Strong Random Correlations in Complex Systems

Imre Kondor

Collegium Budapest and Eötvös University, Budapest

Parmenides Foundation, Munich

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This is a report on work in progress.

Collaborators:

Albert-László Barabási, Boston

Alain Billoire, Saclay

Nándor Gulyás, Budapest

Jovanka Lukic, Rome

Enzo Marinari, Rome

Summary

- Complex systems are (nearly) irreducible (incompressible), depend on a large number of variables in a significant way.
- Irreducibility is related to (implies?) large, random correlations
- Illustration on two toy models: on a spin glass and a random cellular automaton
- Some consequences: e.g. simulation of such systems is a delicate issue, the result depends on tiny details, initial and boundary conditions.

Preliminary considerations

- It is plausible that in a system that depends on a large number of variables the correlations between its components must be „long-ranged” in some sense.
- As complex systems are not translationally invariant, „long-ranged” means strong correlations between many pairs, though not necessarily geometrical neighbours.
- The usual behaviour of correlations in simple systems is not like this: correlations fall off typically exponentially – which is why simple systems fall apart into small, weakly correlated subsystems, and have low effective dimension.

The difficulties of defining complexity

There are nearly as many complexity definitions as there are authors in complexity.

The AIT definition: the length of the shortest algorithm that is able to produce a given string is the measure of the complexity of the string.

Some authors emphasize emergence, confluence of scales, nonlinearity, unpredictability, path dependence, historicity, multiple equilibria, a mixture of sensitivity and robustness, learning and adaptability, and, ultimately, self-reflection, self-representation, consciousness as characteristics of complex systems.

When trying to formulate a common policy of sponsoring complexity research in Europe Complexity-NET, a network of European funding agencies, came to the conclusion that finding a compelling definition was a hopeless endeavour on which no more time should be wasted.

A possible alternative is to list examples: complex systems include the living cell, the brain, society, economy, etc.

Irreducibility

G. Parisi at the 1999 STATPHYS Conference in Paris: A system is complex if it depends on many details.

This suggests the idea of using the degree of irreducibility, perhaps the effective dimensionality (the number of variables) of the simplest model one can construct to describe the system to a given level of precision, as a measure of complexity.

NB: This definition shares the shortcomings of the algorithmic complexity concept: it assigns maximal complexity to noise, and it is probably impossible to decide which model is the simplest.

The incompressibility of history ☺

For the want of a nail the shoe was lost;
For the want of a shoe the horse was lost;
For the want of a horse the battle was lost;
For the failure of battle the kingdom was lost;—
And all for the want of a horseshoe nail.

Background: The Battle of Bosworth Field in 1485, between the armies of King Richard III and Henry, Earl of Richmond, that determined who would rule England.

A more serious example

10 days survival probability of patients after a heart attack. Depends on some 40 factors. Such a model cannot be parametrized even on a population of 10 million (overfitting). Yet health policy decisions depend on such analyses.

(Peter Austin at the 2007 AAAS meeting)

The simplest tool to analyze such problems is linear regression.

Linear regression:

- $$y = \beta_0 + \sum_{i=1}^{N-1} \beta_i x_i + \varepsilon \qquad\qquad \min_{\beta_0, \beta_{1,\dots,}\beta_{N-1}} \mathrm{Var}(\varepsilon)$$

$$R(\beta) = \mathrm{Var}(\varepsilon) = E(\varepsilon^2) = E\left(y - \beta_0 - \sum_{i=1}^{N-1} \beta_i x_i \right)^2$$

$$\frac{\partial R(\beta)}{\partial \beta_j} = -2\left[\sum_{i=1}^{N-1} \mathrm{Cov}(x_j, x_i)\beta_i - \mathrm{Cov}(x_j, y) \right] = 0$$

$$\frac{\partial R(\beta)}{\partial \beta_0} = 2\left[\beta_0 + \sum_{i=1}^{N-1} E(x_i) - E(y) \right] = 0$$

Ideally, the number of dimensions N is small and the length of the available time series T is long. Then the estimation error is small, and the model works fine.

If the system is complex, however, we will have a very large N, and that raises serious estimation error and convergence problems.

When a huge number of regression coefficients are roughly equal, we do not have structure, the model produces noise.

It may happen, however, that the regression coefficients are not equal, but do not have a cutoff beyond which they would become insignificant either: they may not have a characteristic scale, but fall off like a power.

But: large regression coefficients imply large correlations:

$$\frac{\partial R(\beta)}{\partial \beta_j} = -2\left[\sum_{i=1}^{N-1} \mathrm{Cov}\left(x_j, x_i\right)\beta_i - \mathrm{Cov}\left(x_j, y\right)\right] = 0$$

If the independent variables are uncorrelated then the regression coefficients are proportional to the covariances between the dependent variable and the idependent variables

This suggests the idea to look into some toy models and see if large correlations may indeed be a characteristic feature of complex systems.

Two toy models will be studied here:

The +/-J „long range” spin glass
and a
Random cellular automaton

The spin glass: A model of cooperation and competition

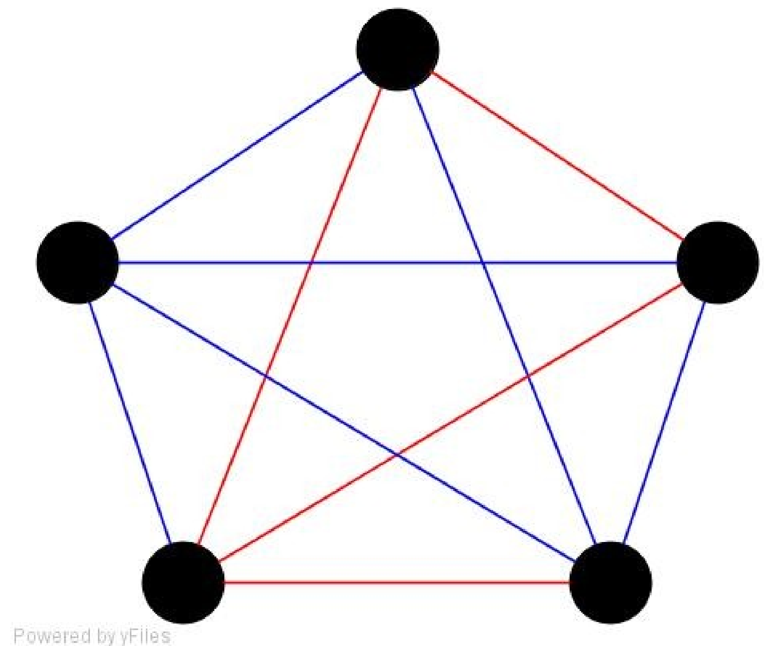
An Ising-like model with random couplings:

$$H = -\sum_{i} J_{ij} \cdot s_i \cdot s_j \qquad\qquad s_i = \pm 1$$

where $J_{ij} = \pm 1$ are randomly scattered over the lattice or graph. For simplicity we keep to the complete graph in the following.

