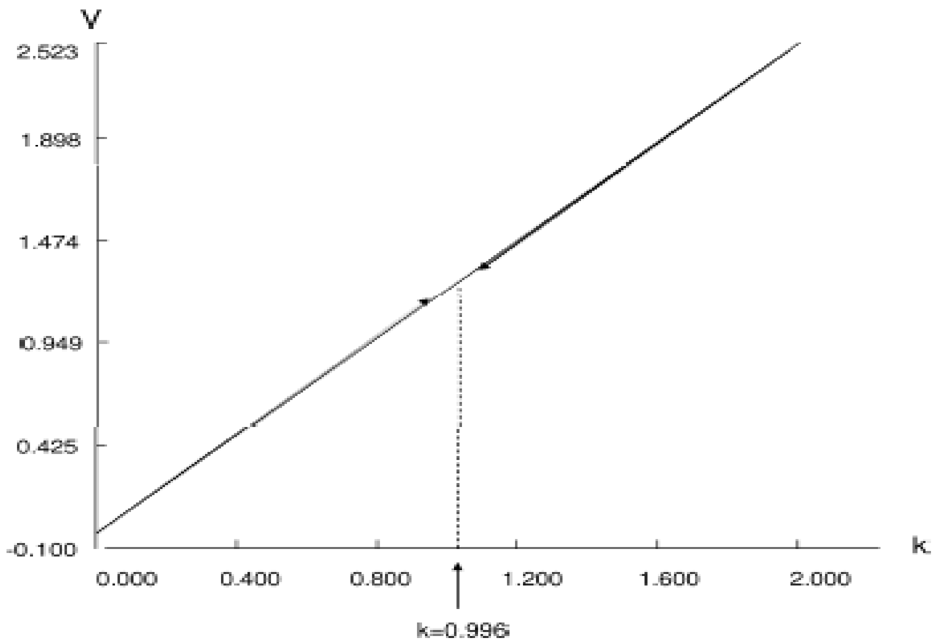
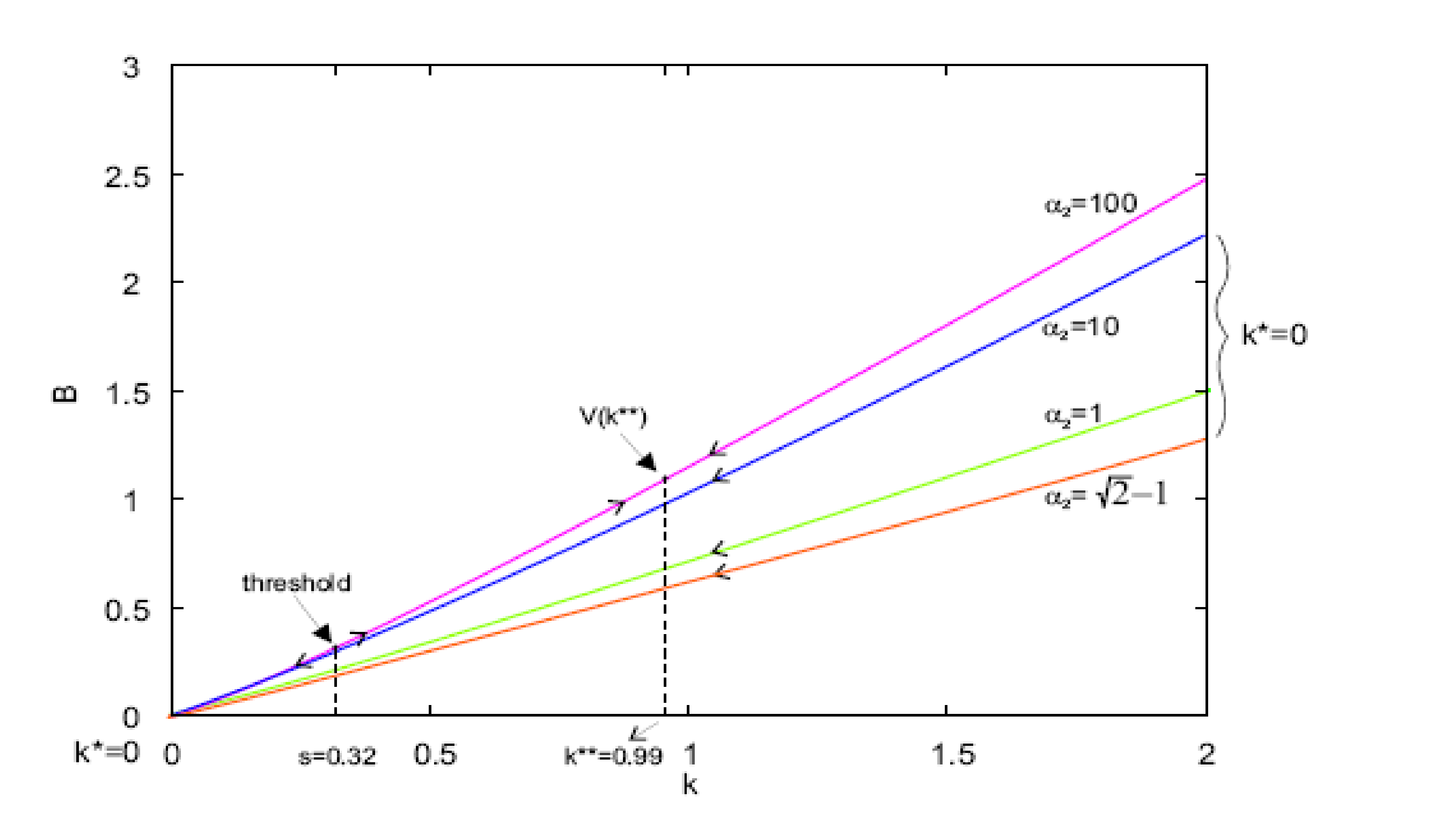


