Polycyclic configurations – Lecture 7

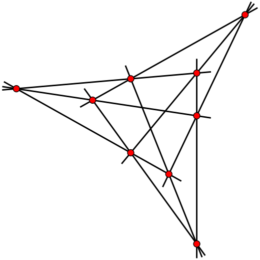
Marko Boben

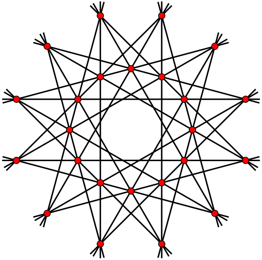
Discrete Mathematics 2 + Configurations

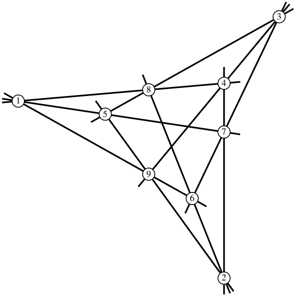
Polycyclic configurations

Definition

A (v_r) configuration (combinatorial, geometric) is *polycyclic* if there exists an automorphism α such that all orbits on points and on lines are of the same size.







Pappus configuration is policyclic configuration (3-cyclic);
The corresponding automorphism is

$$\alpha = (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)$$

Problems. . .

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- which combinatorial polycyclic configurations admit realizatons as (geometric) polycyclic configurations

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Classification of polycyclic configurations can be done via their *quotient graphs*.

- Let G be the Levi (or incidence) graph of the k-cyclic configuration $\mathcal{C}$ for the automorphism α.
- The *quotient graph* $\bar{G}$ of $\mathcal{C}$ for α is the (multi)graph, which is obtained from G by the identification of the vertices and lines from the same orbits of α.

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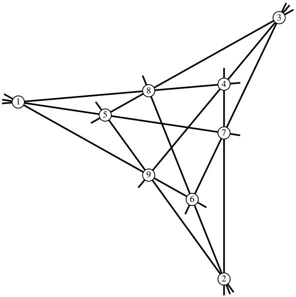
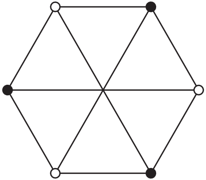
Example

The Pappus configuration and its quotient graph for
$\alpha = (1\,2\,3)(4\,5\,6)(7\,8\,9)$





Quotient graphs and Levi graphs

- Levi graph of a polycyclic configuration is a $\mathbb{Z}_n$ *covering graph* over its quotient graph (with appropriate voltages attached to its edges).

- *Voltage graph:* triple (G, Γ, ξ) where:
  - G is directed graph
  - Γ is a *voltage group*
  - $\xi : E(G) \to \Gamma$; $\xi(e)$ is *voltage*.
- *Covering graph G_ξ over* (G, Γ, ξ) is a graph with
  - $V(G_\xi) = V(G) \times \Gamma$
  - $E(G_\xi) = \{(u, g)(v, g\xi(uv)) : uv \in E(G), g \in \Gamma\}$

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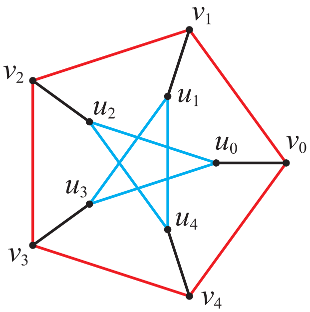

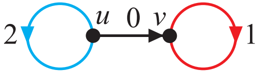

Petersen graph (left) is a covering graph over the voltage graph on the right with $\Gamma = \mathbb{Z}_5$.

Example – cont.

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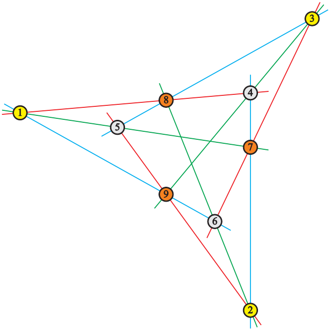
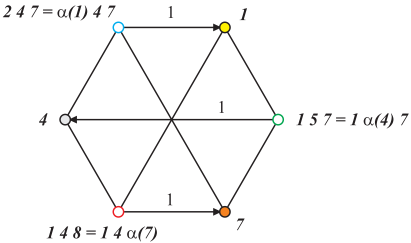
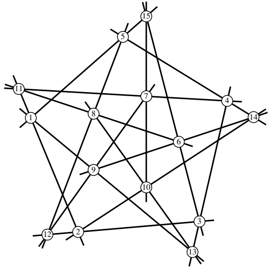
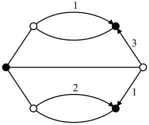




