Maps

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1 Basic definitions

A finite map on a closed compact surface S is an embedding of a finite connected graph G on S, where $S \setminus G$ consists of connected parts homeomorphic to disks (faces).

1.1 The triangulations

Each such map can be triangulated in the following way. By subdividing each edge and by putting one point in each face we obtain additional vertices called *edge-centers* and *face-centers*, respectively. By embedding additional edges we connect each face center with the original vertices and edge centers around the face. This gives us a triangulation of the map where each triangle has as corners a vertex, an edge center and a face center. Therefore, each triangle can be uniquely labeled by the triple (v, e, f) of the incident vertex v, the edge e and the face f. The triangles are called *flags*.

1.2 Flags and incidences

Since we have a compact closed surface, each such triangle is incident to three other triangles over the sides. The three incident triangles differ in exactly one component of (v, e, f). The incidences of triangles can be encoded into three involutions on flags, s_0, s_1 and s_2, where s_i changes the i-th component of (v, e, f) (counting on from 0) in order to obtain the corresponding incident flag. Each such incidence is called a s_i-incidence and the corresponding two flags are *s_i-incident*. By taking flags as vertices and s_i incidences as edges we obtain a connected cubic 3-colored graph (each type of incidence one color) called a *flag graph*.

1.3 The involutions

Note that in such a setting, the three involutions labeled by s_0, s_1 and s_2 are fixed-point-free involutions, where (due to edges) s_0 and s_2 commute and the product $s_0 s_2$ is also a fixed-point-free involution (since a fixed point would imply a semi-edge). The group $\langle s_0, s_1, s_2\rangle$ is a transitive permutation group (since the map is connected), a subgroup of the symmetric group on flags.

1.4 Walks and words

Consider a flag Φ and a word in labels s_0, s_1, s_2. In a natural way, the pair represents the walk in the flag graph starting in Φ. In any case, the walks s_i^2, $i = 0, 1, 2$, and $(s_0 s_2)^2$ starting from any flag bring us back to the initial flag. Therefore we decide to reduce all such walks to trivial walk. Then the set of all reduced words in alphabet s_0, s_1 and s_2 for the operation concatenation is exactly the finitely presented group

$$F = \langle s_0, s_1, s_2 \mid s_0^2, s_1^2, s_2^2, (s_0 s_2)^2\rangle.$$

Note that we (ab)use the labels s_i in two ways, as letters of an alphabet (and generators of F) and as labels for the three involutions. Observe that the mapping assigining the label $s_i \in F$ to the corresponding permutation on flags extends to the group epimorphism $\varphi : F \to \langle s_0, s_1, s_2\rangle$.

2 Review of basics on actions

For a finite set Z, denote by $\mathrm{Sym}_R(Z)$ the symmetric group of permutations on the set Z, where the permutations are composed from the left to the right. Similarly, denote by $\mathrm{Sym}_L(Z)$ the set of permutations on Z, where the permutations are composed as functions, from the right to the left. Obviously, the groups are isomorphic, but the distinction is in how permutations are applied on an element of Z, in the first case from the right and in the second from the left. In what follows we will define right actions (left actions have analogous definition). The application of $g \in \mathrm{Sym}_R(Z)$ on $z \in Z$ will be denoted by z^g.

2.1 Definition

There are two equivalent definitions of right actions.

Definition 1 (I) *A right action of a group G on the set Z is an operation $\cdot : Z \times G \to Z$ defined by a homomorphism $f : G \to \mathrm{Sym}_R(Z)$, where for each $g \in G$ and $z \in Z$, $z \cdot g = z^{f(g)}$. The homomorphism f is called the* action homomorphism*. The image $f(G)$ is called the* image of the action*.*

Definition 2 (II) *A right action of a group G on the set Z is an operation $\cdot : Z \times G \to Z$ such that for each $z \in Z$ and $g, h \in G$*

(1) $z \cdot 1 = z$, and

(2) $(z \cdot g) \cdot h = z \cdot (gh)$.

Homework 1 *Show that the definitions (1) and (2) are equivalent.*

We intend to use right actions only, and we will call them simply “actions”. An action is often denoted by a triple $(Z, G, \cdot)$, or when the operation is clear from the context, just (Z, G). Often we omit writing the operation and instead of $z \cdot g$ we write zg.

2.2 Basic action properties

An action (Z, G) is *transitive*, if for every $z, z' \in Z$ there exists $g \in G$, such that $zg = z'$. A stabilizer of $z \in Z$ of the action is

$$\mathrm{Stab}_G(z) = \{g \in G \mid zg = z\}.$$

Note that $\mathrm{Stab}_G(z)$ is a subgroup of G. An action is *semi-regular* if all the stabilizers are trivial. If a semi-regular action is in addition transitive, it is called *regular action*.

The kernel of the action is $Z(G) = \{g \in G \mid zg = z \text{ for all } z \in Z\}$ and equals the intersection of all the stabilizers. Considering the first definition of a right group action (1) the kernel is exactly the kernel of the homomorphism f in the definition. An action is *faithful* if the kernel $Z(G)$ is trivial.

