Graph embeddings and symmetries
1. Vertex-transitive maps

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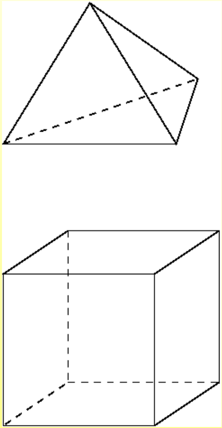

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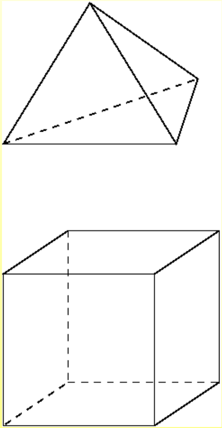

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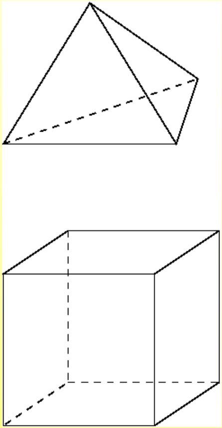

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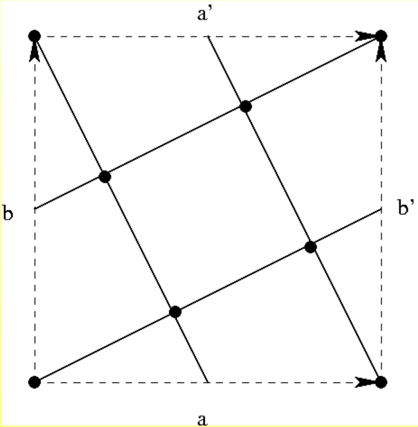

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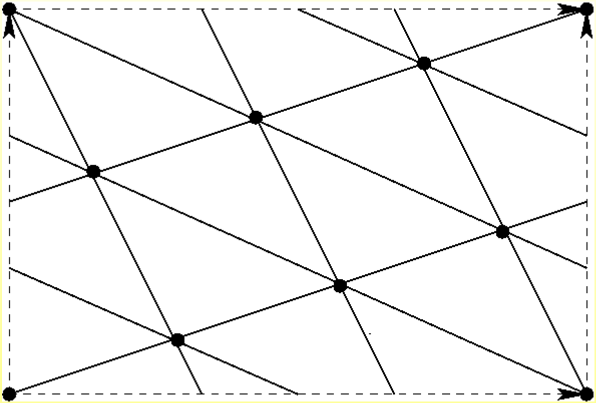

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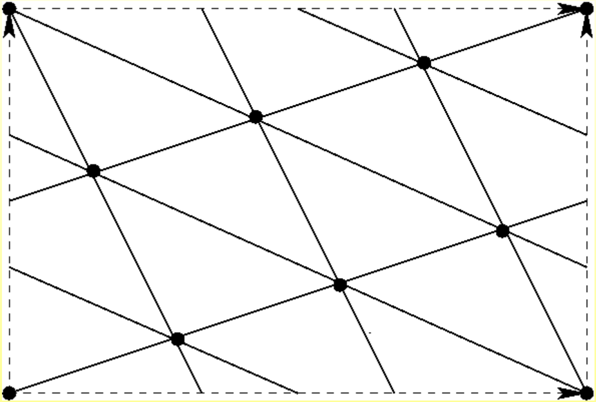

Exercise. Find vertex-transitive embeddings of K_6 and $K_{3,3}$ on a torus.

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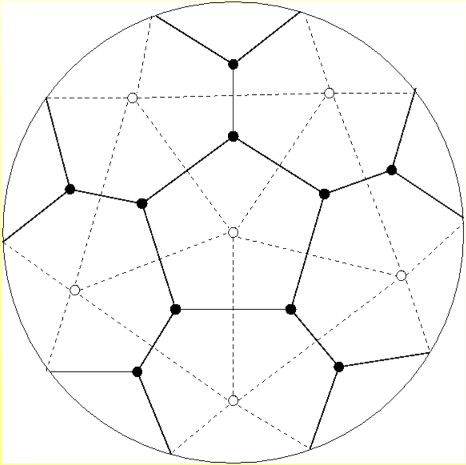