On a small complete graph, e.g…

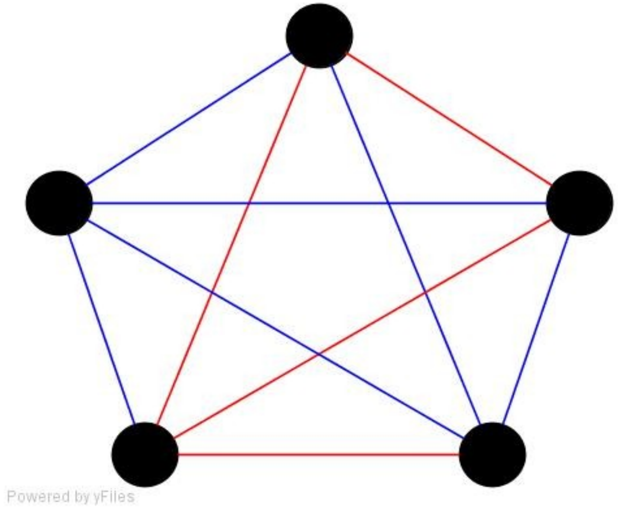
The red edges represent negative („antiferromagnetic”) couplings. Spins linked by such a negative coupling would like to point in opposite directions.



The optimal arrangement of the spins is a distribution of plus-minus ones, correlated with the distribution of couplings in a complicated manner. Even the optimal arrangement can contain a lot of tension: not all the couplings can be satisfied simultaneously.

Frustration

The presence of negative couplings leads to „frustration”: one may have two friends who hate each other. Such a trio cannot be made happy. In the little example



the triangles containing an odd number of red edges are frustrated.

Frustration makes the overall bonding much weaker: the ground state energy is higher than for a pure system. At the same time the degeneracy of low lying states (the multiplicity of states with the same energy) is much enhanced.

For large N, the low temperature structure of such a model can be extremely complicated, with several nearly degenerate minima and their basins of attraction cutting up the set of microscopic states into a set of „pure states” or „phase space valleys”.

A central concept in the characterization of this structure is that of the overlap:

$$q_{\alpha\beta} = \frac{1}{N} \cdot \sum_{i} s_i^{\alpha} \cdot s_i^{\beta}$$

which measures the degree of similarity between two states.

Correlations in ordinary lattice models

Normally, correlations fall off exponentially except

- at the critical point
- in the ordered phase of models with a broken continuous symmetry.

An Ising spin glass does not have any continuous symmetry, there is no a priori reason to expect long range correlations.

Correlations in spin glasses

Due to the random structure of the model, the correlations $\langle s_i \cdot s_{i+r} \rangle$ behave in a chaotic, random manner as a function of distance. When averaged over the random distribution of the couplings they become a trivial Kronecker delta:

$$\overline{\langle s_i \cdot s_{i+r} \rangle} = \delta_{r,0}$$

For this reason it has been customary to study higher order average correlations, often defined for a given average overlap.

Some of these correlation functions:

$$C_2(r) = \frac{1}{N} \cdot \overline{\sum_i \langle s_i \cdot s_{i+r} \rangle_\alpha \cdot \langle s_i \cdot s_{i+r} \rangle_\beta}$$

$$C_3(r) = \frac{1}{N} \cdot \overline{\sum_i \langle s_i \rangle_\alpha \cdot \langle s_{i+r} \rangle_\alpha \cdot \langle s_i \cdot s_{i+r} \rangle_\beta}$$

$$C_4(r) = \frac{1}{N} \cdot \overline{\sum_i \langle s_i \rangle_\alpha \cdot \langle s_{i+r} \rangle_\alpha \cdot \langle s_i \rangle_\beta \cdot \langle s_{i+r} \rangle_\beta}$$

Correlation in one phase space valley

A natural combination of the above correlation functions, computed in the Gaussian approximation via replica field theory, turned out to be long-ranged (De Dominicis, Temesvári, I.K.):

$$\overline{\left(\langle s_i \cdot s_{i+r}\rangle - \langle s_i\rangle \cdot \langle s_{i+r}\rangle\right)^2} \propto \frac{1}{r^{d-2}} \qquad q = q_{\max} \qquad \forall T < T_c$$

Correlations between distant valleys

Remarkably, the overlap between correlation functions belonging to phase space valleys with zero overlap also was found long-ranged:

$$\overline{\langle s_i \cdot s_{i+r} \rangle_\alpha \cdot \langle s_i \cdot s_{i+r} \rangle_\beta} \propto \frac{1}{r^{d-4}} \qquad q_{\alpha\beta} = 0$$

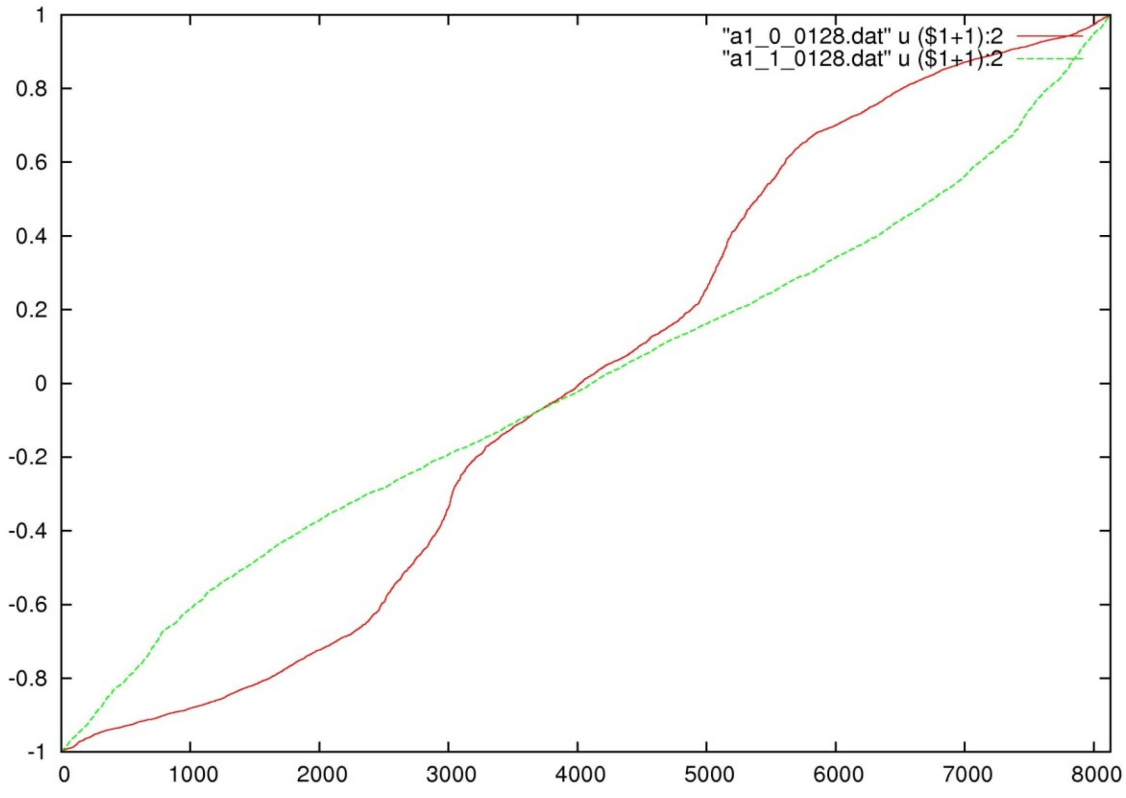
So the average correlations are long-ranged in spin glasses.