2.3 Action morphisms

An *action morphism* of two actions $(Z, G, \cdot)$ and $(Z', G', *)$ is a pair of mappings (p, q), $p : Z \to Z'$ and $q : G \to G'$ respecting both actions. That is, for each $z \in Z$ and $g \in G$

$$p(z \cdot g) = p(z) * q(g).$$

Our particular interest will be in *action epimorphisms*, where both p and q are surjective, and *action isomorphisms*, where both p and q are bijections.

Example 3 *Let (Z, G) be a faithful action and $f : G \to \mathrm{Sym}_R(Z)$ be the action homomorphism. Since* $\ker f = \{1\}$*, $f(G)$ is an isomorphic image of G. The pair $(Id, f) : (Z, G) \to (Z, f(G))$ is the action isomorphism. The group G is said to be* permutation isomorphic *to $f(G)$.*

2.4 Transitive actions

In what follows we will focus on transitive actions. The *natural transitive actions* of a group G are obtained by taking any subgroup $N \leqslant G$ and considering the action of G on the factor set G/N. The following two lemmas tell us that each transitive action is isomorphic to some natural transitive action.

Lemma 4 *Let (Z, G) be a transitive action, $z \in Z$ and $N = \mathrm{Stab}_G(z)$. Then (Z, G) is isomorphic to the natural action of G on the cosets of N, namely $(G/N, G)$. The action isomorphism is (p, Id), where $Id : G \to G$ is the identity automorphism and $p : Z \to G/N$ is defined by $p : z \cdot g \mapsto Ng$, for every $g \in G$.*

Proof. Define $p : z \mapsto N$. Let $z' \in Z$. Transitivity implies the existence of $g \in G$ such that $z' = z \cdot g$. Define $p : z' \mapsto Ng$. If $z' = z \cdot h$ for some other $h \in G$ then $gh^{-1} \in N$ and $Ng = Nh$. Therefore p is well defined and injective. Obviously it is also surjective. Let $x \in Z$ and $h \in G$ such that $x = zh$. Let $g \in G$, then $p(x \cdot g) = p(zhg) = Nhg = Nh\ Id(g) = p(x)Id(g)$. □

2.4.1 Interpretation of the lemma

A simple interpretation of the lemma is that in a transitive action (Z, G) one can label bijectively the elements of Z by the cosets in $G/\mathrm{Stab}_G(z)$ in such a way that the natural action of G on labels (i.e. cosets) matches the action on the elements of Z. Note that the different choice of z implies a different stabilizer.

It is not hard to see that the stabilizers of a transitive action correspond to exactly all conjugates of any particular stabilizer. Hence we have the corollary.

Corollary 5 *Let G be a group and N a subgroup. For each $w \in G$ the actions $(G/N, G)$ and $(G/w^{-1}Nw)$ are isomorphic.*

2.5 The core

The kernel $Z(G)$ of a transitive action (Z, G) is the intersection of all the stabilizers and therefore the intersection of all the conjugates of $\mathrm{Stab}_G(z)$, for any $z \in Z$. The consequence is that $Z(G)$ is the largest normal subgroup in G which is contained in $\mathrm{Stab}_G(Z)$. The largest normal subgroup $H \lhd G$ contained in a subgroup $K \leqslant G$ is usually called the *core* of K and denoted by $\mathrm{Core}_G(K)$. Therefore, $Z(G) = \mathrm{Core}_G(\mathrm{Stab}_G(z))$, for any $z \in Z$. Each two elements from the same coset in $G/Z(G)$ have the same action on all elements in Z. Hence, the action $(G/\mathrm{Stab}_G(z), G/Z(G))$ is well defined and the kernel of this action is trivial – the action is faithful and according to the Example 3, the action can be considered as an action of a permuatation group.

3 Algebraic definition of a map and generalizations

3.1 Rooted maps

A map is an action of the group

$$F = \langle s_0, s_1, s_2 \mid s_0^2, s_1^2, s_2^2, (s_0 s_2)^2 \rangle$$

on a finite set Z of elements called flags. If $f : F \to \mathrm{Sym}_R(Z)$ is the action homomorphism, then $G := f(F)$ together with the distinguished generators $f(s_i)$ (see the three involutions in Section

1) is called the *monodromy group*. If we additionally choose a distinguished flag denoted by $\underline{\mathrm{id}}$ and call it a *root*, then the quadruple $(f, Z, G, \underline{\mathrm{id}})$ represents a *rooted map*.

3.2 Comments on rooted maps

Rather then insisting on G to be a subgroup of $\mathrm{Sym}_R(Z)$ we will be often satisfied with G being an abstract group acting faithfully on Z (which is by Example 3 permutation isomorphic to some subgroup of $\mathrm{Sym}_R(Z)$).