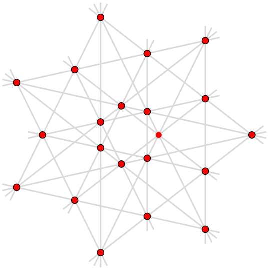
Combinatorial description of a geometric structure.

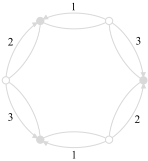
Examples





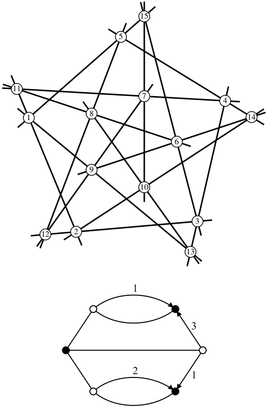
Cremona-Richmond configuration, (15_3).





Configuration (21_4).

Examples



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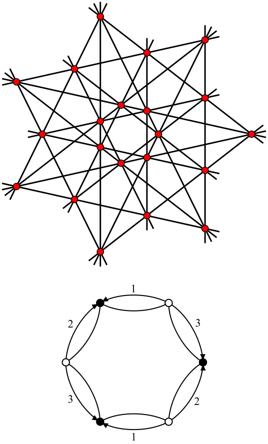
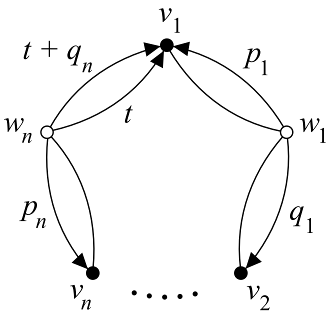


Configuration (21_4).

- Configuration $\mathcal{C}_4(k,(p_1,\ldots,p_n),(q_1,\ldots,q_n),t)$ is a polycyclic configuration with its incidence graph being a $\mathbb{Z}_k$ covering graph over



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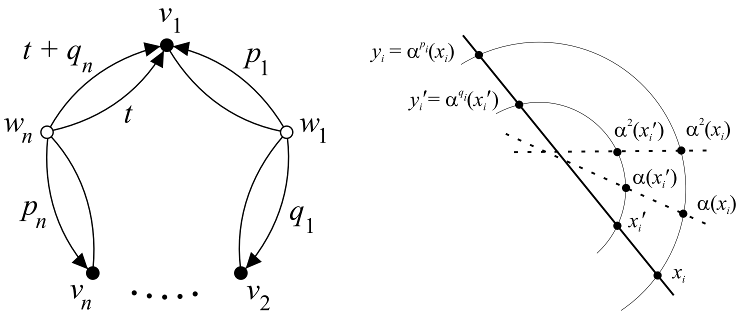
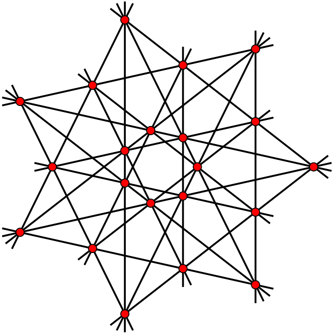
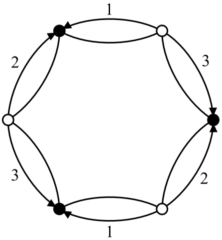


$\mathcal{C}_4$ configurations
Example





$$\mathcal{C}_4(7,(1,2,3),(3,1,2),0)$$

B. Grünbaum, J. F. Rigby, *The real configuration* (21_4), J. London Math. Soc. **41** (2) (1990), 336–346.

Which sequences p, q and number t are good?