Correlations in a given sample

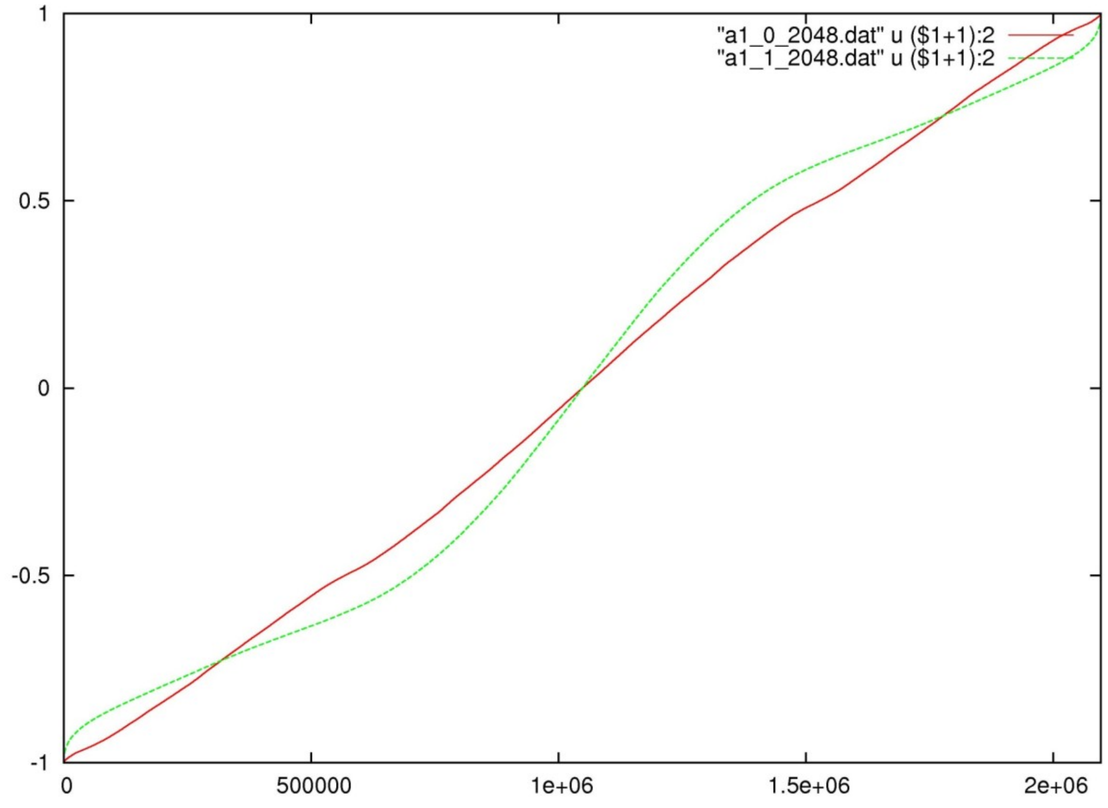
It may also be of interest to look into the distribution of correlations as random variables *in a given sample*. In order to do this, we measured all the $N(N\text{-}1)/2$ correlations $\langle s_i \cdot s_{i+r} \rangle$ and ranked them according to magnitude. Exact enumeration on small systems (up to N=20) and numerical simulations up to N = 2048 indicate that the correlations are anomalously large in the low temperature spin glass.

Some preliminary results follow.

Sorted correlations for two samples of size N=128, at low temperature (T=0.4), averaging over all microstates



The same for two samples of size N= 2048, at T=0.4, averaging over all microstates



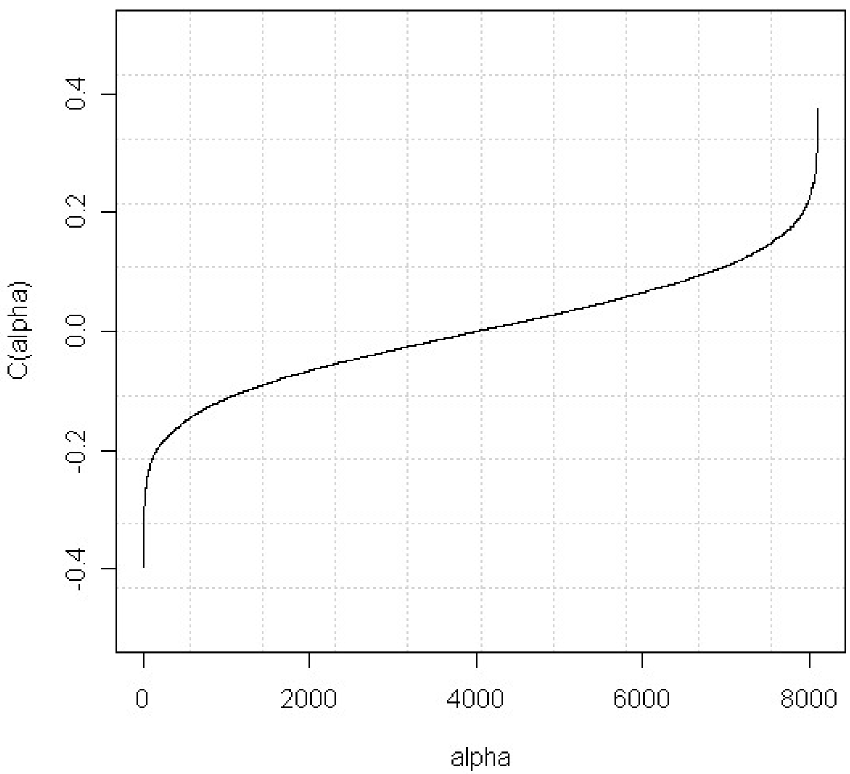
The sorted distribution is the inverse of the cumulative distribution function.

For large N the sample to sample fluctuations disappear, the distribution can, in principle, be calculated via replicas.