3.2.1 Gluing according to the action

Note that this definition is slightly more general then the one in Section 1. It basically says that for any action (Z, F) we can take the set Z as a set of triangles with sides labeled by all the three labels s_0, s_1 and s_2. Two triangles Φ and Ψ are glued over the side labeled by s_i if and only if $\Phi s_i = \Psi$. Hence, the action is a set of rules for gluing.

Rooting maps will later turn out to be helpful in developing the theory.

3.2.2 Surfaces with boundary

The conditions giving us a compact closed surface (namely, the permutations $f(s_i)$, $i = 0, 1, 2$, and $f(s_0 s_2)$ are fixed point free) are equivalent to that the elements s_i, $i = 0, 1, 2$, and $s_0 s_2$ are not contained in any of the stabilizers. In the definition of a rooted map we omit these conditions. When $\Phi \cdot s_i = \Phi$ for some flag Φ, the corresponding side of the triangle is on the boundary of the surface. This gives us a class of maps embedded on more general surfaces, the ones with boundary.

3.2.3 Vertices, edges, faces

Beside the gluing, the action of F also defines the vertices, edges and faces. A vertex is defined by a set of all the flags containing it. Algebraically, vertices correspond to orbits of $\langle s_1, s_2 \rangle$. Similarly edges corresponds to orbits of $\langle s_0, s_2 \rangle$ and faces to orbits of $\langle s_0, s_1 \rangle$.

3.2.4 Orientability

The existence of an odd length word of F in a stabilizer of a flag implies an existence of an odd length cycle in the flag graph. The triangles glued along the odd length cycle form a Möbius band and therefore the underlying surface is non-orientable. If no such case occurs, the surface is orientable.

In the orientable case only the even length words can bring us back to the initial flags. Therefore all the stabilizers (i.e. the sets of words which bring us back) are contained in the index two subgroup $F^+ \leqslant F$ generated by even words. Relative to the root flag, the flags can be divided into the two partitions, namely the ones reachable from $\underline{\mathrm{id}}$ only by even-length walks and the the ones that can be reachable from $\underline{\mathrm{id}}$ only by odd-length walks. The group F^+ has two orbits on the flags. The flag graph is in the orientable case bipartite.

If this is not the case, we have at least one odd-length cycle in the flag graph. Each flag can be reached by even-length walk and therefore F^+ is transitive on the flags. This case represents non-orientable rooted maps.

4 Quotients and covers of maps

4.1 Map morphism

A *rooted map morphism* of two rooted maps $M = (f, Z, G, \underline{\mathrm{id}})$ and $N = (f', Z', G', \underline{\mathrm{id}}')$ is an action epimorphism $(p, q) : (Z, G) \to (Z', G')$, where $p(\underline{\mathrm{id}}) = \underline{\mathrm{id}}$ and $f' = q \circ f$. Therefore, $(p, Id) : (Z, F) \to (Z', F)$, where $Id : F \to F$ is the identity isomorphism, is an action epimorphism.

Homework 2 *Prove that at most one rooted map morphism exists between any two rooted maps.*

4.2 Canonical representation of a rooted map

Homework 3 *Let $M = (f, Z, G, \underline{\mathrm{id}})$ be a rooted map. Denote by $N = \mathrm{Stab}_F(\underline{\mathrm{id}})$ and $K = \mathrm{Core}_F(N)$. Then M is isomorphic to the rooted map $(q, F/N, F/K, N)$, where $q : F \to F/K$ is the natural epimorphism.*

The corrolary of the Homework 3 is that each rooted map $M = (f, Z, G, \underline{\mathrm{id}})$ can be assigned a subgroup $N = \mathrm{Stab}_F(G) \leqslant F$. The assignment is denoted by the mapping S taking a rooted map M to the subgroup $S(M) = \mathrm{Stab}_F(\underline{\mathrm{id}}) \leqslant F$.

4.3 Quotients and stabilizers

Proposition 6 *Let M and N be rooted maps. Then there is a map morphism from M to N if and only if $S(M) \leqslant S(N)$. In particular, $M \cong N$ if and only if $S(M) = S(N)$.*

Proof. Denote $H = S(M)$, $K = \mathrm{Core}_F(H)$, $H' = S(N)$, $K' = \mathrm{Core}_F(H')$, where $q : F \to F/K$ and $q' : F \to F/K'$ are the natural epimorphisms. We may assume that $M = (q, F/H, F/K, H)$ and $N = (q', F/H', F/K', H')$.

($\Leftarrow$) If $H \leqslant H'$, then $K \leqslant K'$ and the mapping $f : F/K \to F/K'$, defined by $f : Kg \mapsto K'g$ is a group epimorphism, such that $f \circ q = q'$. It suffices to show that f is well defined as the rest is clear. If $Ka = Kb$, then $ab^{-1} \in K \leqslant K'$ and therefore $K'a = K'b$. Hence, f is well defined.