Figure 8: Value function

The debt control problem is solved whenever debt is bounded by the firm's asset value, so that we have $V - B \geq 0$.

Numerical Solution: With Risk Premium (Multiple Attractors and Threshold Dynamics)



Numerical Solution: With credit constraints (defined by banks` lending standards)

- Model with credit constraints: $c = 1.2$ and $c = 0.6$

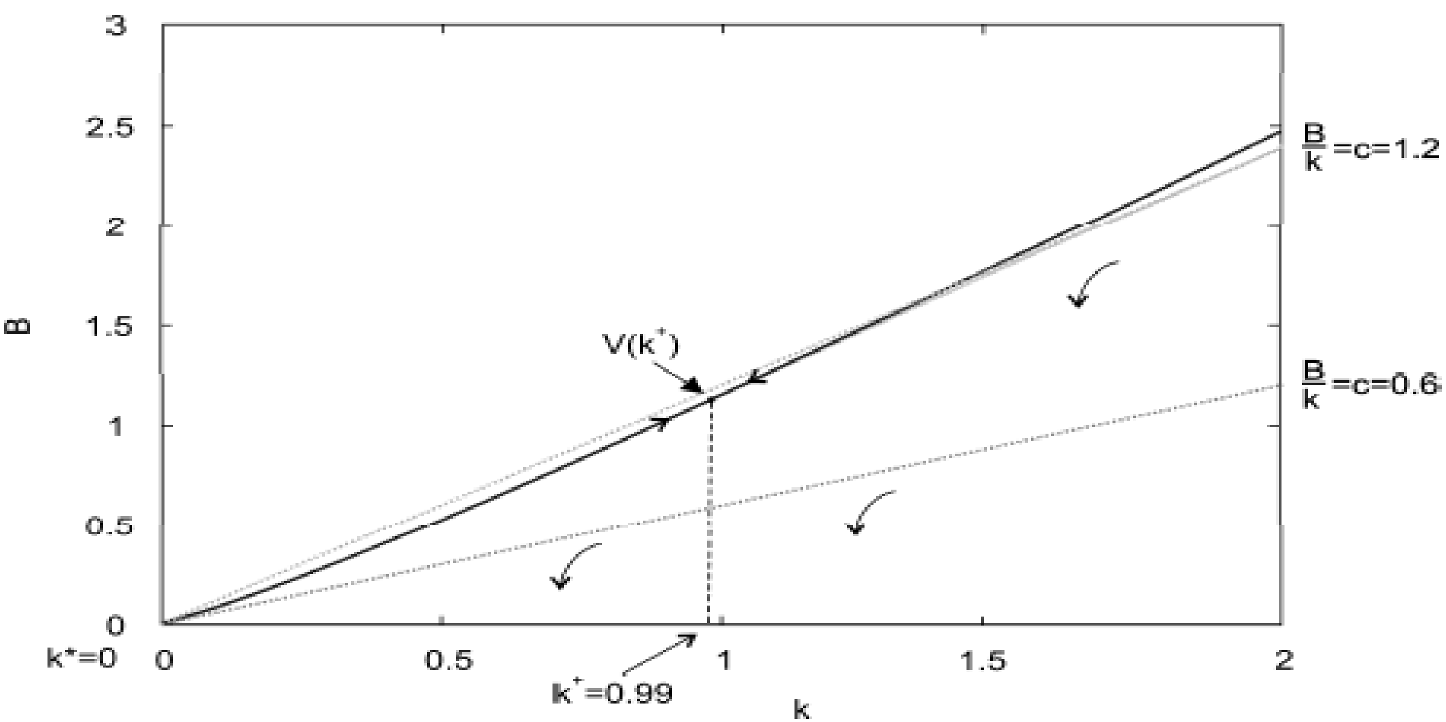


Figure 12: Present value curve $V(k)$ for different debt ceilings, $H(k, B)$ from (4)

Stochastic Case: with additive Shock

- With additive disturbances,

$$\dot{k}(t) = (j(t) - \delta_k k(t))dt + \delta_k k(t)dw(t) \tag{15}$$

$$\dot{B}(t) = (H(k(t), B(t))) - f(k(t), j(t))dt \tag{16}$$

We use again our standard parameters of section 5, but $\alpha_2 = 100$, $\alpha_1 = (\alpha_2 + 1)^2$ and $\mu = 2$. Details of the numerical procedure are given in the appendix.

