(v_3) configurations – Lecture 3

Marko Boben

Discrete Mathematics 2 + Configurations

Definition

An *Incidence structure* is a triple $\mathcal{S} = (P, \mathcal{B}, I)$ where P and $\mathcal{B}$ are disjoint sets and I is a binary relation between P and B, i.e. $I \subseteq P \times \mathcal{B}$.

The elements of P are called *points*, those of $\mathcal{B}$ *blocks* and those of I *flags*.

Instead of $(p, B) \in I$ we write $p\ I\ B$ and use such geometric language as "the point p lies in the block B", "B passes through p", "p and B are incident" etc.

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Example

Incidence structure 1.

$$P = \{1, 2, 3, 4, 5\}$$
$$\mathcal{B} = \{\{1, 2, 3, 4\}, \{1, 5\}, \{2, 5\}, \{3, 5\}, \{4, 5\}\}$$

Example

Incidence structure 2 (Fano plane)

$$P = \{0, 1, 2, 3, 4, 5, 6\}$$
$$\mathcal{B} = \{\{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 0\}, \{5, 6, 1\}, \{6, 0, 2\}\}$$

Fano plane figure

Example

Incidence structure 1.

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Incidence structure 2 (Fano plane)

$$P = \{0, 1, 2, 3, 4, 5, 6\}$$
$$\mathcal{B} = \{\{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\},$$
$$\{4, 5, 0\}, \{5, 6, 1\}, \{6, 0, 2\}\}$$

Fano plane figure

Connection between incidence structures and graphs.

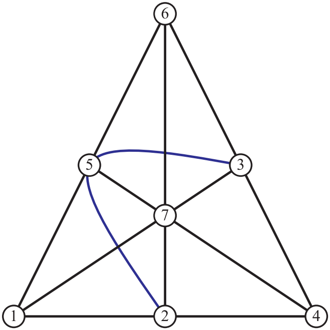
Definition

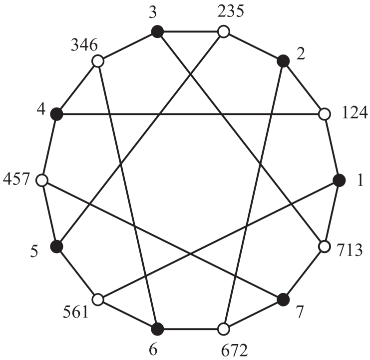
Incidence graph or *Levi graph* $G(\mathcal{C})$ of an incidence structure $\mathcal{S} = (P, \mathcal{B}, I)$ is

- bipartite graph with
- v "black" vertices (representing points of $\mathcal{S}$),
- b "white" vertices (representing blocks of $\mathcal{S}$),
- an edge joining two vertices if and only if the corresponding point and line are incident in $\mathcal{S}$.

Example

Fano plane and its incidence graph (Heawood graph)





A well known family of incidence structures are t-designs.

Definition

Let v, k, t, λ be integers such that $v \geq k \geq t \geq 0$ and $\lambda \geq 1$. A *t-design* with *blocks of size k* and *index λ* is an incidence structure $\mathcal{D} = (P, \mathcal{B}, I)$ with the following properties

1. $|P| = v$
2. $|B| = k$ for each $B \in \mathcal{B}$
3. For each t-subset (i.e. subset of size t) T of P there are exactly λ blocks containing T.

Notation: $t - (v, k, \lambda)$ or $S_\lambda(t, k, v)$.

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Example

Fano plane is $2-(7,3,1)$ design.

Fano plane figure

In general, a $2-(v,k,\lambda)$ design with $\lambda = 1$ and $v = |P| = |\mathcal{B}|$ (if such exists) is called a *(finite) projective plane* of order $k-1$.

Every $2-(n^2,n,1)$ design (if exists) is called a *(finite) affine plane*.

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Another example of incidence structures are *configurations*.