The sorted correlations suggest that their probability distribution is very broad, maybe uniform, or even bimodal!

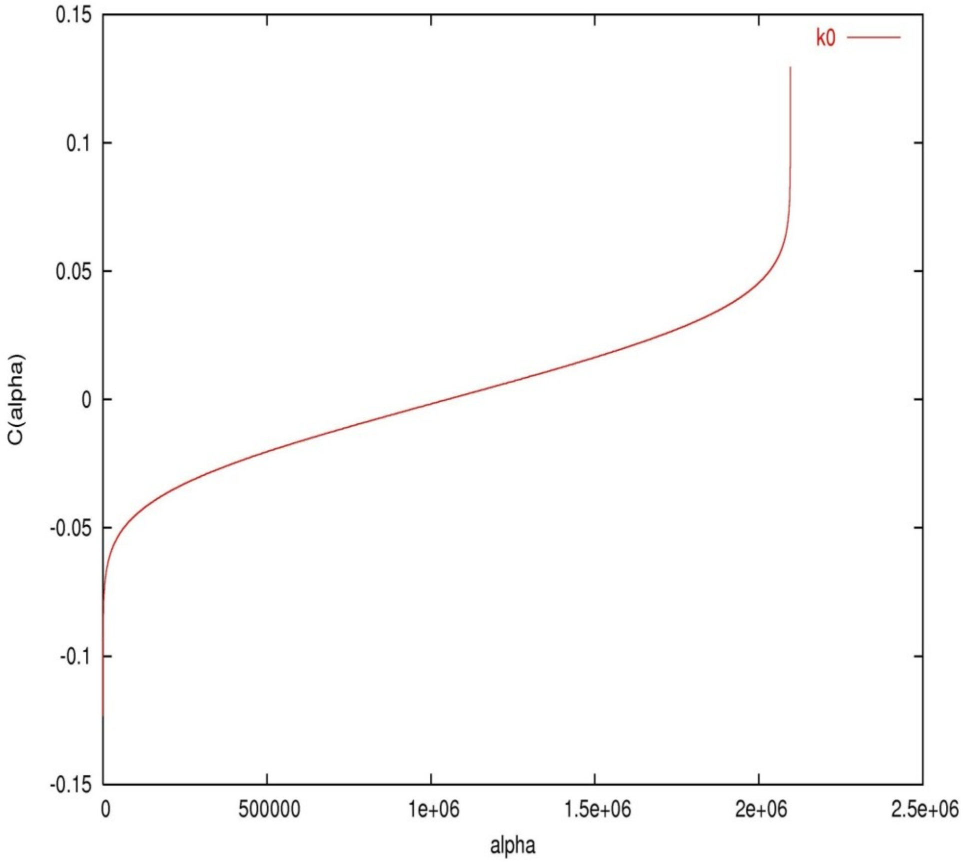
Note the apparent symmetry of the sorted distribution which does not correspond to any exact symmetry of the system.

When we go above the critical temperature T=1: N=128, T=1.3



Note the change of scale! Most correlations are very small now.

N=2048, T=1.3



For this large system
the correlations
even smaller.
for N large
number of
correlations
which is
on

The main points

There is a marked difference between the high and low temperature phases. Correlations are strong all through the low temperature phase.

A replica calculation shows that the distribution of $\langle s_i \cdot s_{i+r} \rangle$ is symmetric, and its second moment is the same as the second moment of the overlap distribution $P(q)$.

A random cellular automaton RCA

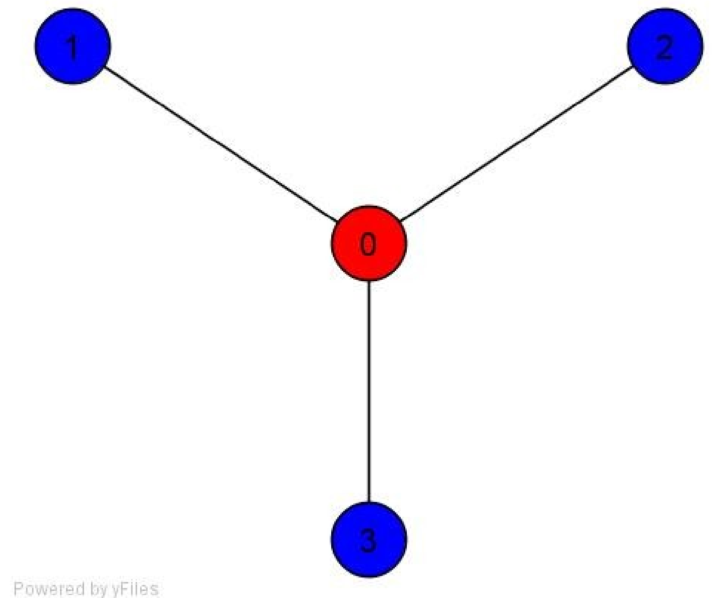
The model is a finite T variant of the Kauffman automaton.

It is a collection of binary variables again, this time living on a 2d lattice. They update their state according to the configuration of their neighbours.

RCA update rule

Table of interactions

s_1	s_2	s_3	K
1	1	1	0.3
1	1	-1	-0.6
1	-1	1	0.4
1	-1	-1	0.5
-1	1	1	-0.9
-1	1	-1	0.01
-1	-1	1	0.5
-1	-1	-1	-1.2