Stochastic Case: no additive Shock, delta_k=0

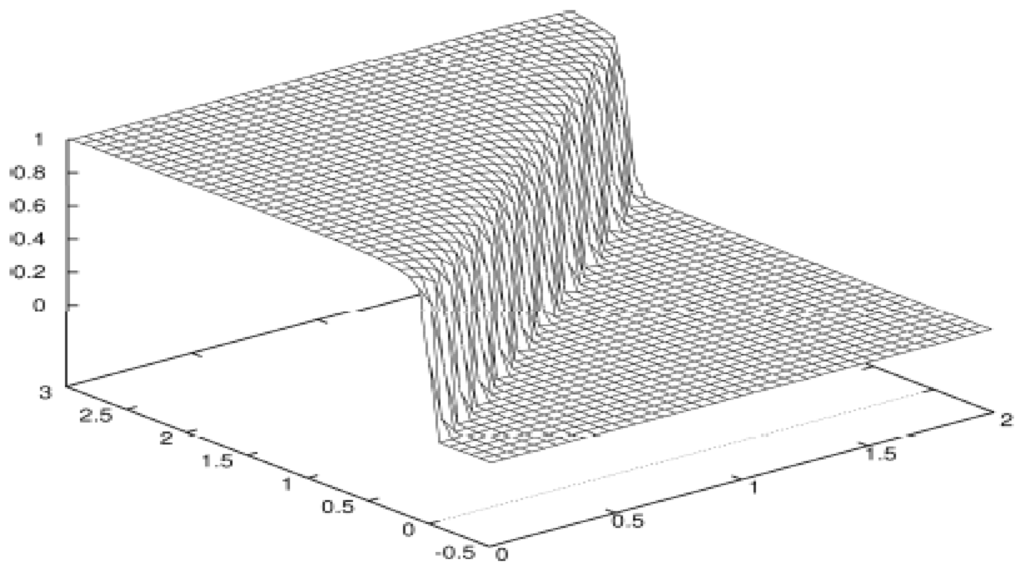


Figure 13: Numerically determined probabilities for $\delta_k = 0$

Stochastic Case: with additive Shock, delta_k=0.1

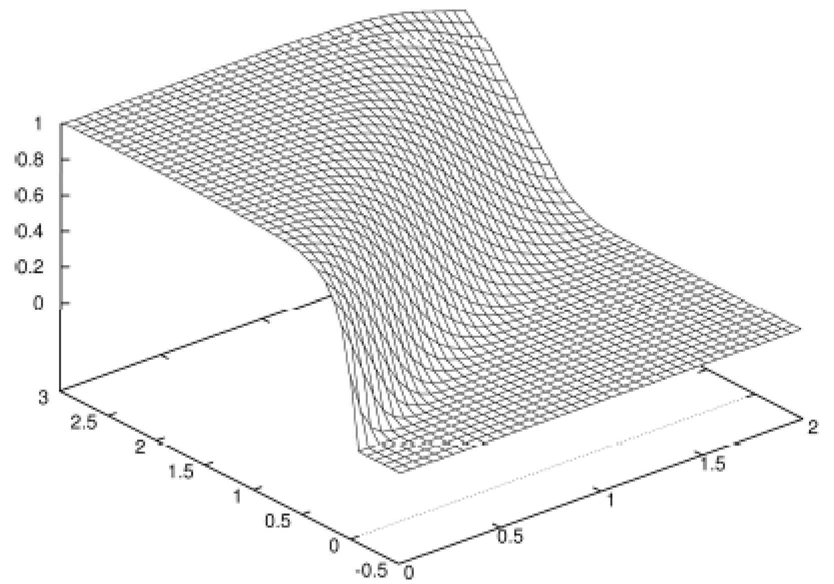


Figure 14: Numerically determined probabilities for δ_k =0.1

Stochastic Case: with additive Shock, delta_k=0.5

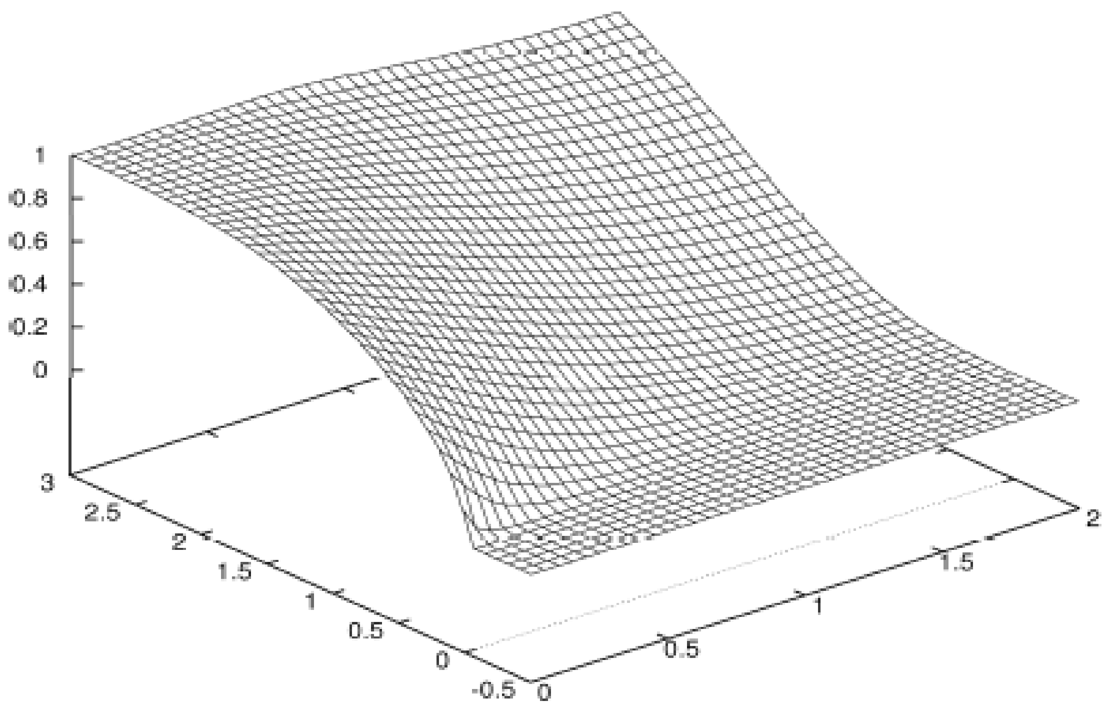


Figure 15: Numerically determined probabilities for δ_k =0.5

VI: Multiple Attractors, Heterogeneity and Empirics

- Many of the models admit multiple attractors, thresholds and intricate global dynamics which can be solved by DP with adaptive grid