Definition

A *(combinatorial) configuration* (v_r, b_k) is an incidence structure of points and lines (blocks) with the following properties.

1. There are v points and b lines.
2. There are r lines through each point and k points on each line.
3. Two different points are connected by at most one line and two lines intersect in at most one point.

Configurations with $v = b$ (and hence $r = k$) are called *symmetric* and denoted by (v_r).

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Proposition

If there exists a (v_r, b_k) *configuration, then*

$$vr = bk \qquad \text{and} \qquad v \geq r(k-1)+1.$$

Proof. For the first equality count the flags (edges of the Levi graph) in two ways. For the second equality take one point and count all points on incident lines. □

Remark

The conditions above are not sufficient. For example:

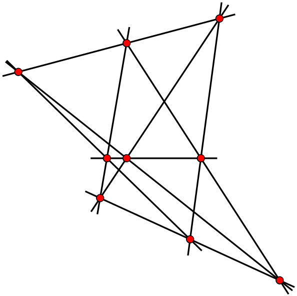
- they are sufficient for (v_3) configurations (exist for $v \geq 7$)
- for (v_r, b_3) configurations (exist iff $v \geq 2r+1$ and $vr = 3b$).
- there is no (43_7) configuration although $7(7-1)+1 = 43$.

Example

Fano plane or projective plane of order 2 is the only (7_3) configuration.

Example

Pappus configuration – one of three (9_3) configurations.



Example

Each 2-design with $\lambda = 1$, $S_1(2, k, v)$, is a (v_r, b_k) configuration with

$$r = \ldots \qquad \text{and} \qquad b = \ldots$$

Example

Projective plane of order k is $((k^2 + k + 1)_{k+1})$ configuration.

Definition

Let $a, b \in \mathbb{Z}_v$, $a \neq b$, $a, b \neq 0$ and

$$\mathcal{B} = \{\{0, a, b\}, \{1, a+1, b+1\}, \ldots, \{v-1, a+v-1, b+v-1\}\}.$$

If the incidence structure $\mathcal{C} = (\mathbb{Z}_v, \mathcal{B}, \in)$ is a (v_3) configuration, then we call it a *cyclic* (v_3) *configuration* with *base block* $B = \{0, a, b\}$ and denote it with $Cyc(v, B)$.

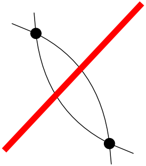
Example

Fano plane is $Cyc(7, \{0, 1, 3\})$.

The characterization of configurations in terms of their incidence graphs:

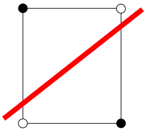
Proposition

An incidence structure is a (v_r, b_k) configuration if and only if its incidence graph is (r, k)-regular and has girth ≥ 6.
(girth$(G) =$ the length of the shortest cycle in G.)



Two lines intersecting in two points

$\longleftrightarrow$



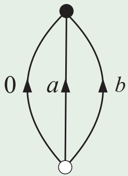
4-cycle

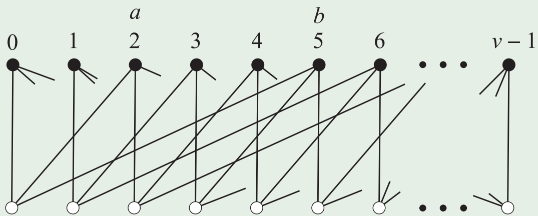
Remark

Every r-regular bipartite graph with girth at least 6 gives one (v_r) configuration or two **dual** (v_r) configurations.

Example

Levi graph of $Cyc(v, \{0, a, b\})$ is a $\mathbb{Z}_v$-covering graph over





- Combinatorial construction of non-isomorphic (v_r, b_k) configurations.
  - construction of (r, k)-regular bipartite graphs with girth ≥ 6.
  - iterative construction from smaller configurations (for (v_3) configurations).
- *Realization* of configurations in Euclidean plane with "points" and "lines".