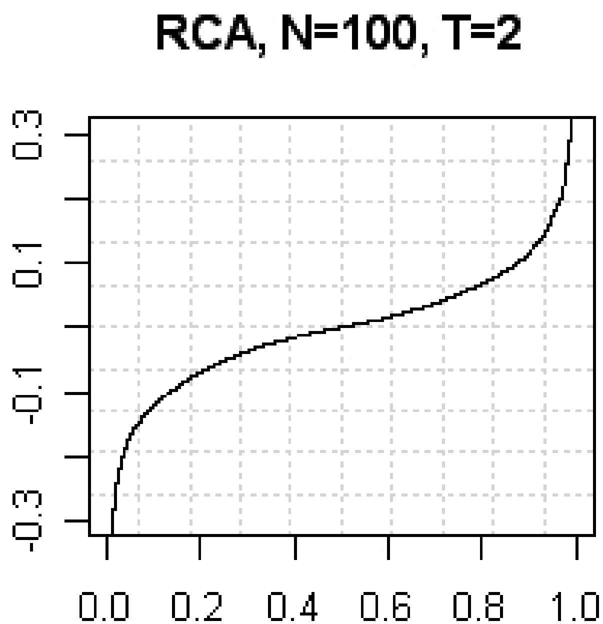


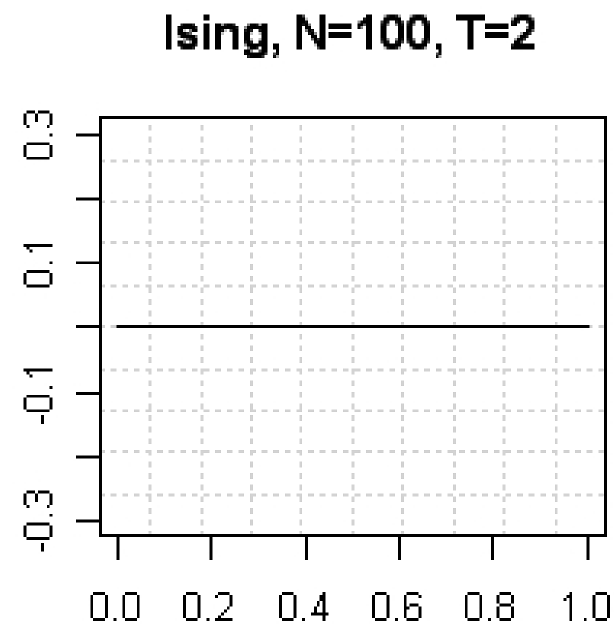
„Hamilton function" with the *K*'s N(0,1) distributed:

$$H = -\sum_i K_i \cdot s_i$$

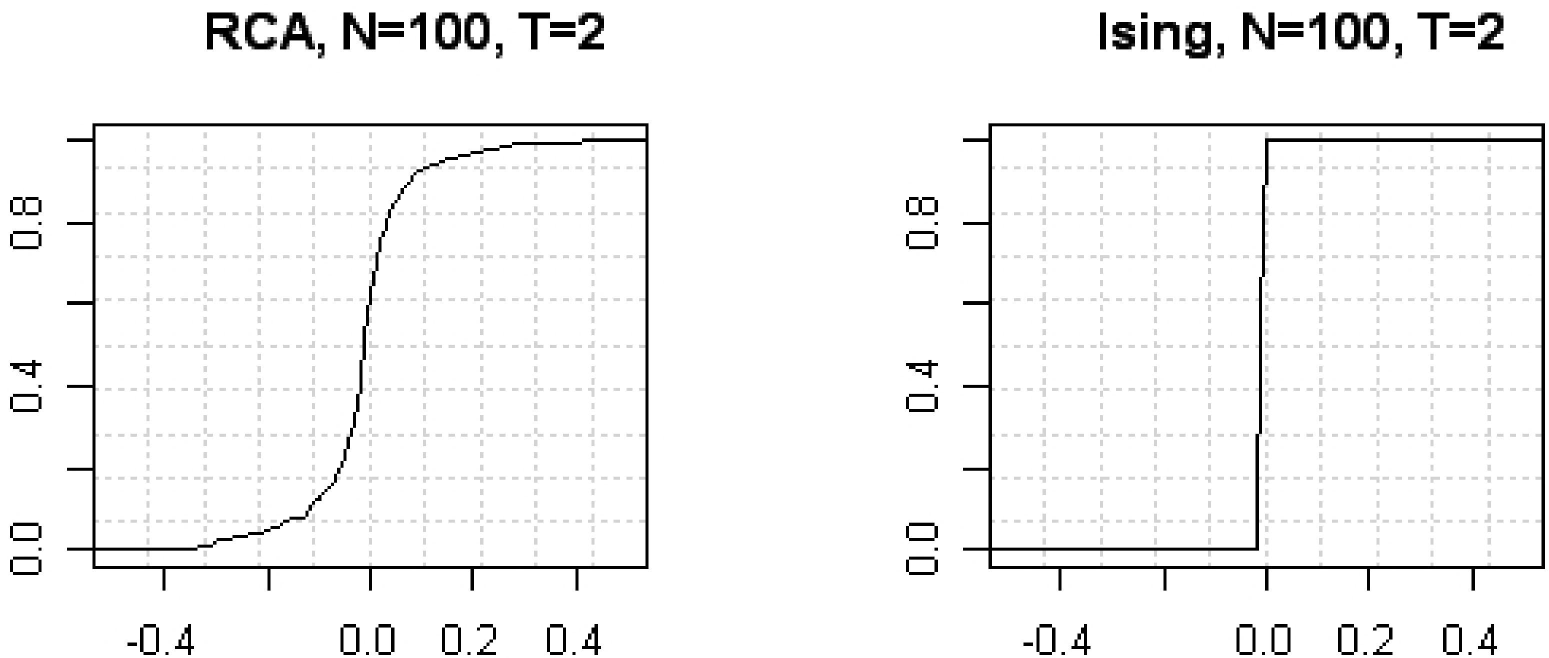
- From this point on the simulation of the model runs along the usual Monte Carlo path
- The results are shown parallel to the same for the Ising model.
- Note that this is a lattice model, so there is a geometry behind it (distance, neighbourhood, etc. make sense), and we can ask questions about the geometric distribution of large correlations.