Let $p : F/H \to F/H'$ be defined by $p(Hw) = H'w$. Similarly as for f, p is well defined and obviously a surjection. We claim that (p, f) is the rooted map morphism. Let $a \in F/K$. Then there exists $v \in F$, such that $q(v) = a$ and

$$\begin{aligned} p(Hw \cdot a) &= p(Hw \cdot q(v)) = p(HwKv) = p(HwKw^{-1}wv) \\ &= p(HKwv) = p(Hwv) = H'wv. \end{aligned}$$

On the other hand

$$p(Hw)f(a) = p(Hw)f(q(v)) = H'w \cdot q'(v) = H'wK'v = H'wv.$$

Obviously, $p(H) = H'$.

($\Rightarrow$) Let $(p, f) : (F/H, F/K) \to (F/H', F/K')$ be the rooted map morphism. Then $p(H) = H'$ and $f \circ q = q'$. This implies that

$$f(\mathrm{Stab}_{F/K}(H)) \leqslant \mathrm{Stab}_{F/K'}(H') = H'/K' = q'(H').$$

On the other hand,

$$f(\mathrm{Stab}_{F/K}(H)) = f(H/K) = f(q(H)) = q'(H).$$

Since $q'(H) \leqslant q'(H')$ it follows $H \leqslant q'^{-1}(q'(H')) = H'$. $\square$

4.4 Isomorphism classes of rooted maps and subgroups of F

The direct consequence of the proposition is the following important theorem.

Theorem 7 *The isomorphism classes of rooted maps are for the relation $M \to N$ (i.e. there exists a rooted map morphism from M to N) an algebraic lattice anti-isomorphic to the lattice of finite index subgroups of F.*

The theorem has far reaching consequences. It tells us that each two rooted maps have the unique common cover and the unique common quotient.

4.5 Classification of all quotients

Proposition 8 *Let $M = (f, Z, G, \underline{\mathrm{id}})$. Then all maps N such that there is a rooted map morphism from M to N are isomorphic to one of $(q, F/K, F/\mathrm{Core}_F(K), K)$, where $K \geqslant \mathrm{Stab}_F(\underline{\mathrm{id}})$ and $q : F \to F/\mathrm{Core}_F(K)$ is the natural epimorphism.*

From now on, each rooted map N, such that there is an epimorphism from the rooted map M to N will be called a *quotient* of M.

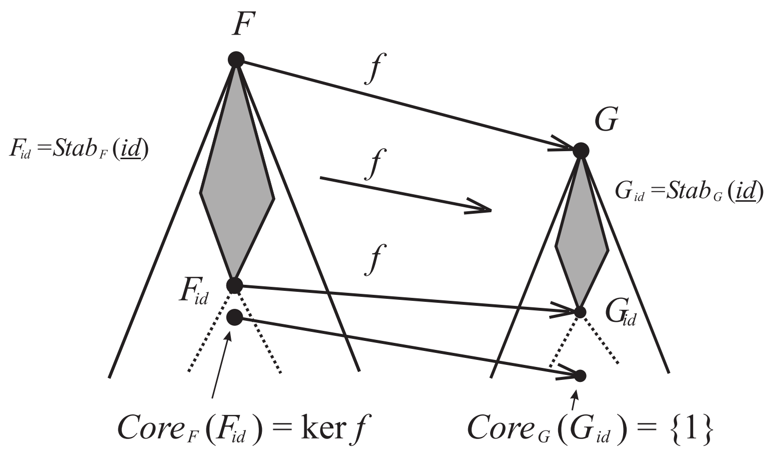
4.5.1 Calculation issues

The fact that F and all K in Proposition 8 are infinite groups makes any calculation hard. In order to enable us to perform any calculation on $M = (f, Z, G, \underline{\mathrm{id}})$, like find the smallest common cover or find all quotients, it would be useful, if we could bring the Theorem 7 and Proposition 8 down into the permutation group G.

4.6 The correspondence theorem

For that we will use the following well known theorem in group theory, so called the Correspondence theorem or the 4th isomorphism theorem for groups.

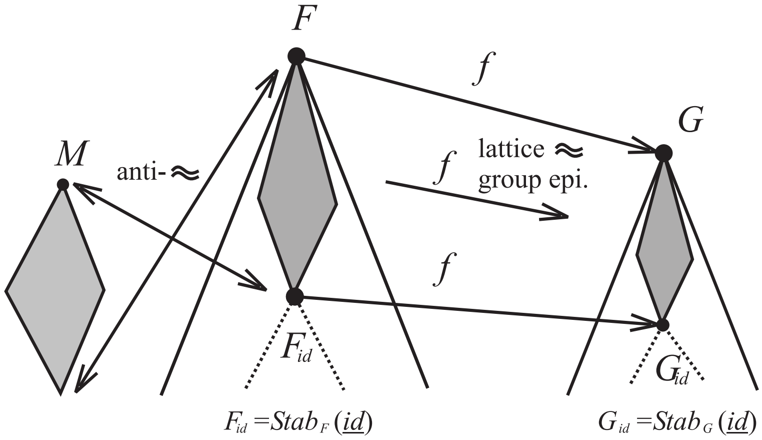
Theorem 9 *Let G and G' be groups, $f : G \to G'$ an epimorphism, $\mathcal{A} = \{K : \ker f \leqslant K \leqslant G\}$ and $\mathcal{B} = \{K' : K' \leqslant G'\}$. The mapping $\Theta : \mathcal{A} \to \mathcal{B}$ defined by $\Theta : K \mapsto f(K)$ is a bijection. Under this bijection normal subgroups correspond to normal subgroups. If $K \in \mathcal{A}$ and $K' \in \mathcal{B}$, $\Theta(K) = K'$, then for any $w \in G$, $\Theta(w^{-1}Kw) = f(w)^{-1}K'f(w)$, and for any $v \in G'$ and any $z \in f^{-1}(v)$, $\Theta^{-1}(v^{-1}K'v) = z^{-1}Kz$. Furthermore, $[G : K] = [G' : K']$ and for any two groups $K, H \in \mathcal{A}$, if $K \leqslant H$, then $F(K) \leqslant F(H)$.*