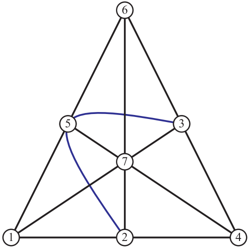
v	a	b	c	d	e
7	1	1	1	1	0
8	1	1	1	1	0
9	3	3	2	1	0
10	10	10	2	1	0
11	31	25	1	1	0
12	229	95	4	3	0
13	2 036	366	2	2	0
14	21 399	1 433	3	3	1
15	245 342	5 802	5	4	1
16	3 004 881	24 105	6	4	4
17	38 904 499	102 479	2	2	13
18	530 452 205	445 577	9	5	47
19	7 640 941 062	1 992 044	3	3	290

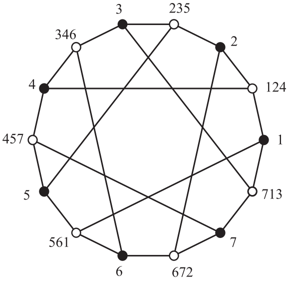
In the previous table:

$a =$ number of all configurations,
$b =$ number of self-dual configurations,
$c =$ number of point-transitive configurations,
$d =$ number of cyclic configurations,
$e =$ number of non-connected configurations.

$$124,\ 235,\ 346,\ 457,\ 561,\ 672,\ 713$$

or Fano plane is projective plane of order 2 and therefore the smallest (v_3) configuration. Its Levi graph is 6-cage, the smallest cubic graph of girth 6 – Heawood graph. It is also cyclic configuration $Cyc(7, \{0, 1, 3\})$.

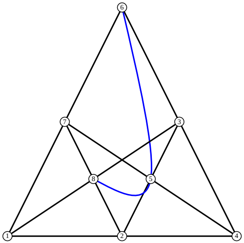


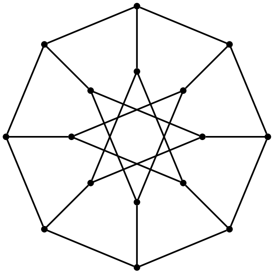


124, 235, 346, 457, 568, 671, 782, 813,

$Cyc(8, \{0, 1, 3\})$, is the only (8_3) configuration. Its levi graph is Generalized Petersen graph $G(8, 3)$.

It can be constructed from the affine plane of order 3 by removing one point and all incident lines.

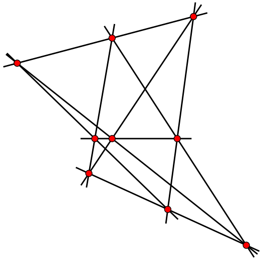




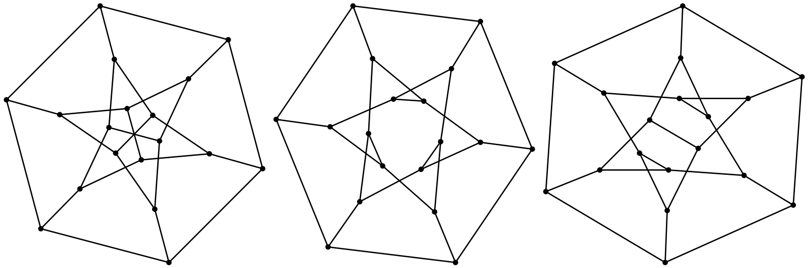
Pappus configuration

$$127,\ 149,\ 168,\ 238,\ 259,\ 347,\ 369,\ 458,\ 567$$

represents Pappus theorem:
Let x, y, z and x', y', z' be two sets of three different collinear points on two different lines in the plane (such that none of these points lies on the intersection of both lines). Then the points $u = xy' \cap x'y$, $v = xz' \cap x'z$, $w = yz' \cap y'z$ are collinear.