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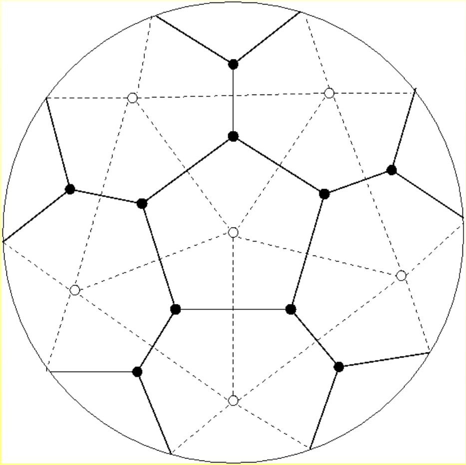

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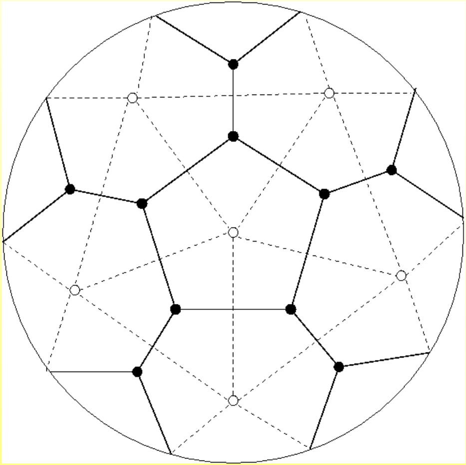

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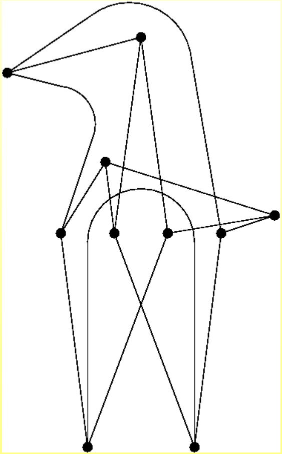

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Exercise. Do not try to prove the above classification theorem.

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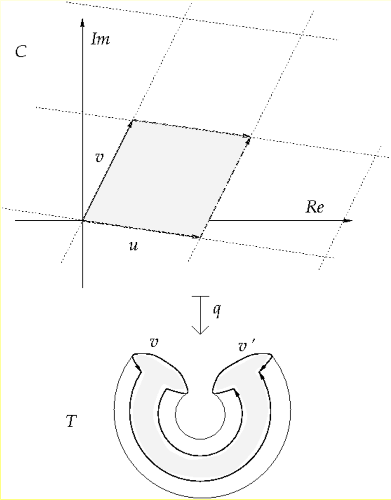

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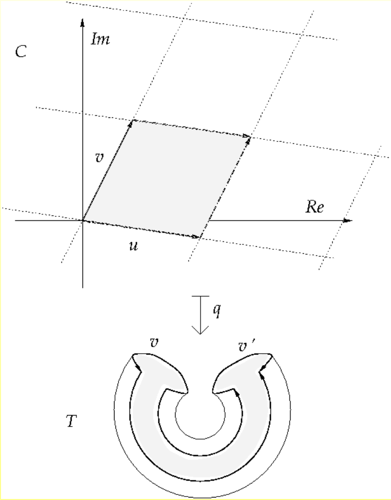

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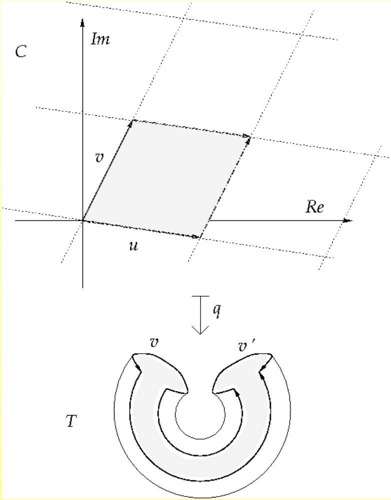

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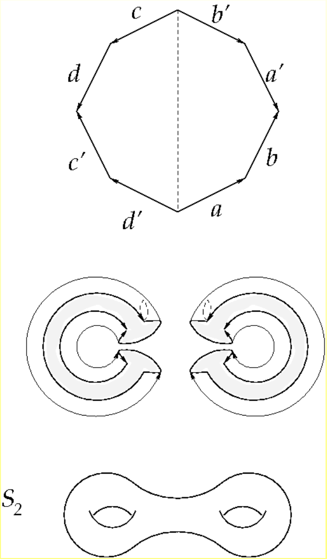