Homework 4 *Prove all the claims in the Theorem 9.*

4.7 Classification of quotients through the monodromy group

Proposition 10 *Let $M = (f, Z, G, \underline{\mathrm{id}})$ be a rooted map and $S = \mathrm{Stab}_G(\underline{\mathrm{id}})$. Then for each subgroup K, where $S \leqslant K \leqslant G$, the map $M/K = (q \circ f, G/K, G/\mathrm{Core}_G(K), K)$, where $q : G \to G/\mathrm{Core}_G(K)$ is the natural epimorphism, is a quotient of M. Furthermore, for any K', where $S \leqslant K' \leqslant G$, M/K' is a quotient of M/K if and only if $K \leqslant K'$. In particular $M/K \cong M/K'$ if and only if $K = K'$.*



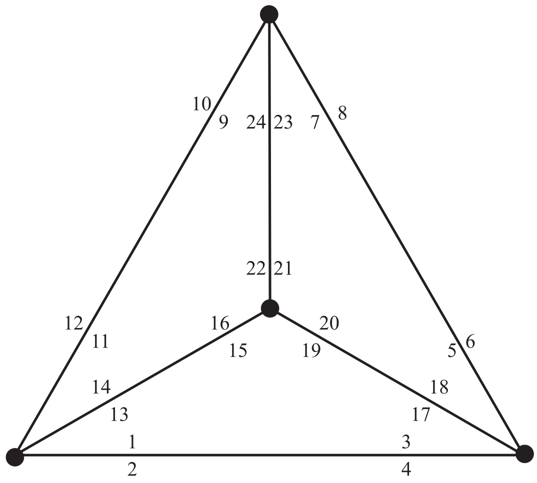
Proof. To verify that M/K is a rooted map, it suffices to check that $(G/K, G/\mathrm{Core}_G(K))$ is a faithful action (which obviously is) and that $(q \circ f)(F) = q(G) = G/\mathrm{Core}_G(K)$. To show that M/K is a quotient of M we have to show that $S(M/K) \leqslant S(M)$. Note that

$$\begin{aligned} S(M/K) &= (q \circ f)^{-1}(\mathrm{Stab}_{G/\mathrm{Core}_G(K)}(K)) \\ &= f^{-1}(q^{-1}(K/\mathrm{Core}_G(K))) \\ &= f^{-1}(K). \end{aligned}$$

On the other hand $S(M) = f^{-1}(G)$. Since $G \leqslant K$, it follows $S(M) \leqslant S(M/K)$. Note that by Theorem 9, the groups K, where $S \leqslant K \leqslant G$ and the groups N, where $S(M) \leqslant N \leqslant F$, are in one-to-one correspondence. □

4.8 Application of the theory

Example 11 *We will use* MAGMA *to determine all the quotients of the tetrahedron map.*



The three involutions are elements of $\mathrm{Sym}_R(24)$*. Calculate the symmetric group on 24 vertices:*

```
S24 := SymmetricGroup(24);
```

Let

```
s0 := S24!(1,3)(2,4)(5,7)(6,8)(9,11)(10,12)(13,15)(14,16)(17,19)(18,20)(21,23)(22,24);
s1 := S24!(1,13)(2,12)(3,17)(4,6)(5,18)(7,23)(8,10)(9,24)(11,14)(15,19)(16,22)(20,21);
s2 := S24!(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24);
```

Calculate the monodromy group as the subgroup of `S24` *generated by* `s0`*,* `s1` *and* `s2`*.*

```
Mon := sub<S24| s0, s1, s2>;
```

The group is of order 24 (check with `Order(Mon);`*. For the root flag we will choose the flag numbered 1. Calculate the stabilizer:*

```
Stab := Stabilizer(Mon, 1);
```

We se, that the stabilizer is trivial. This means that the action of the monodromy group is regular. Calculate the lattice of subgroups for `Mon`*.*

```
sgl := SubgroupLattice(Mon);
```