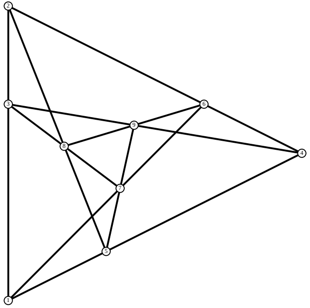
Levi graph of the Pappus configuration can be obtained from $G(6, 2)$ by subdividing two triangles and connecting the new points with 3 new edges. This can be done in three different ways (to get a bipartite graph). Each of them gives a Levi graph of one of the three (9_3) configurations.

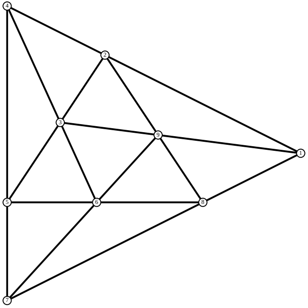


Configuration $(9_3)_2$ is given by the lines

$$123,\ 145,\ 167,\ 246,\ 258,\ 349,\ 378,\ 579,\ 689.$$

Configuration $(9_3)_3$ is the cyclic configuration $Cyc(9, \{0, 1, 3\})$.

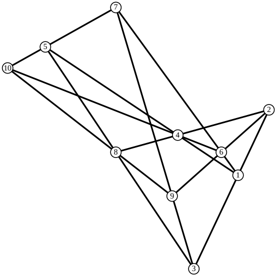


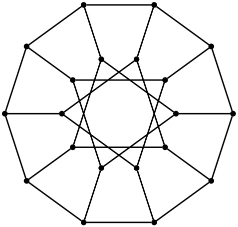


Desargues configuration

123, 145, 167, 248, 269, 358, 379, 460, 570, 890

Resembles the Desargues theorem: *Let* c, x, y, z, x', y', z' *be different points where* c, x, x', c, y, y' *and* c, z, z' *are collinear,* x, y, z *and* x', y', z' *determine two triangles. It follows that* $u = xy \cap x'y'$, $v = xz \cap x'z'$, $w = yz \cap y'z'$ *are collinear*.
Levi graph is $G(10, 3)$.

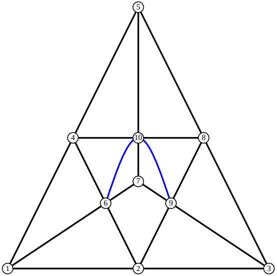


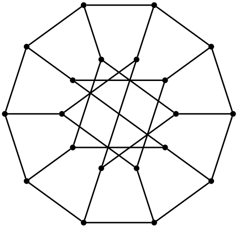


The non-realizable (10_3) configuration

$$123,\ 145,\ 167,\ 246,\ 289,\ 358,\ 379,\ 480,\ 570,\ 690.$$

Levi graph of the configuration $(10_3)_5$ can be obtained from $G(10,3)$ by removing two antipodal edges uv in $u'v'$ and replacing them with uv' and $u'v$ (in such way that the graph remains bipartite).





Definition

The smallest cubic graph with girth g is called a g-cage.

Example

- The only 3-cage is K_4,
- The only 4-cage is $K_{3,3}$,
- The only 5-cage is Petersen graph $G(5,2)$,
- The only 6-cage is Heawood graph,
- One 7-cage (McGee graph),
- One 8-cage (Tutte graph),
- 18 9-cages,
- 3 10-cages, 1 11-cage, 1 12-cage,
- ?

Definition

An *n-gon* in a configuration is a sequence

$$p_1 B_1 p_2 B_2 \dots p_n B_n$$

of n pairwise different points $p_i \in P$ and n pairwise different blocks $B_j \in \mathcal{B}$ such that $(p_i, B_i) \in I$, $(p_i, B_{i-1}) \in I$, and $(p_1, B_i) \in I$.

The existence of an n-gon in the configuration $\mathcal{C}$ is equivalent to the existence of cycle of length $2n$ in Levi graph of $\mathcal{C}$.