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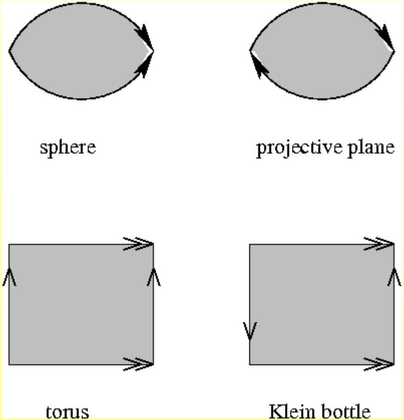

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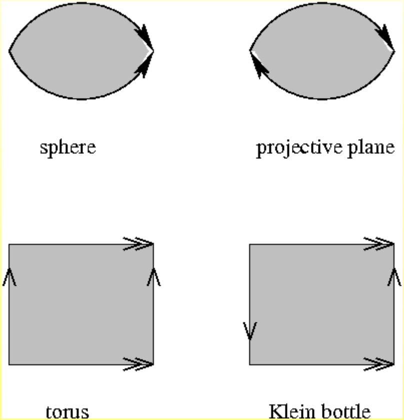

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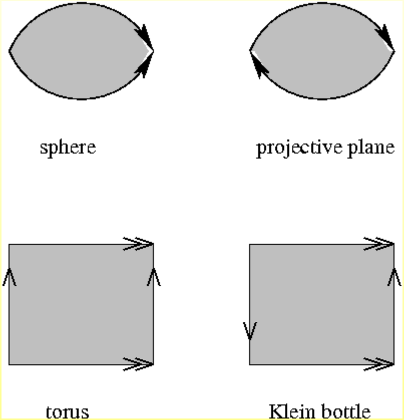

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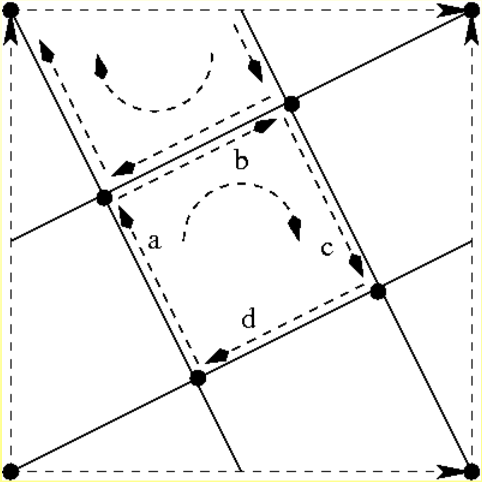

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Theorem 1. *A connected graph Γ has an oriented vertex-transitive embedding if and only if $Aut(\Gamma)$ contains a vertex-transitive subgroup with free cyclic vertex stabilisers.*

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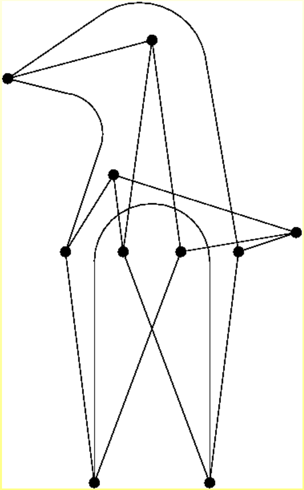

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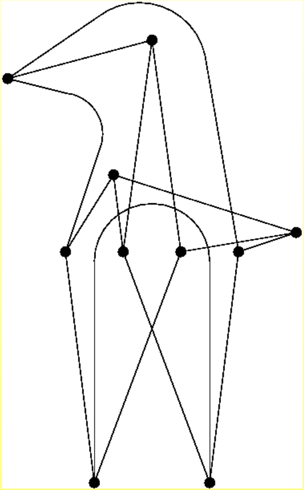

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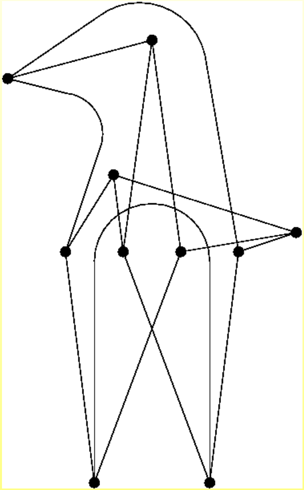