The result returned is as follows:

```
Partially ordered set of subgroup classes
-----------------------------------------

[ 1]  Order 1   Length 1  Maximal Subgroups:
---
[ 2]  Order 2   Length 3  Maximal Subgroups: 1
[ 3]  Order 2   Length 6  Maximal Subgroups: 1
[ 4]  Order 3   Length 4  Maximal Subgroups: 1
---
[ 5]  Order 4   Length 1  Maximal Subgroups: 2
```

```
[ 6]  Order 4   Length 3  Maximal Subgroups: 2
[ 7]  Order 4   Length 3  Maximal Subgroups: 2 3
[ 8]  Order 6   Length 4  Maximal Subgroups: 3 4
---
[ 9]  Order 8   Length 3  Maximal Subgroups: 5 6 7
[10]  Order 12  Length 1  Maximal Subgroups: 4 5
---
[11]  Order 24  Length 1  Maximal Subgroups: 8 9 10
```

Therefore, the tetrahedron has exactly 11 non-isomorphic quotients, including itself (corresponding to [1]*) and the trivial map on a single flag (corresponding to* [11]*). Consider one of the quotients, let's say the one corresponding to* [2]*. The group K for taking the quotient is of order 2. If T denotes the tetrahedron map, then we will now calculate T/K.*

```
proj, MonTK := CosetAction(Mon, sgl[2]);
```

Here we calculate the monodromy group of the quotient `MonTK` *and the corresponding epimorphism* `proj`*. Now calculate the new three involutions.*

```
ss0 := proj(s0);
ss1 := proj(s1);
ss2 := proj(s2);
```

And print them out:

```
> ss0; ss1; ss2;
(1, 2)(3, 7)(4, 6)(5, 10)(8, 11)(9, 12)
(1, 3)(2, 5)(4, 9)(6, 11)(7, 10)(8, 12)
(1, 4)(2, 6)(3, 8)(5, 10)(7, 11)(9, 12)
```

Lets count the orbits:

```
> Orbits(sub<MonTK| ss1, ss2>);
[
    GSet{ 1, 3, 4, 8, 9, 12 },
    GSet{ 2, 5, 6, 7, 10, 11 }
]
> Orbits(sub<MonTK| ss0, ss2>);
[
    GSet{ 5, 10 },
    GSet{ 9, 12 },
    GSet{ 1, 2, 4, 6 },
    GSet{ 3, 7, 8, 11 }
]
> Orbits(sub<MonTK| ss0, ss1>);
[
    GSet{ 1, 2, 3, 5, 7, 10 },
    GSet{ 4, 6, 8, 9, 11, 12 }
]
```

The map has 2 vertices of degree 3, 2 edges, 2 semi-edges 2 faces.

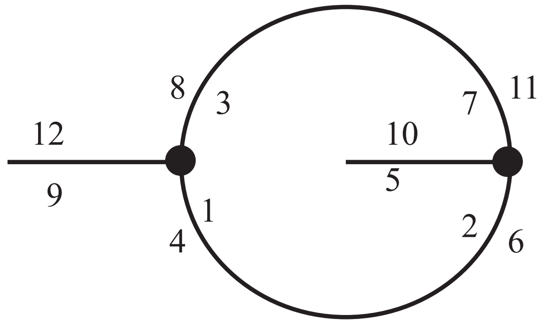
Now we will determine the orientability of the map. The image of the even word subgroup F^+ *can be generated by* `ss0*ss1` *and* `ss1*ss2`. *Let us count the orbits:*

```
> #Orbits(sub<MonTK| ss0*ss1, ss1*ss2>);
2
```

Since we have two orbits the map embedded into an orientable surface. Note that `ss0`, `ss1` *and* `ss2` *are fixed-point-free which means that the underlaying surface is compact closed. The euler formula for orientable compact closed surfaces is*

$$v - e + f = 2 - 2g.$$

Note that in this formula we omit semiedges. Therefore $2 - 2g = 2 - 2 + 2$ *and* $g = 0$. *The map is embedded into the sphere. By using the three involutions we can obtain the image of the embedding:*



4.9 Parallel product

One of interesting questions is also how to calculate the smallest common cover of two rooted maps.

In 1994 S. E. Wilson proposed the operation called the *parallel product*. Let $M = (f, Z, G, \underline{\mathrm{id}})$ and $N = (f', Z', G', \underline{\mathrm{id}}')$ be two rooted maps. Then $G \times G'$ acts naturally on $Z \times Z'$. The subgroup $P \leqslant G \times G'$ generated by $\{(f(s_i), f'(s_i))\}_{i=0}^{2}$ acts on $Z \times Z'$ but not necessarily transitively. Denote by $(Z \times Z')_{(\underline{\mathrm{id}},\underline{\mathrm{id}}')}$ the orbit of P containing $(\underline{\mathrm{id}}, \underline{\mathrm{id}}')$. Then

$$((f, f'), (Z \times Z')_{(\underline{\mathrm{id}},\underline{\mathrm{id}}')}, P, (\underline{\mathrm{id}}, \underline{\mathrm{id}}'))$$

is a rooted map (according to Homework 5) which is the unique smallest map that covers both M and N. The parallel product of M and N is denoted by $M \parallel N$.