Sorted correlations

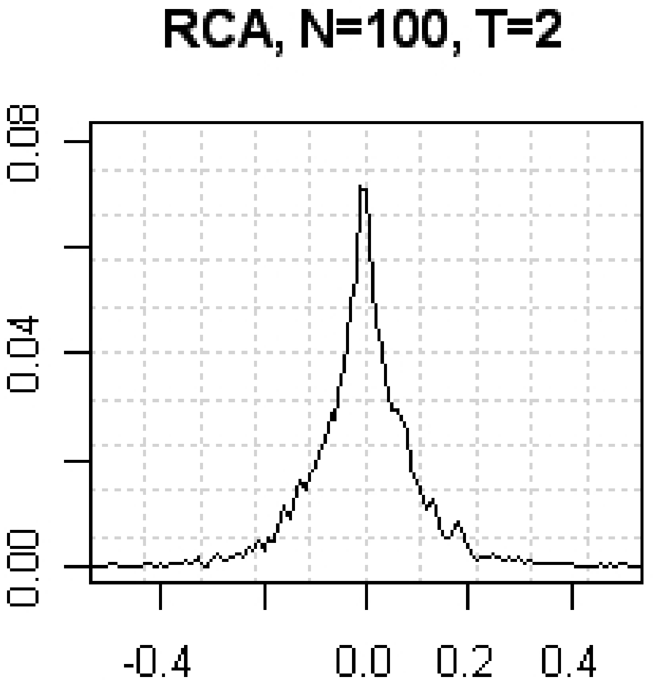


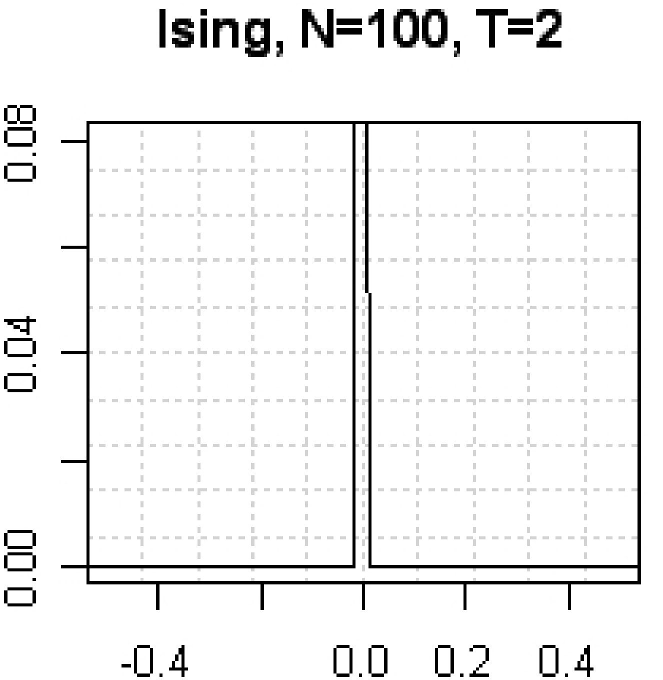


Distribution functions

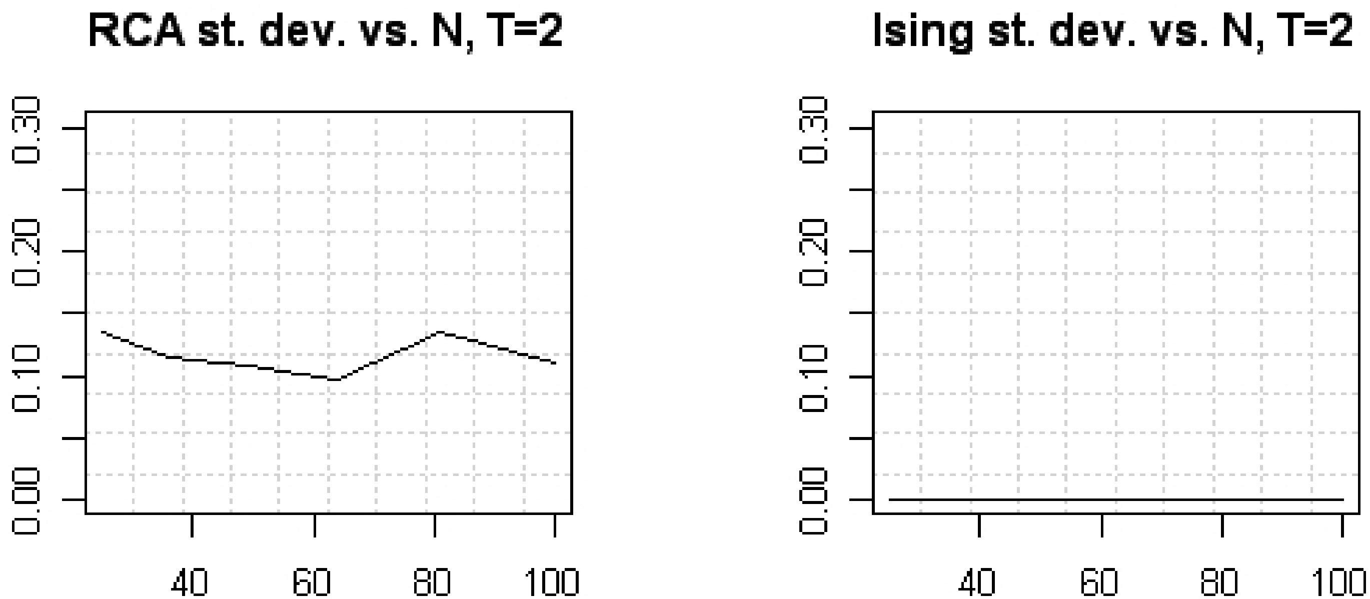


Density functions

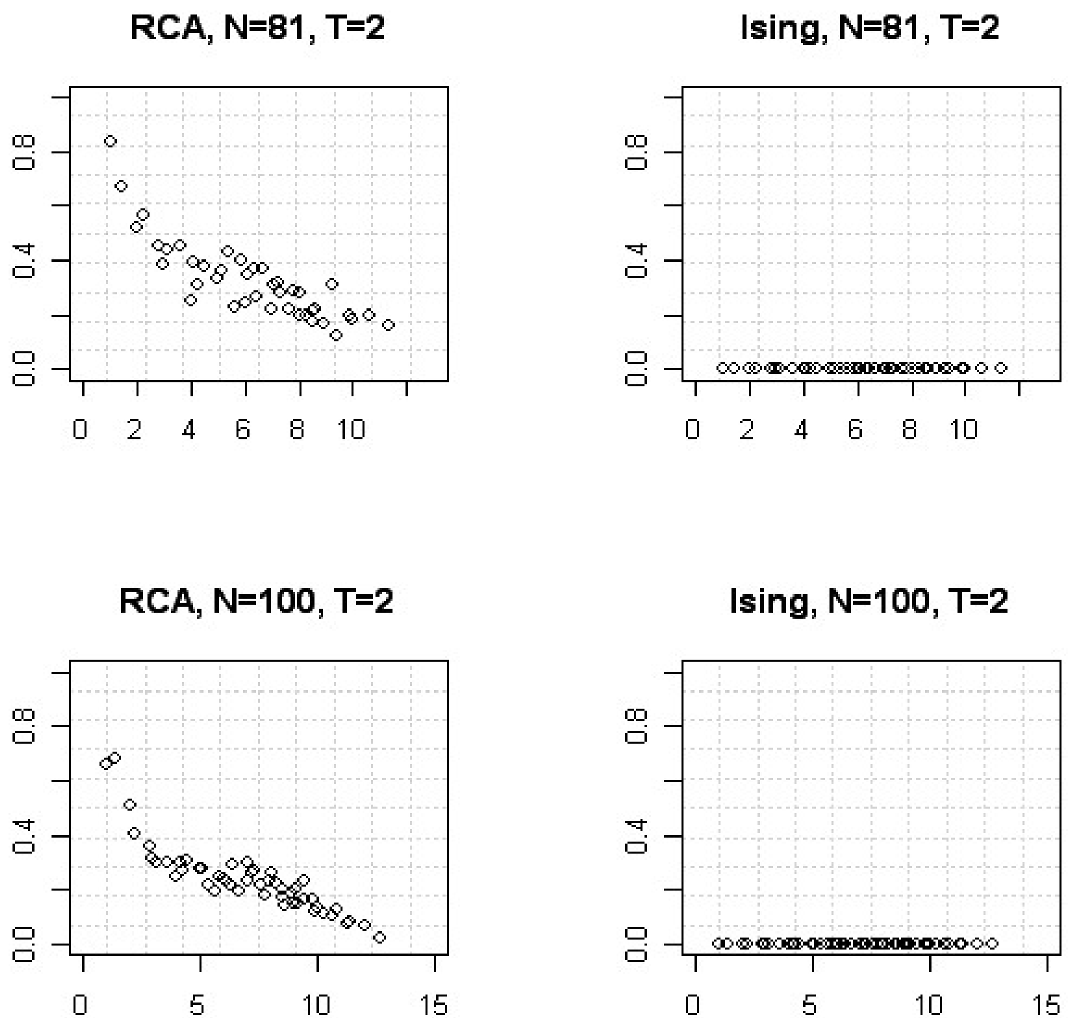




RCA vs. Ising model (standard deviation of correlations)