- We can allow heterogeneity of economic agents: a distribution of agents along the relevant state space (1dim, 2 dim)
- We can undertake empirical work to test whether the data support multiple attractors

Example 1: Investment model with multiple attractors and heterogeneity (Kato, Semmler and Ofori (2006), Firms 1960-1991)

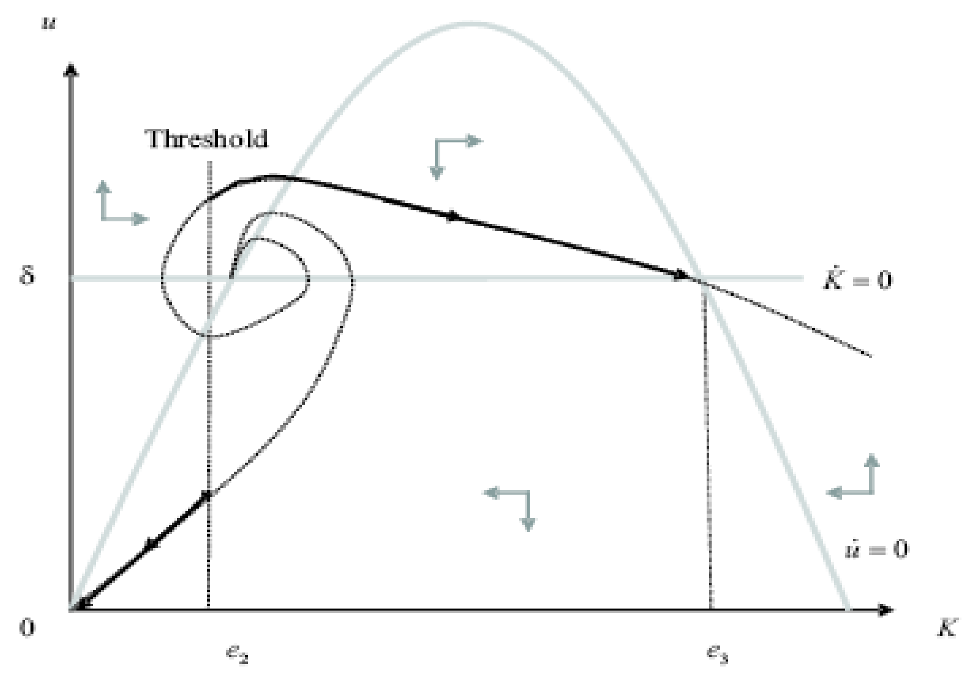


Figure 3: Discontinuous Policy Function in $\{u, K\}$ Space

Empirics with Markov Transition Matrices (transition matrice for firm data, 1960-1991)

For purpose of illustration the fluctuation matrices for the time period 1973-74 is presented in table 1:

	$k \leq \frac{1}{4}$	$\frac{1}{4} < k \leq \frac{1}{2}$	$\frac{1}{2} < k \leq 1$	$1 < k \leq 2$	$k > 2$
$k \leq \frac{1}{4}$	837	12	0	0	0
$\frac{1}{4} < k \leq \frac{1}{2}$	12	177	12	0	0
$\frac{1}{2} < k \leq 1$	1	8	99	11	0
$1 < k \leq 2$	0	0	5	78	7
$k > 2$	0	0	0	2	129

Table 1: Example of a fluctuation matrix

The Average Markov Transition Matrix: Thinning Out of the Middle

$$P = \begin{pmatrix} 0.951 & 0.023 & 0.008 & 0.005 & 0.013 \\ 0.228 & 0.692 & 0.067 & 0.005 & 0.007 \\ 0.162 & 0.069 & 0.700 & 0.062 & 0.010 \\ 0.155 & 0.008 & 0.085 & 0.674 & 0.078 \\ 0.158 & 0.006 & 0.006 & 0.032 & 0.799 \end{pmatrix}$$

Example 2: On Poverty Traps, Thresholds and Take-Offs (Semmler and Ofori, Journal of Economic Dynamics and Structural Change, 2006, per capita income across the world)

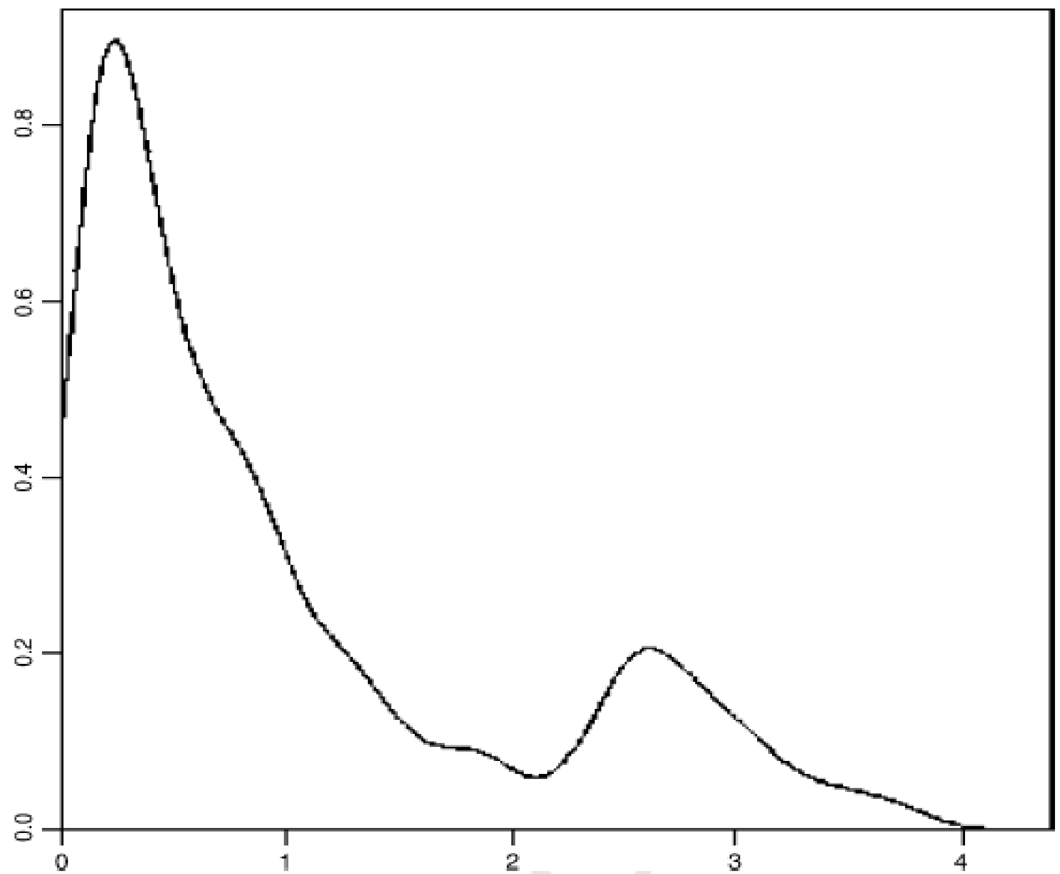


Fig. 10. Density of relative income in 1985.

Average Markov Transition Matrix for per capita Income (1960-1985)

Transition matrix M

	$Y \leq \frac{1}{4}$	$\frac{1}{4} < Y \leq \frac{1}{2}$	$\frac{1}{2} < Y \leq 1$	$1 < Y \leq 2$	$Y > 2$
$Y \leq \frac{1}{4}$	0.9412	0.0587	0.0000	0.0000	0.0000
$\frac{1}{4} < Y \leq \frac{1}{2}$	0.0693	0.9307	0.0420	0.0000	0.0000
$\frac{1}{2} < Y \leq 1$	0.0000	0.0345	0.9243	0.0411	0.0000
$1 < Y \leq 2$	0.0000	0.0000	0.0369	0.9433	0.0197
$Y > 2$	0.0000	0.0000	0.0000	0.0076	0.9923