Theorem

For given $n \geq 2$, $k \geq 7$ the sequences $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, q_2, \ldots, q_n)$, $1 \leq p_i, q_i < k/2$, and the number t determine a combinatorial $((nk)_4)$ configuration $\mathcal{C}_4(k, p, q, t)$ if and only if

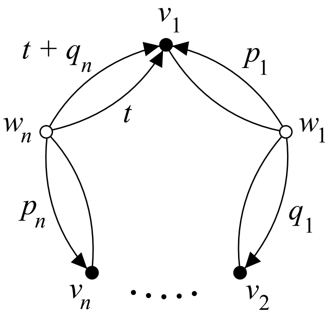
$$p_i \neq q_i, \qquad p_i \neq q_{i-1}, \qquad i = 1, 2, \ldots, n. \tag{1}$$

For $n = 2$, in addition to (1), there are conditions

$$a - b + c - d \not\equiv 0 \pmod{k}. \tag{2}$$

for any possible choice of a, b, c, d, where $a \in \{0, p_1\}$, $b \in \{0, q_1\}$, $c \in \{0, p_2\}$, $d \in \{t, t + q_2\}$.

Proof: Conditions (1) and (2) prevent the existence of 4-cycles in the covering graph over



Proposition

The dual of $\mathcal{C}_4(k,(p_1,\ldots,p_n),(q_1,\ldots,q_n),t)$ *is the configuration*

$$\mathcal{C}_4(k,(q_1,\ldots,q_{n-1},q_n),(p_2,\ldots,p_n,p_1),\\ q_1+\cdots+q_n-p_1-\cdots-p_n+t).$$

Proof. First we reverse the direction of edges by substituting voltages v with $-v$, then we return them back to their original values by rotating them around vertices. The values which help us rotate the voltages accumulate on the last edge giving us the new value for t. $\square$

Proposition

The configuration $\mathcal{C}_4(k, (p_1, p_2, \ldots, p_n), (q_1, q_2, \ldots, q_n), t)$ *is connected if and only if* $\gcd(k, p_1, \ldots, p_n, q_1, \ldots, q_n, t) = 1$.

Proof: The Levi graph of $\mathcal{C}_4$, which is a $\mathbb{Z}_k$ covering graph on $G/_{\approx}$, is connected precisely when gcd of all voltages and k is 1. $\square$

Proposition

The configuration $\mathcal{C}_4(k, (p_1, p_2, \ldots, p_n), (q_1, q_2, \ldots, q_n), t)$ *is isomorphic to*

$$\begin{aligned} &\mathcal{C}_4(k, (p_i, p_{i+1}, \ldots, p_n, p_1, \ldots, p_{i-1}), \\ &\quad (q_i, q_{i+1}, \ldots, q_n, q_1, \ldots q_{i-1}), t) \quad \textit{and} \\ &\mathcal{C}_4(k, (q_i, q_{i-1}, \ldots, q_1, q_n, \ldots, q_{i+1}), \\ &\quad (p_i, p_{i-1}, \ldots, p_1, p_n, \ldots, p_{i+1}), -t) \end{aligned}$$

for each $i = 1, 2, \ldots, n$.

Proof: The first claim is true since the voltage t on $G/_{\approx}$ can be moved around the cycle to any non-adjacent pair of double edges. To prove the second set of identities we have to read the voltages in reverse order. $\square$

Necessary conditions on p, q, t to give a geometric geometric $\mathcal{C}_4(k, p, q, t)$ configuration.

Theorem

If a polycyclic realization of $\mathcal{C}_4(k, p, q, t)$ exists then the equation

$$\cos\frac{p_1\pi}{k}\cos\frac{p_2\pi}{k}\cdots\cos\frac{p_n\pi}{k} = \cos\frac{q_1\pi}{k}\cos\frac{q_2\pi}{k}\cdots\cos\frac{q_n\pi}{k}.$$

holds and

$$t = \frac{1}{2}\sum_{i=1}^{n}(p_i - q_i) \tag{3}$$

is an integer.