Definition

A configuration $\mathcal{C}$ is n-gonal if the Levi graph of $\mathcal{C}$ has girth $2n$, i.e. it contains no m-gon for $m < n$.

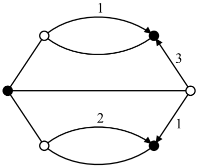
If there is a bipartite $2n$-cage, then it is the Levi graph of the smallest n-gonal configuration(s).

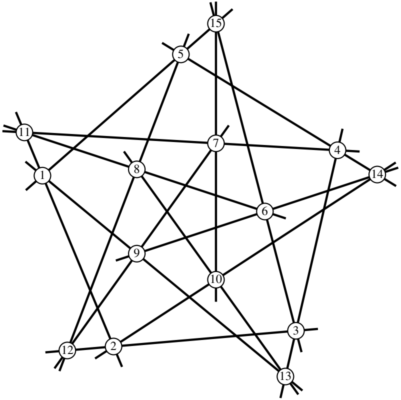
Conjecture

All $2n$-cages are bipartite graphs.

Heawood graph $\Longrightarrow$ Fano plane

Tutte cage (30 vertices) $\Longrightarrow$ Cremona-Richmond (15_3) cfg.

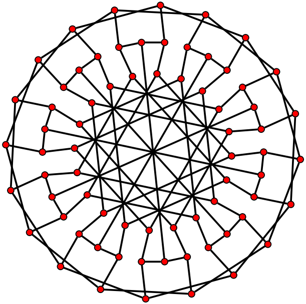


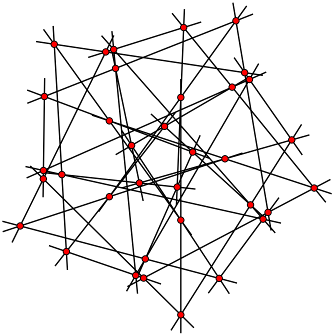


(Tutte cage is $\mathbb{Z}_5$ covering graph over the graph on the left)

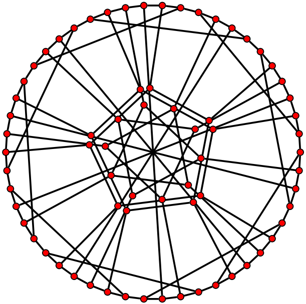
(v_3) configurations and cages
Examples

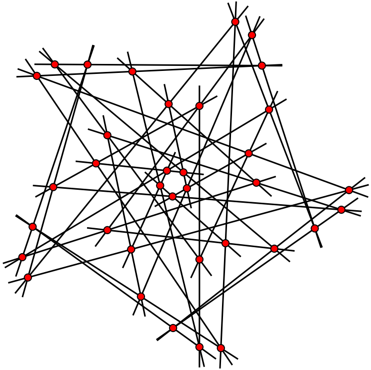
The 1^{st} 10-cage (Balaban cage) on 70 vertices $\Longrightarrow$ The 1^{st} 5-gonal (35_3) cfg.

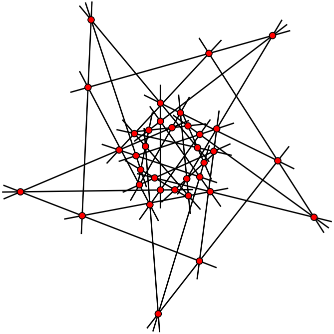




The 2^{nd} 10-cage $\Longrightarrow$ Two 5-gonal (35_3) cfgs.