Example 3: Heterogeneity, Portfolio Choice and Wealth Distribution (Grüne, Öhrlein and Semmler (2007), 2 dim problem)

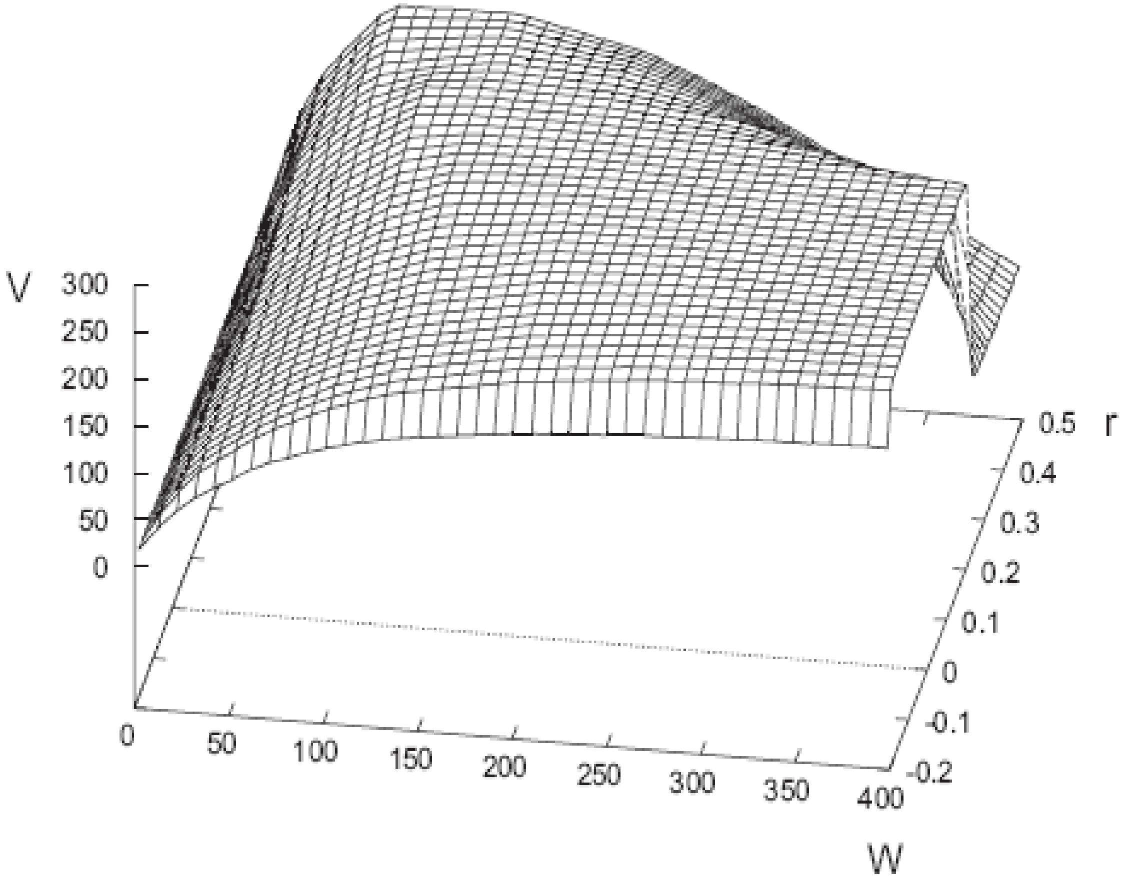
$$\max_{\alpha,C} \int_0^\infty e^{-\delta t} \frac{C^{1-\gamma}}{1-\gamma} dt$$

s.t.