Proof: Let R_1 be the radius of the first orbit $\mathcal{O}_1$ (orbit on which a line has "span" p_1) and let R_2 be the radius of the second orbit $\mathcal{O}_2$ (orbit on which the same line has span q_1). Then

$$R_2 = \frac{\cos \frac{p_1 \pi}{k}}{\cos \frac{q_1 \pi}{k}} R_1$$

When we continue the construction we get

$$R_m = \frac{\cos \frac{p_m \pi}{k}}{\cos \frac{\pi q_m}{k}} \cdot \frac{\cos \frac{p_2 \pi}{k}}{\cos \frac{q_2 \pi}{k}} \cdot \frac{\cos \frac{p_1 \pi}{k}}{\cos \frac{q_1 \pi}{k}} R_1$$

Since the construction "closes" we get

$$R_1 = R_n = \frac{\cos \frac{p_n \pi}{k}}{\cos \frac{q_n \pi}{k}} \cdot \frac{\cos \frac{p_2 \pi}{k}}{\cos \frac{q_2 \pi}{k}} \cdot \frac{\cos \frac{p_1 \pi}{k}}{\cos \frac{q_1 \pi}{k}} R_1$$

which gives the Cosine equation in the theorem.

Parameter t is the "combinatorial difference" between the first point of orbit $\mathcal{O}_1$ and the point where the first line of the last line-orbit hits $\mathcal{O}_1$.

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We notice that each first point of orbit $\mathcal{O}_{i+1}$ is shifted for

$$\frac{1}{2}(p_i - q_i)\frac{2\pi}{k}$$

with respect to the first point of $\mathcal{O}_i$.

Geometrically, this last constructed line must hit a point on $\mathcal{O}_1$ and this difference, expressing it in the number of skipped points w.r.t. to the first point of $\mathcal{O}_1$, is uniquely determined – it is

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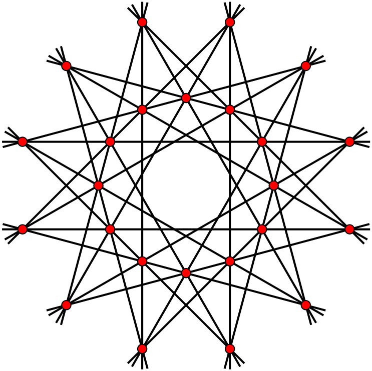
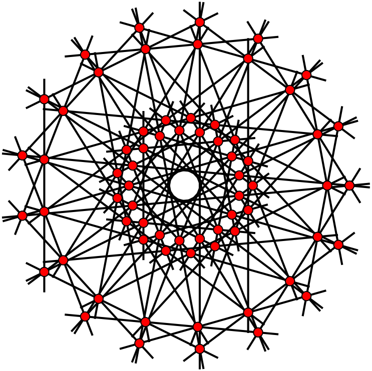


$\mathcal{C}_4(12, (1, 5), (4, 4), -1)$



$\mathcal{C}_4(17, (1, 8, 4, 2), (3, 7, 5, 6), -3)$

Which non-trivial sequences p, q satisfy the “cos” equation?

k	p, q
12	$(1, 5), (4, 4)$
15	$(\mathbf{1}, 4), (\mathbf{3}, \mathbf{3}); (2, \mathbf{7}), (6, 6); (\mathbf{3}, 6), (\mathbf{5}, \mathbf{5})$
18	$(1, 6), (4, 5); (1, 8), (7, 6); (2, 7), (5, 6)$
24	$(1, 11), (8, 10); (2, 8), (5, 7); (2, 10), (8, 8); (3, 9), (6, 8)$
...	...
9	$(1, 2, 4), (3, 3, 3)$
10	$(1, 1, 4), (2, 3, 3)$
15	$(1, 4, 6), (3, 5, 5); (2, 3, 7), (5, 5, 6)$
14	$(1, 1, 6), (3, 4, 5); (1, 2, 5), (3, 3, 4); (1, 3, 6), (2, 5, 5)$
16	$(1, 2, 7), (3, 5, 6); (1, 4, 7), (2, 6, 6); (2, 2, 6), (3, 4, 5)$
...	...
17	$(1, 2, 4, 8), (3, 5, 6, 7)$
...	...