Max correl vs. distance



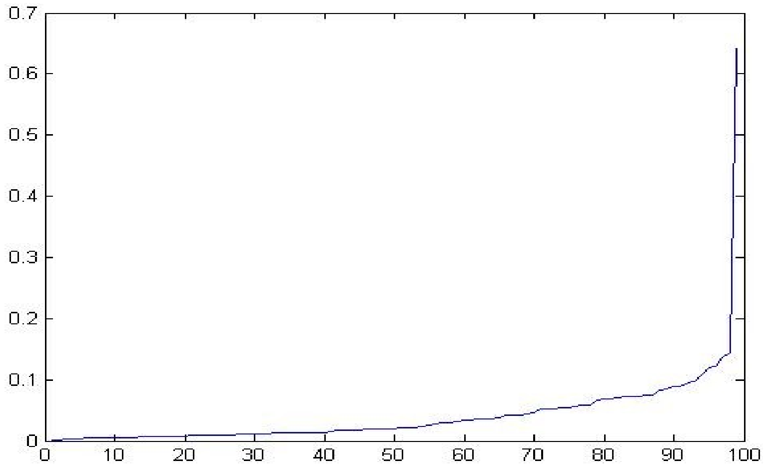
Strong and weak correlations are randomly scattered about the system. Two strongly correlated spins may not be connected by strongly correlated paths.

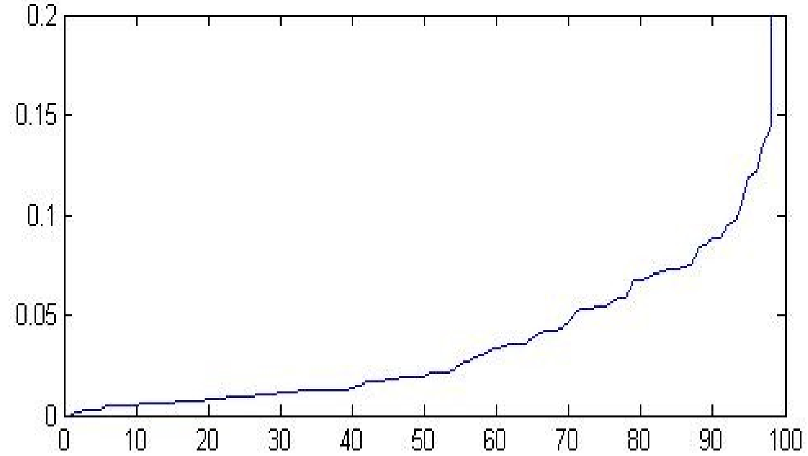
See little demo

Linear regression (RCA model)

$$s_k - \langle s_k \rangle = \sum_{i=1, i \neq k}^{N} a_i \cdot \left(s_i - \langle s_i \rangle \right) + \varepsilon$$

The sorted coefficients (N=100, T=2)





Concluding remarks

Randomly distributed large correlations may be a general characteristic of complex systems. In this sense complex systems may be regarded as critical in a wide region of parameter space.

This property may explain their sensitivity to changes in the control parameters, boundary conditions, initial conditions and other details, even for large sizes (e.g. chaos in spin glasses).

It also calls for caution when doing, and drawing conclusions from, simulations of such systems.

Strong random correlations redefine the geometry of the system. Problems of the RG and the thermodynamic limit.

Thank you!

Appendix

The Ising model: a model of cooperation

N „spins” $i = 1,2,\ldots,N$, having a binary choice $s_i = \pm 1$. The spins are coupled by ferromagnetic interaction, they want to minimize the energy

$$H = -\sum_i s_i \cdot s_j - h \cdot \sum_i s_i \qquad h > 0$$

The „magnetic field” h wants to align all the spins with itself.

This is a simple description of magnetism and a host of other cooperative phenomena.

The total number of microscopic arrangements of the spins is 2^N. The model has two optimal states (ground states): All spins +1 (up), or -1 (down).

„Finite temperature”: some spins fail to comply.

Averaging

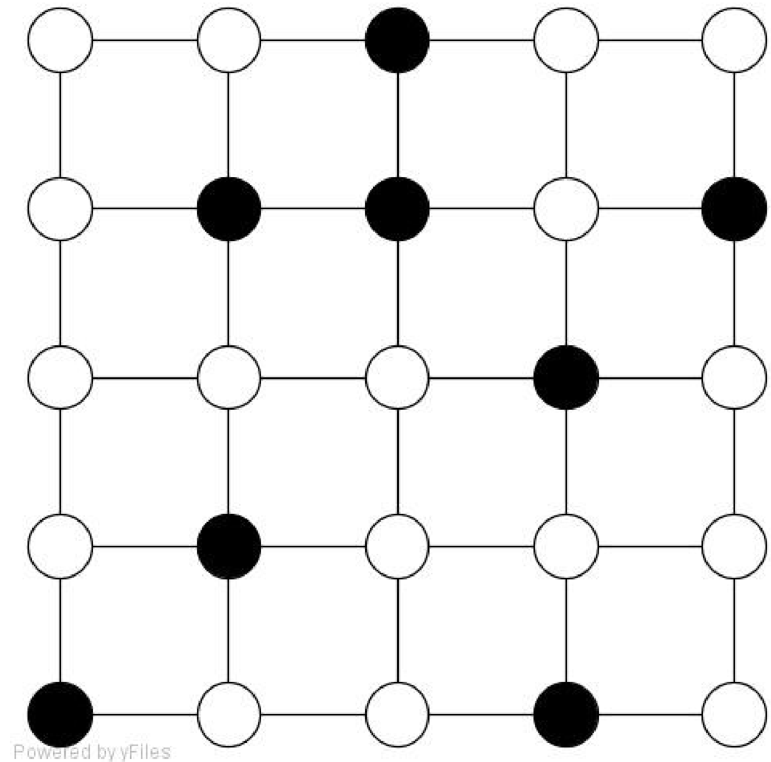
Averages at temperature T are calculated over the whole ensemble of microscopic states, with the Boltzmann-weight $\sim exp\{-H/T\}$.

Alternatively, we define a Monte Carlo dynamics on the system, and measure time averages.