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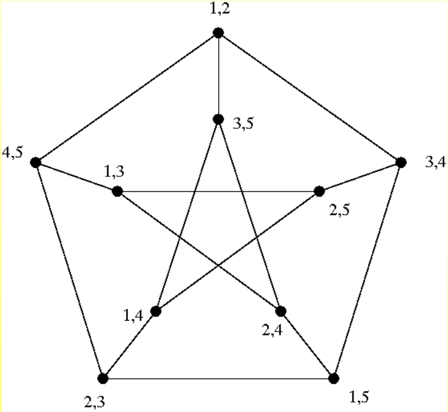

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2. If a suitable subgroup of $Aut(\Gamma)$ exists, how to get an embedding?

Illustration

So, we have the Petersen graph P, with vertex set “encoded” by two-element subsets of $\{1, 2, 3, 4, 5\}$, two vertices being adjacent if the corresponding sets are disjoint. It can be shown that $Aut(P) \cong S_5$.

By Theorem 1, P has an oriented vertex-transitive embedding if and only if $Aut(P) \cong S_5$ contains a subgroup transitive on vertices such that the vertex stabilisers are either trivial or isomorphic to Z_3. But the first case is impossible since P is not a Cayley graph. The second case is excluded either, as S_5 is known to contain no subgroup of order 30 at all. Therefore the Petersen graph has no orientably vertex-transitive embedding.

This raises the following three questions:

1. Is the analysis of subgroups of $Aut(\Gamma)$ unavoidable?
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3. What are the supporting surfaces of such embeddings?

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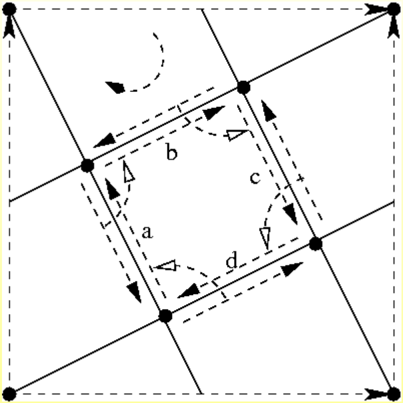

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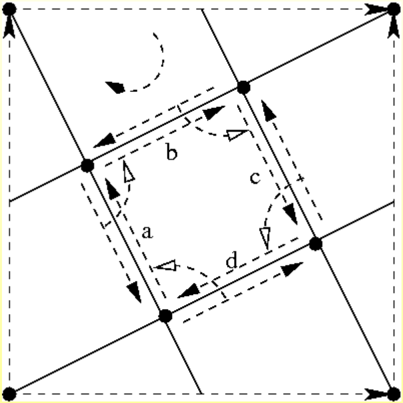

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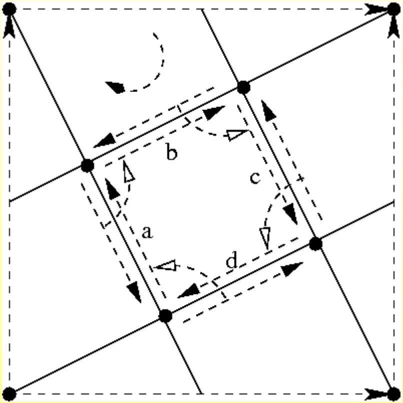

Note: $\rho\lambda(a) = b$, $\rho\lambda(b) = c$, etc., so (a, b, c, d) is indeed a cycle of $\rho\lambda$.

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$\lambda(x) = x'$,$x \in \{a, \ldots, f\}$, $\lambda^2 = id$;
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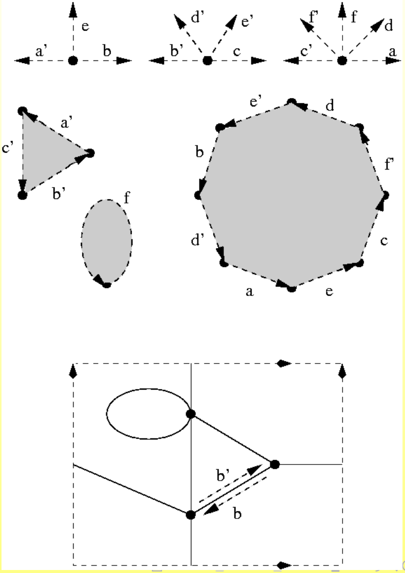
Solution. Vertices and faces of the map correspond to orbits of ρ and $\rho\lambda$ where
$\rho\lambda = (a, e, c, f', d, e', b, d')(a', c', b')(f)$:



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This means that $Aut(M)$ is the centraliser of the group $\langle \lambda, \rho \rangle$ in the full symmetric group $Sym(D)$ of all permutations of D.