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k	p, q
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18	$(1,6),(4,5);(1,8),(7,6);(2,7),(5,6)$
24	$(1,11),(8,10);(2,8),(5,7);(2,10),(8,8);(3,9),(6,8)$
$\ldots$	$\ldots$
9	$(1,2,4),(3,3,3)$
10	$(1,1,4),(2,3,3)$
15	$(1,4,6),(3,5,5);(2,3,7),(5,5,6)$
14	$(1,1,6),(3,4,5);(1,2,5),(3,3,4);(1,3,6),(2,5,5)$
16	$(1,2,7),(3,5,6);(1,4,7),(2,6,6);(2,2,6),(3,4,5)$
$\ldots$	$\ldots$
17	$(1,2,4,8),(3,5,6,7)$
$\ldots$	$\ldots$

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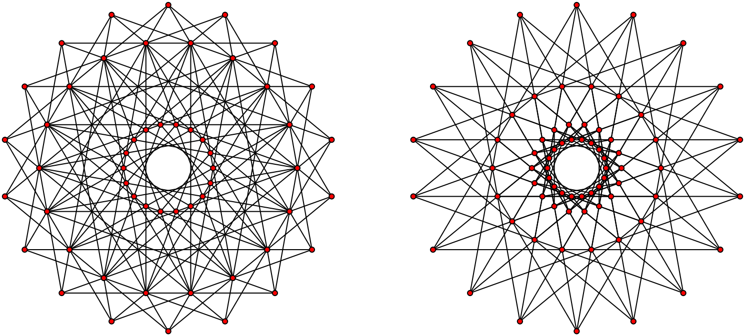
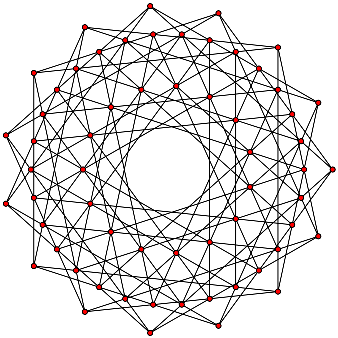
To obtain a weak rotational realization we must avoid the coincidence of points and lines.

Proposition

Given a rotational representation of a $\mathcal{C}_4(k, p, q)$ configuration satisfying "Cos equation" and (3), there are points which geometrically coincide if and only if there exist proper subsequences $p' = (p_i, \dots, p_j)$ and $q' = (q_i, \dots, q_j)$, $0 < |j - i| < n - 1$, of p and q which already satisfy the Cos equation and (3)



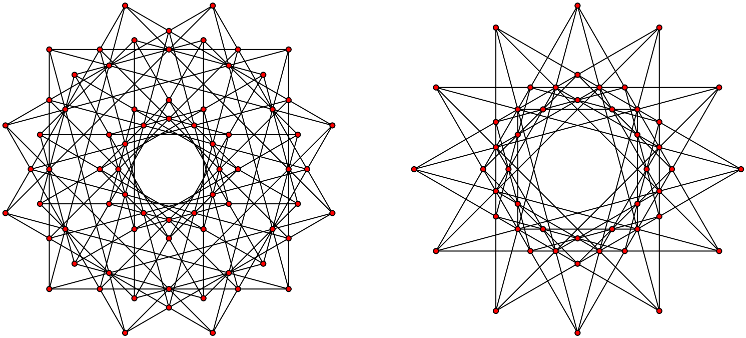
Coincidence of points occurs in a rotational drawing of configuration $C_4(18, (1, 6, 7, 6), (4, 5, 1, 8))$ (left) while coincidence of lines occurs in a rotational drawing of is dual, $\mathcal{C}_4(18, (4, 5, 1, 8), (6, 7, 6, 1))$ (right).



Configuration $\mathcal{C}_4(15, (1, 4, 5, 5), (3, 3, 3, 6))$ has the property that there are four vertex orbits but only three radii. (Cos equation satisfied for a subvectors, but (3) is not.)

$\mathcal{C}_4$ configurations

Accidental incidences



Weak rotational realizations of
$\mathcal{C}_4(12, (3, 3, 5, 1, 4, 2), (4, 2, 3, 3, 5, 1))$ (left) and
$\mathcal{C}_4(12, (1, 2, 5, 3), (3, 4, 4, 2))$ (right) containing additional incidences, lines with five or six points on them.

$\mathcal{C}_4$ configurations

- See M. B., T. Pisanski, *Polycyclic configurations*, European J. Combin. **24** (2003) 431–457.
- Remark: geometric $\mathcal{C}_4(k, (p_1, p_2), (q_1, q_2))$ configurations – *astral* (v_4) *configurations* (B. Grünbaum, L. Berman).

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