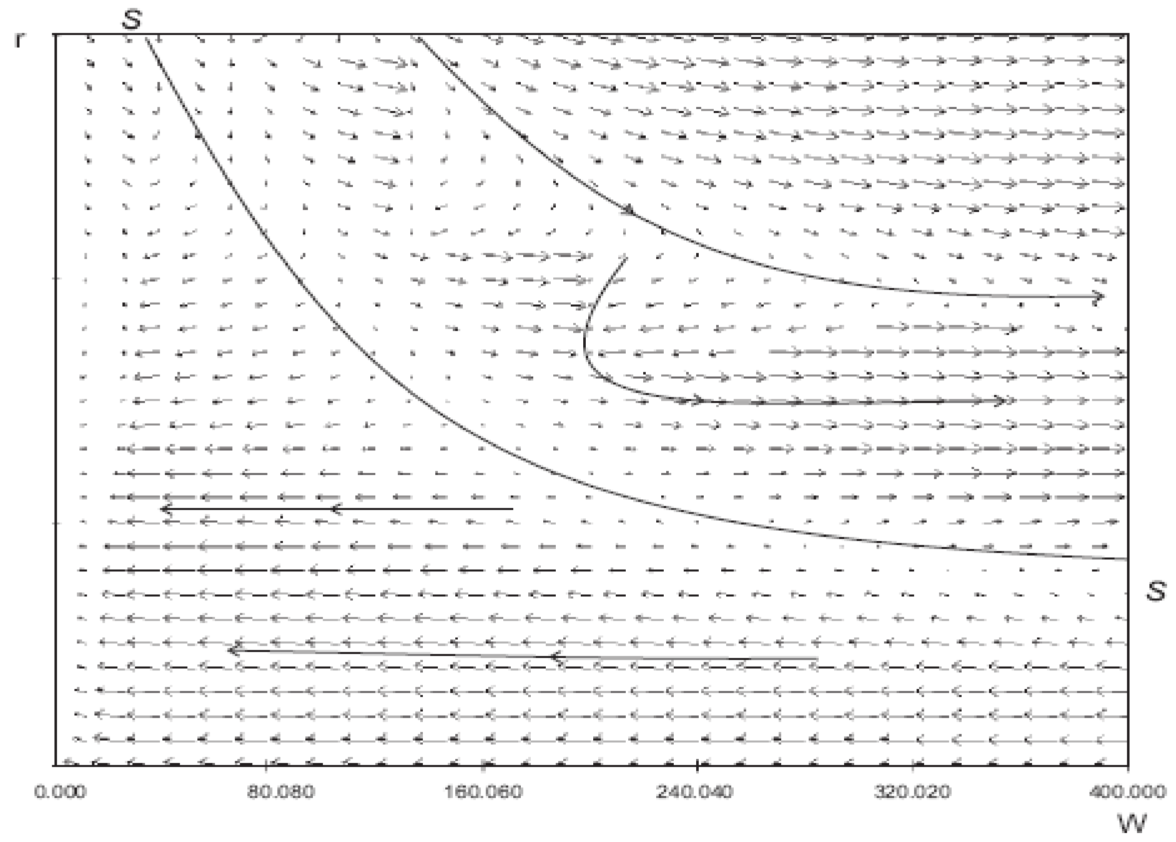
$$dW = \{[\alpha_t(r_t + \overline{x}_t) + (1-\alpha_t)r_t]W_t - C_t\}dt + \sigma_w dz_t \quad W$$

$$dr_t = \kappa(\theta - r_t)dt + \sigma_r dz_t$$

Wealth distribution and value function



Domains of Attraction and Threshold Line



VII. Policy Implications: Enlarging Domains of Attraction

(lowering the interest rate enlarges the domain of attraction of the higher equilibrium, see Example 1)