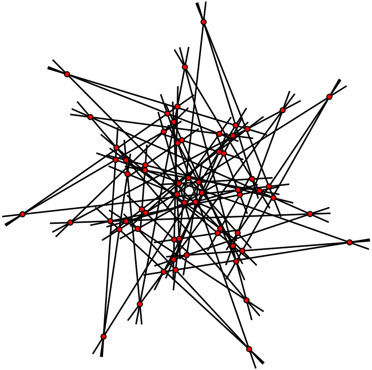


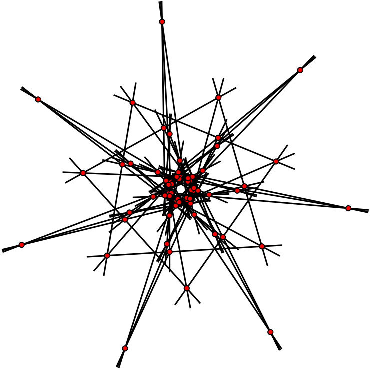




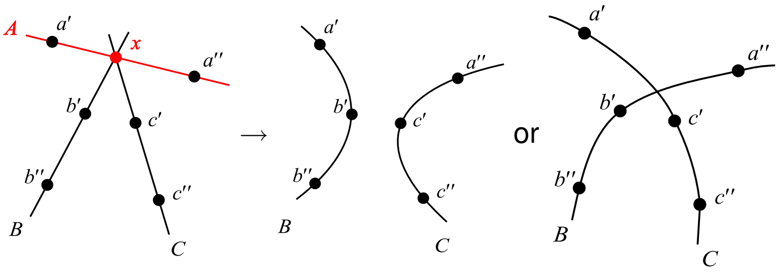
The 3^{nd} 10-cage $\Longrightarrow$ Two 5-gonal (35_3) cfgs.

12-cage (126 vertices) $\Longrightarrow$ Two 6-gonal (63_3) cfgs.





Martinetti's reduction of line A and point x:

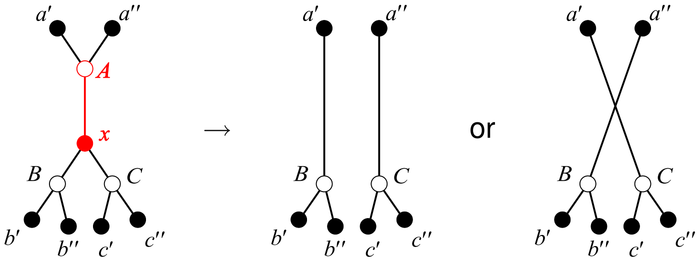


(v_3) configuration $\rightarrow$ $((v-1)_3)$ configurations, if the reduced structure is a configuration.

Definition

If a (v_3) configuration $\mathcal{C}$ does not admit reduction of any line, then $\mathcal{C}$ is called *irreducible* configuration. Otherwise it is *reducible*.

The same story on Levi graphs (we will call them (v_3) graphs)...



Definition

If a (v_3)-graph G does not admit reduction of any edge such that the resulting graph is again a (v_3)-graph, then G is called *irreducible*. Otherwise G is *reducible*.

The smallest (v_3)-graph, the Heawood graph, is clearly irreducible.

Question

Are there other irreducible (v_3)-graphs or is every (v_3)-graph reducible to the Heawood graph?

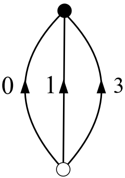
Yes, there are other irreducible (v_3)-graphs.

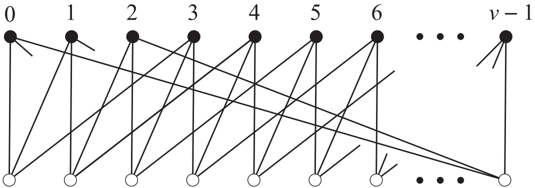
Martinetti's reduction of (v_3) configurations

The first family

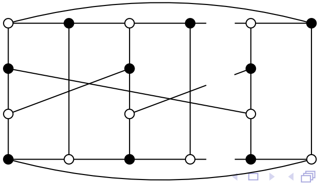
Proposition

Levi graphs of cyclic configurations $Cyc(n, \{0, 1, 3\})$, $n \geq 7$, *are irreducible* (v_3)*-graphs on* $2n$ *vertices. (We will denote them by* $C(n)$*.)*