Homework 5 *Show that the definition of the parallel product yields a rooted map. In particular you have to show that the action* $((Z \times Z')_{(\underline{\mathrm{id}},\underline{\mathrm{id}}')}, P)$ *is faithful. Show that* $S(M \parallel N) = S(M) \cap S(N)$ *and conclude that* $M \parallel N$ *is the unique smallest common cover of* M *and* N.

The construction proposed by Wilson is suitable for computational purposes.

4.9.1 Parallel product on isomorphism classes

Note that the algebraic lattice of subgroups has the two operations, meet of two groups $H, K \leqslant F$, defined simply by intersection $H \cap K$, and join, defined by the smallest group generated by H and K. By Theorem 7, the parallel product is join in the algebraic lattice of isomorphism classes of rooted maps and is in the correspondence with operation meet in the subgroup lattice (due to the anti-isomorphism). The correspondence could be written as:

$$[M] \parallel [N] = [(q, F/K, F/\mathrm{Core}_F(K), K)],$$

where $K = S(M) \cap S(N)$, $q : F \to F/\mathrm{Core}_F(K)$ is the natural epimorphism, and the notation $[M]$ denotes the isomorphism class of M.

5 Automorphisms of maps

An automorphism of a rooted map $M = (f, Z, G, \underline{\mathrm{id}})$ is any bijection $\alpha \in \mathrm{Sym}_L(Z)$, such that for any $z \in Z$ and any $w \in F$, it follows $\alpha(z \cdot w) = \alpha(z) \cdot w$.

5.1 Semi-regularity of $\mathrm{Aut}(M)$

Due to transitivity of F, there can be at most one map automorphism taking the flag $\underline{\mathrm{id}}$ to some flag Φ. Namely, the assignment $\alpha : \underline{\mathrm{id}} \mapsto \Phi$ implies that for any flag Ψ, there exists $v \in F$, such that $\Psi = \underline{\mathrm{id}} \cdot v$ and therefore $\alpha(\Psi) = \alpha(\underline{\mathrm{id}} \cdot v) = \alpha(\underline{\mathrm{id}}) \cdot v = \Phi \cdot v$.

5.2 The automorphism naming convention

This fact we can use in labeling automorphism according to where they take the root flag. The notation α_w, for $w \in F$ will denote the automorphism which maps $\underline{\mathrm{id}}$ to $\underline{\mathrm{id}} \cdot w$. As usual, we will denote the group of automorphisms of a rooted map by $\mathrm{Aut}(M)$.

5.3 (Non)categorical remark

Note that rooted map automorphisms are not map morphisms in a strict categorical sense. But in the theory of maps this mathematical blasphemy turns out to be convenient and therefore we use it.

5.4 Characterization of automorphisms

Note that in general α_w does not exist for every $w \in F$. The following theorem characterizes when this is the case. Before that, recall the definition of the normalizer. Let K be a subgroup of G. Then the normalizer of K equals

$$\mathrm{Norm}_G(K) = \{g \in G \mid g^{-1}Kg = K\}.$$

Theorem 12 *Let* $M = (f, Z, G, \underline{\mathrm{id}})$, $N = \mathrm{Stab}_F(\underline{\mathrm{id}})$ *and* $\mathcal{N} = \mathrm{Norm}_F(N)$. *Then* $\mathrm{Aut}(M) = \{\alpha_w \mid w \in \mathcal{N}\}$. *In particular* $\alpha_w = \alpha_v$, *if and only if* $Nw = Nv$. *The mapping* $\Theta : \mathcal{N} \to \mathrm{Aut}(M)$, *defined by* $\Theta : w \mapsto \alpha_w$ *is a group epimorhism, with* $\ker \Theta = N$ *and induces the isomorphism of* $\mathcal{N}/N$ *and* $\mathrm{Aut}(M)$.

Proof. Let $\alpha_w \in \mathrm{Aut}(M)$ (therefore α_w exists) and let $n \in N$. It is clear that $\alpha_{w^{-1}} \in \mathrm{Aut}(M)$ and $\alpha_{w^{-1}} = (\alpha_w)^{-1}$ Therefore,

$$\begin{aligned}\underline{\mathrm{id}} \cdot w^{-1}nw &= \alpha_{w^{-1}}(\underline{\mathrm{id}}) \cdot nw = \alpha_{w^{-1}}(\underline{\mathrm{id}} \cdot nw) \\ &= \alpha_{w^{-1}}(\underline{\mathrm{id}} \cdot w) = \alpha_{w^{-1}}(\alpha_w(\underline{\mathrm{id}})) = \underline{\mathrm{id}},\end{aligned}$$

implying that $w^{-1}Nw \leqslant N$ and since $[F : N] = [F : w^{-1}Nw]$, it follows $w^{-1}Nw = N$ and $w \in \mathcal{N}$.