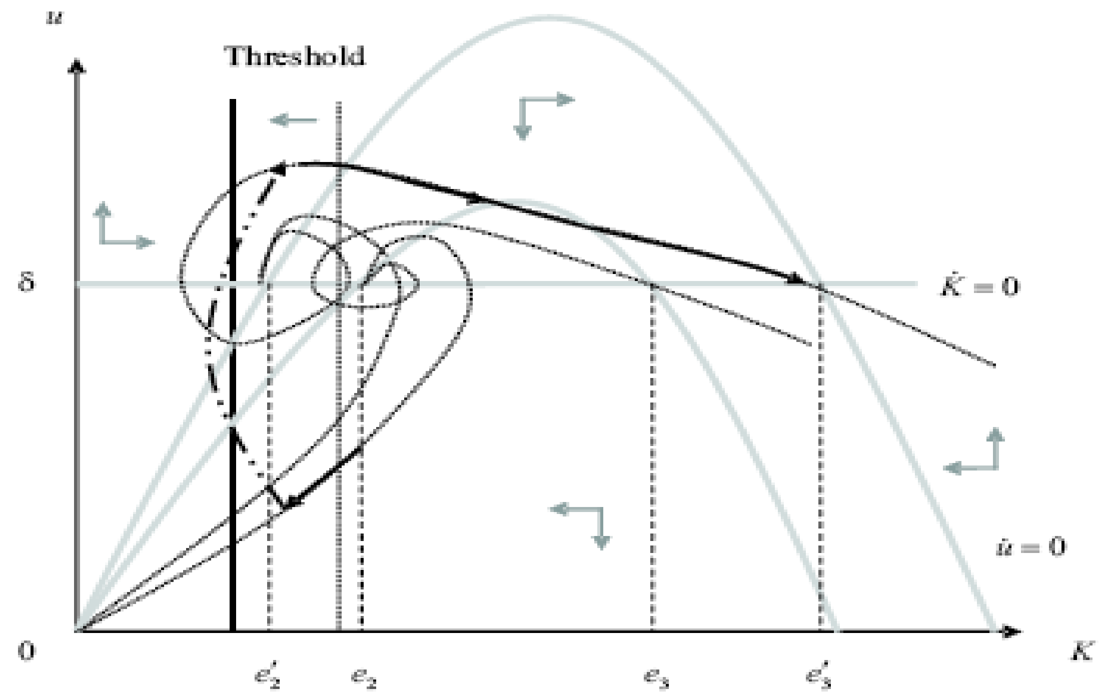
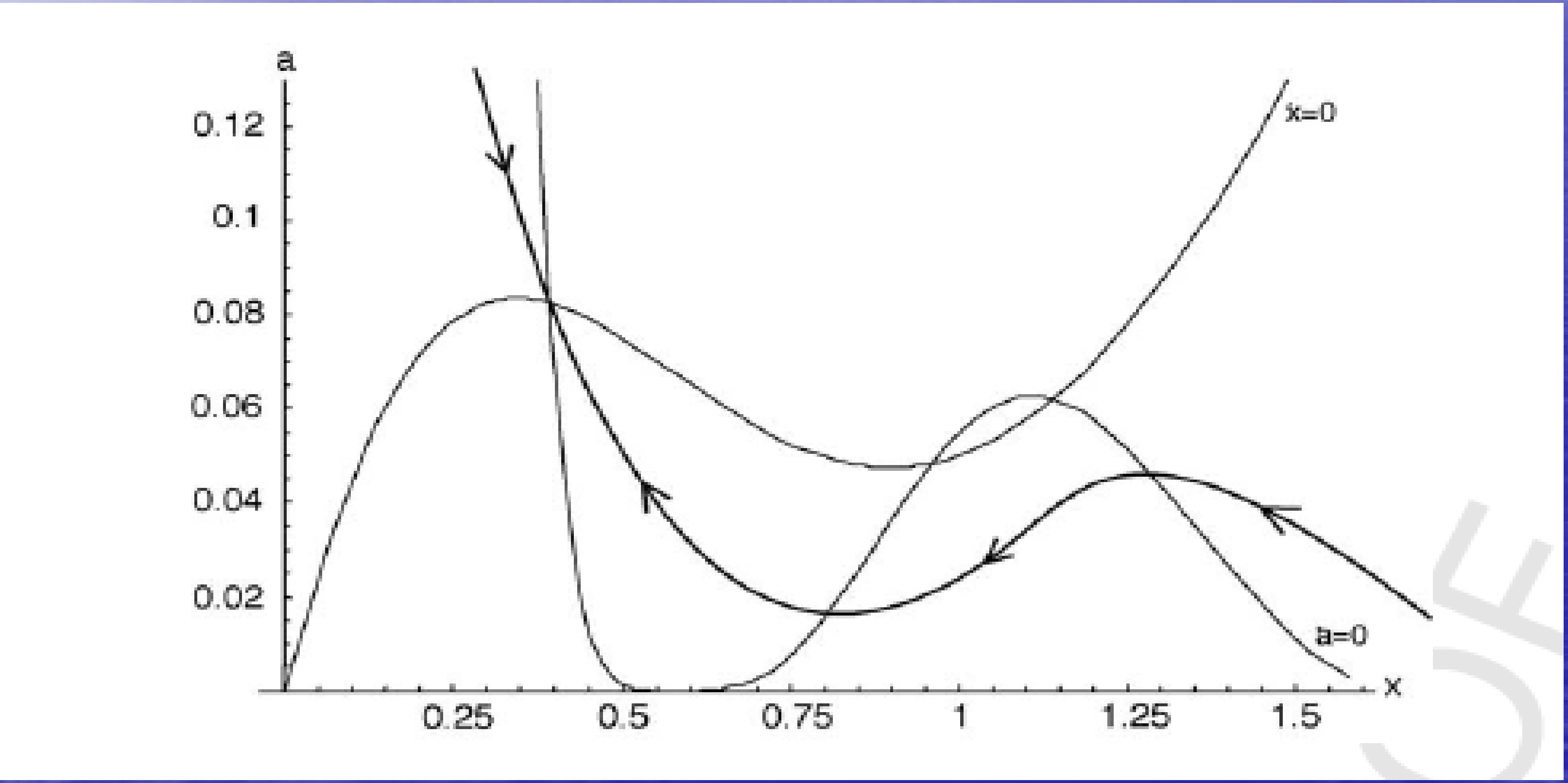


Figure 4: Comparative dynamic results

VII. Policy Implications: Enlarging Domains of Attraction

(Ecological Management Problem, Grüne, Kato and Semmler (2005), tax rates create the low equilibrium as sole attractor)



VII. Policy Implications: Enlarging Domains of Attraction

- Poverty traps in income distribution, thresholds and domains of attraction (Kato and Semmler 2007), transfers
- Currency and financial crises, thresholds and domains of attraction (Kato, Proano and Semmler 2007)
- Growth, global warming and thresholds, Greiner and Semmler, JEBO article (2005), and Greiner and Semmler, 2008 Book (OUP)

VIII. Conclusions

- We give a large number of examples of models with multiple equilibria from different areas of economics
- We show that the dynamic decisions of agents add to the intricacy of the dynamics (multiple attractors can arise)
- The global dynamics can be studied through DP with adaptive grid size
- Our approach allows for heterogeneity and empirical studies (Markov transition matrices)
- Importance for policies: Policy can change domains of attraction (enlarging the domain of attraction of preferable attractors)

Thank you

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„The Global Environment, Natural Resources, and Economic Growth", with A. Greiner (Oxford University Press, 2008)