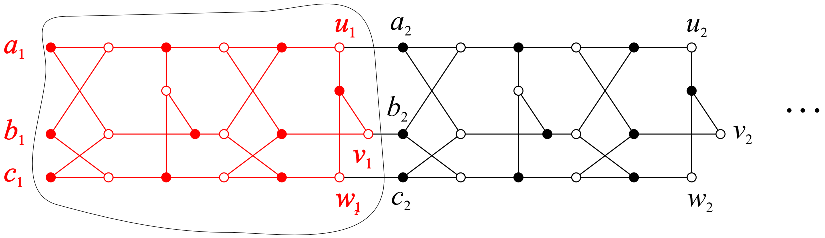


Another picture:



Family 2.

Consider a graph $D(n)$ on $20n$ vertices which is constructed from n segments in the following way

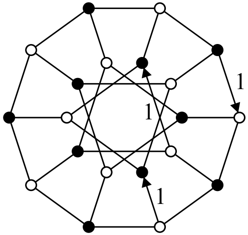


Vertices a_1, b_1, c_1 from the first segement can be connected with vertices u_n, v_n, w_n from the last segment in 6 ways. But we only get 3 non-isomorphics graphs. We denote them by $D_1(n)$, $D_2(n)$ and $D_3(n)$.

Proposition

Graphs $D_1(n)$, $D_2(n)$, $D_3(n)$, $n \geq 1$ are irreducible (v_3)-graphs on $20n$ vertices.

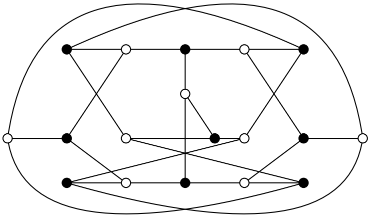
Graph $D_1(n)$ is $\mathbb{Z}_n$-covering graph over

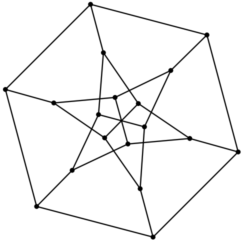


$D_1(1) = GP(10, 3)$ is a graph of the Desargues configuration.

Proposition

The Pappus graph (incidence graph of the Pappus configuration) is irreducible (v_3)-graph on 18 vertices.





Theorem

The only irreducible (v_3)*-graphs are*

- *graphs* $C(n)$, $n \geq 7$ *(Family 1)*
- *graphs* $D_1(n)$, $D_2(n)$, $D_3(n)$, $n \geq 1$ *(Family 2)*
- *The Pappus graph.*

Remark

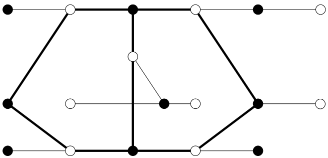
In the original paper of Martinetti (and in the citations of this result) graphs $D_2(n)$, $D_3(n)$ are missing for $n \geq 2$! (configurations arising from these graphs.)

Lemma

A (v_3)-graph G is irreducible if and only if for each edge e of G one of the following is true:

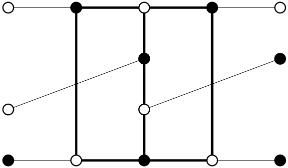
- *Edge e and one of its neighboring edges are in the intersection of two 6-cycles.*
- *There exists a path efg which is the intersection of two 6-cycles.*

Case 1: We assume that in an irreducible (v_3)-graph there exist no 6-cycles which intersect in a path of length 3.
From an initial graph



we can construct, by adding vertices and edges, the Pappus graph and graphs $D_i(n)$.

Case 2: We assume that there exist two 6-cycles intersecting in a path of length 3.
From an initial graph



we can construct, by adding vertices and edges, graphs $C(n)$.

Examples
Fano plane