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6. So, $G < Aut(M(\lambda, \rho))$, and the map $M(\lambda, \rho)$ is vertex-transitive. □

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A Cayley map can be loosely described as an oriented embedding of a Cayley graph in which the cyclic order of darts (in terms of generators) at any vertex is "the same".

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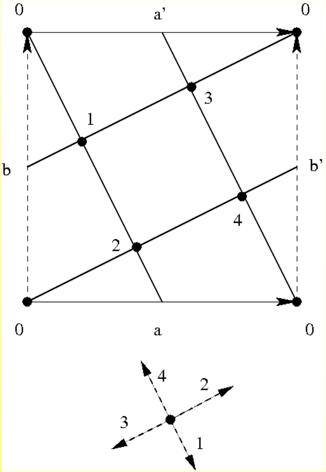

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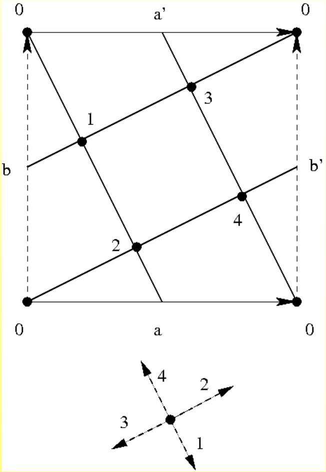

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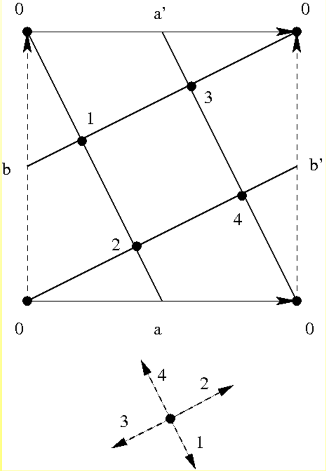

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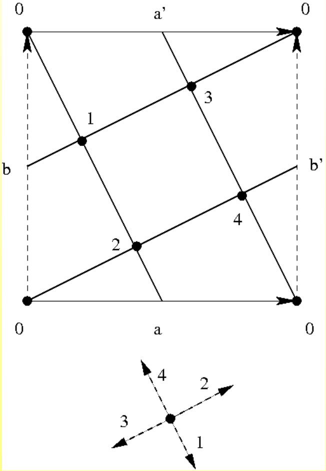

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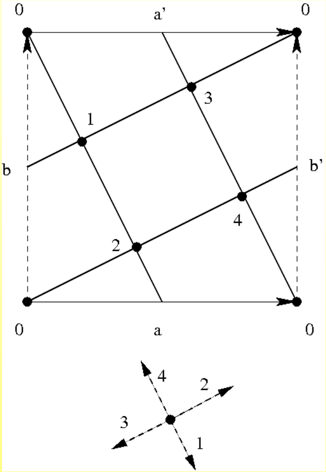

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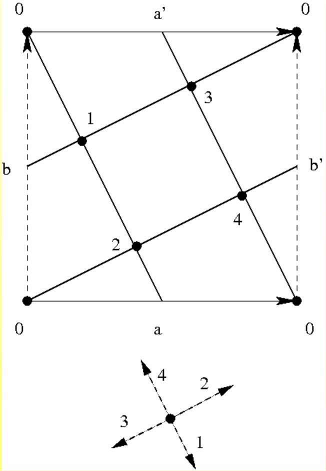

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If there are no other automorphisms, such Cayley maps can be viewed as vertex-transitive maps with the lowest "level of symmetry".

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We have encountered vertex-transitive embeddings, among which the Cayley maps had the "lowest level of symmetry". Similar theory has been developed for edge-transitive maps. In the second lecture we will continue with considering the "highest level of symmetry" in oriented maps.

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Potential thesis: *Develop a theory for (oriented as well as unoriented) face-transitive embeddings of graphs.*