Breakdown of the mean-field approximation in a wealth distribution model

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International Workshop on Challenges and Visions in the Social Sciences

Introduction

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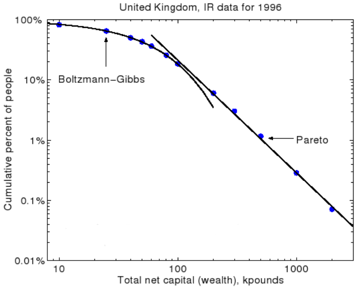
- 1st empirical observation: Vilfredo Pareto, 1897
- wealth and income also follow power-law distributions



(Drăgulescu, Yakovenko, 2001)

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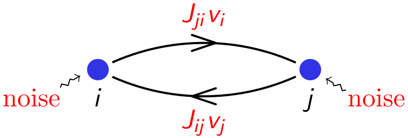

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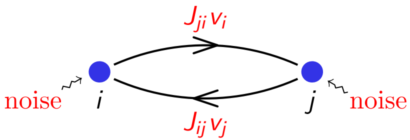

- simplest case: $J_{ij} = 1/(N-1)$

$$\mathrm{d}v_i(t) = \underbrace{\Big(\tilde{v}_i(t) - v_i(t)\Big)\,\mathrm{d}t}_{\text{trading}} + \underbrace{\sqrt{2}\sigma\, v_i(t)\,\mathrm{d}W_i(t)}_{\text{speculations}}$$

Mean-field solution

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Fokker-Planck equation for the wealth distribution $f(v_i)$

$$\downarrow$$

stationary solution

$$f(v_i) = C \exp\left[-\frac{1}{\sigma^2 v_i} \right] v_i^{-2-1/\sigma^2}$$

oh, power-law tail!

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Something is wrong...

- this stationary solution has the fixed variance $\frac{\sigma^2}{1-\sigma^2}$ $(\sigma < 1)$
- by summing all dv_i's we get

$$\mathrm{d}v_T = \sqrt{2}\sigma \sum_{i=1}^{N} v_i \,\mathrm{d}W_i$$

- such dispersion never stops...

What happens for $N < \infty$?

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- without the mean-field approximation, the Fokker-Planck equation cannot be solved
- **the key idea:** let's take a look at averages!
- with the Itô lemma and average $\langle\cdot\rangle$ over noise

$$\boxed{\begin{aligned}\frac{\mathrm{d}\langle v_i^2\rangle}{\mathrm{d}t} &= 2\left[\langle v_i v_j\rangle - (1-\sigma^2)\langle v_i^2\rangle\right]\\ \frac{\mathrm{d}\langle v_i v_j\rangle}{\mathrm{d}t} &= \frac{2}{N-1}\left[\langle v_i^2\rangle - \langle v_i v_j\rangle\right]\\ \langle v_i v_j\rangle(0) &= 1\\ \langle v_i^2\rangle(0) &= 1\end{aligned}}$$

Time scales

- from $\langle v_i v_j \rangle$ and $\langle v_i^2 \rangle$ we obtain $\mathrm{var}[v_i]$ and C_{ij}

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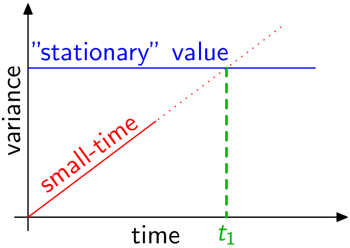

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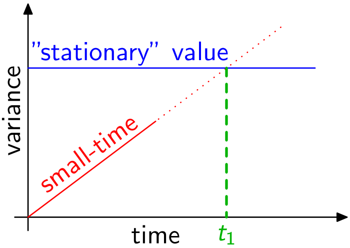

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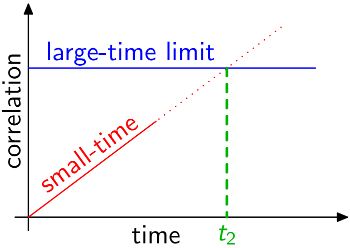
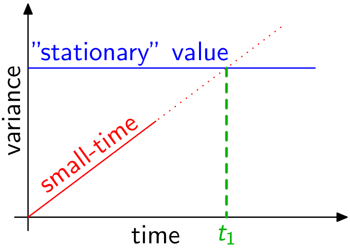
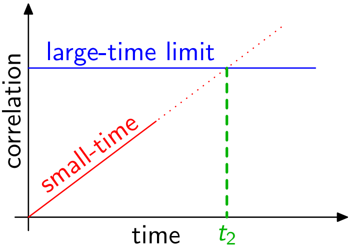


$$t_2 = (1 - \sigma^2)\, N$$

Time scales

- from $\langle v_i v_j \rangle$ and $\langle v_i^2 \rangle$ we obtain var$[v_i]$ and C_{ij}





$$t_1 = \frac{1}{2(1-\sigma^2)}$$

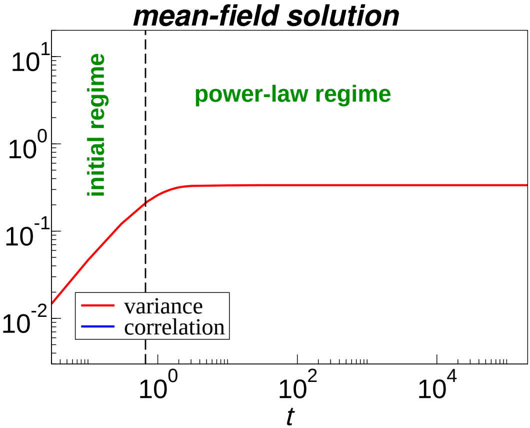
$$t_2 = (1-\sigma^2)\,N$$

- variance contains a term proportional to $\exp[\lambda t]$, $\lambda > 0$

$$t_3 = 1/\lambda = \frac{1-\sigma^2}{2\sigma^2}\,N + O(1)$$

Time evolution

$$N = 10^4,\ \sigma^2 = 0.5$$



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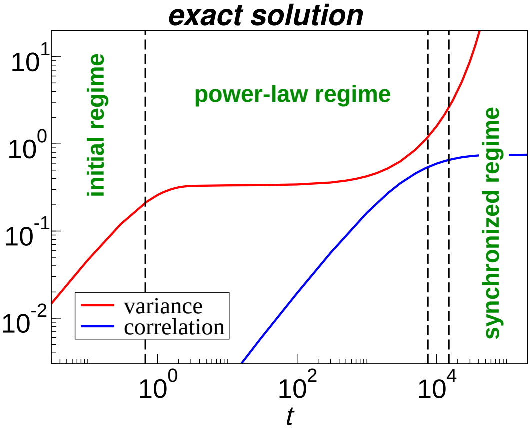

Concluding remarks

- for any $N < \infty$, there is no stationary wealth distribution
- when the average number of neighbours is z, correlations appear in time $O(z)$
- can taxes stabilize the system?
- what is the wealth distribution in the synchronizaed regime?
- the case $\sigma > 1$ must be treated separately
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thank you for your attention