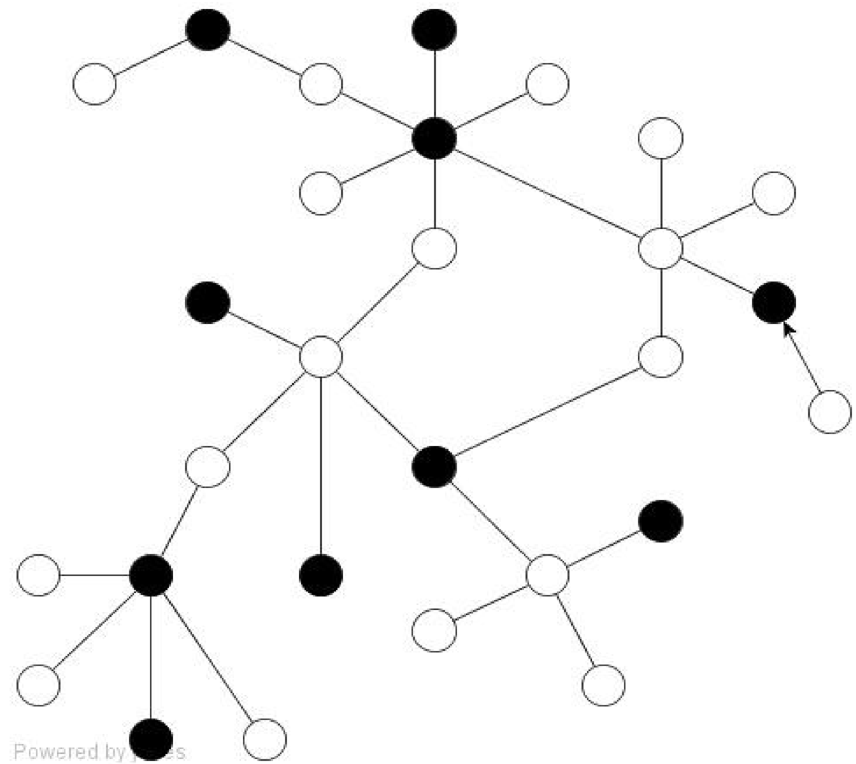
Pick initial state, calculate its energy E_1. Flip randomly chosen spin, calculate new energy E_2. Accept new state if $\Delta E = E_2 - E_1 < 0$, and accept new state with probability $p = e^{-\frac{\Delta E}{T}}$, if $\Delta E = E_2 - E_1 > 0$.

The underlying geometry

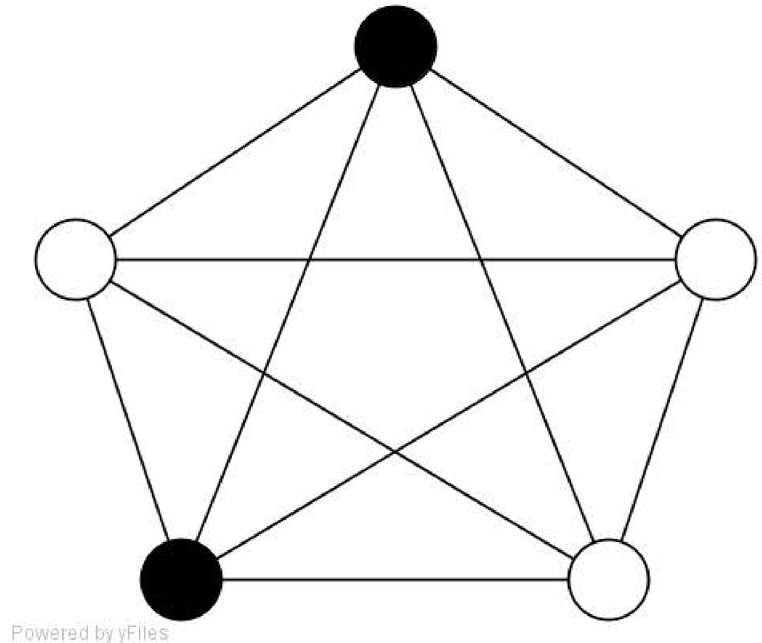
Such a model can be implemented on a regular lattice, like the 2d square lattice shown here



on a random graph



or on a complete graph:



Full circles mean spins +1, empty ones -1.

Phase transition

At high temperatures the acceptance rate of „bad moves" is nearly as large as that of the good moves, the system is totally disordered. As T is lowered, the tendency of cooperation gradually overcomes thermal agitation. If the graph is sufficiently large and connected, at a critical temperature a sharp transition takes place to a spontaneously ordered state, with the majority of spins pointing, say, up, even without the help of the external field h.

For the 2d square lattice the value of this critical temperature is $T_c \approx 2.26$.

Correlations in the Ising model

The correlations $K_{ij} = \langle s_i \cdot s_j \rangle$ between the spins at lattice sites *i* and *j* are short-ranged (fall off exponentially with distance) above the critical temperature. (The angular brackets denote the thermal average.)

A typical formula is

$$G(r) \propto \frac{1}{r^{d-2+\eta}} \cdot e^{-\frac{r}{\xi}}$$

where ξ is the coherence length.

Below the critical temperature the system is polarized, so K tends to a constant, but its „connected part" $C_{ij} = \langle s_i \cdot s_j \rangle - \langle s_i \rangle \cdot \langle s_j \rangle$ is decaying exponentially again.

The critical state

As the temperature goes to its critical value, $T \to T_c$, the coherence length diverges: $\xi \to \infty$.

Right at the critical point correlations in the system become long-ranged. There is no characteristic distance beyond which they would become negligible, they fall off like a negative power of the distance:

$$G(r) \propto \frac{1}{r^{d-2+\eta}}$$

As a direct consequence, the system becomes extremely sensitive to changes in the control parameters, such as the external field: even an infinitesimal h provokes a large response.

Note, however, that in order to reach the critical point one has to fine tune the parameters of the model, this is an exceptional point where even the humble Ising model becomes complex.

Models with broken continuous symmetry

If instead of the binary Ising spins we consider little vectors that can rotate in 3d space and interact via a scalar product-like coupling, we arrive at the Heisenberg model. This has a continuous (rotation) symmetry. When the system orders, it develops a macroscopic magnetization and the rotation symmetry is broken.

We can now define two different correlation functions: the longitudinal one corresponding to fluctuations parallel to the magnetization, and the transverse one that is perpendicular to it.

Goldstone modes

The transverse correlation function can exactly be shown to fall off like a power all through the ordered phase:

$$G_{\perp}(r) \propto \frac{1}{r^{d-2}} \qquad , \qquad T \leq T_c$$

Such long-ranged behaviour always appears when a continuous symmetry is broken.

Complex systems are typically very inhomogeneous, they do not display any symmetry. There seems to be no reason to expect them to have long-range correlations.