Now let $w \in \mathcal{N}$. Define $\delta : Z \to Z$ by the rules $\delta(\underline{\mathrm{id}}) = \underline{\mathrm{id}} \cdot w$ and $\delta(\underline{\mathrm{id}} \cdot u) = \underline{\mathrm{id}} \cdot wu$, for any $u \in F$. We need to show that δ is well defined. If we show that, then $\delta = \alpha_w \in \mathrm{Aut}(M)$. Consider now $u, v \in F$, such that $\underline{\mathrm{id}} \cdot u = \underline{\mathrm{id}} \cdot v$. Then $uv^{-1} \in N$. Since $\mathcal{N}$ is a group and $w \in \mathcal{N}$, the $w^{-1} \in \mathcal{N}$. Hence, $(w^{-1})^{-1}uv^{-1}w^{-1} \in N$. But $(w^{-1})^{-1}uv^{-1}w^{-1} = wu(wv)^{-1}$ and thus $\underline{\mathrm{id}} \cdot wu = \underline{\mathrm{id}} \cdot wv$. It follows that δ is well defined.

We can conclude, that the mapping $\Theta : \mathcal{N} \to \mathrm{Aut}(M)$, defined by $\Theta : w \mapsto \alpha_w$ is well defined. Note that for any $w, v \in \mathcal{N}$, $\Theta(wv) = \alpha_{wv}$. But

$$\begin{aligned}\alpha_{wv}(\underline{\mathrm{id}}) &= \underline{\mathrm{id}} \cdot wv = \alpha_w(\underline{\mathrm{id}}) \cdot v = \alpha_w(\underline{\mathrm{id}} \cdot v) \\ &= \alpha_w(\alpha_v(\underline{\mathrm{id}})) = (\alpha_w \circ \alpha_v)(\underline{\mathrm{id}}).\end{aligned}$$

Hence $\alpha_{wv} = \alpha_w \circ \alpha_w$ and Θ is group epimorphism. It is clear that $\ker \Theta = N$. □

5.5 Intuitive interpretation of the normalizer

The intuitive interpretation of the theorem is that $\mathcal{N}$ contains exactly all the words in F which take us from $\underline{\mathrm{id}}$ to the flags in the same orbit of $\mathrm{Aut}(M)$. Note that the intuitive interpretation of the stabilizer N was: "all the words in F which bring us back to $\underline{\mathrm{id}}$".

Since (Z, F) is a transitive action, we know that the flags in Z can be labeled by the cosets of F/N in such a way, that the natural action of F on labels corresponds to the action of F on flags. Note that $\mathcal{N}$ is build of some of the cosets in F/N. These cosets label exactly the orbit of $\mathrm{Aut}(M)$ containing $\underline{\mathrm{id}}$ (which is labeled by the coset N).

5.5.1 Seems like hard homework, but it is not ...

Homework 6 *Let $M = (f, Z, G, \underline{\mathrm{id}})$, $N = S(M)$ and $\mathcal{N} = \mathrm{Norm}_F(N)$. Then we know that M is isomorphic to $M' = (q, F/N, F/\mathrm{Core}_F(N), N)$, where $q : F \to F/\mathrm{Core}_F(N)$ is the natural epimorphism. Let $(p, r) : M \to M'$ be the isomorphism. Then*

(1) Then for each orbit $\mathcal{O}$ under the action of $\mathrm{Aut}(M)$ on Z, there exists $w \in F$, such that the flags in $\mathcal{O}$ are mapped bijectively on the cosets $\{Nv \mid v \in \mathcal{N}w\}$. We say that w corresponds *to the orbit $\mathcal{O}$.*

(2) If $w \in F$ corresponds to the orbit $\mathcal{O}$, than any $w' \in \mathcal{N}w$ also corresponds to $\mathcal{O}$. Therefore we say that the coset $\mathcal{N}w$ corresponds *to $\mathcal{O}$.*

(3) Different cosets $\mathcal{N}w$ and $\mathcal{N}v$ correspond to different orbits.

(4) All orbits of $\mathrm{Aut}(M)$ on the flags Z are of the same size which equals $|\mathrm{Aut}(M)| = [\mathcal{N} : N]$ and number of orbits equals $[F : \mathcal{N}]$.

5.6 Representation of the action of $\mathrm{Aut}(M)$

For any map $M = (f, Z, G, \underline{\mathrm{id}})$, we know that M is isomorphic to $(q, F/N, F/\mathrm{Core}_F(N), N)$ where $N = S(M)$ and $q : F \to F/\mathrm{Core}_F(N)$ is the natural epimorphism. Let (p, r) be the isomorphism. Therefore the flags can be considered as cosets in F/N.

In this case the natural left action of the group $\mathcal{N}/N$ on the cosets F/N (i.e. labels) is equivalent to the action of $\mathrm{Aut}(M)$ on flags. Let θ be the isomorphism between $\mathcal{N}/N$ and $\mathrm{Aut}(M)$ induced by Θ in Theorem 12.

The natural left action of $\mathcal{N}/N$ on the cosets F/N is defined as follows.

For $Nw \in F/N$ and $Nv \in \mathcal{N}/N$ (that is: $w \in F$ and $v \in \mathcal{N}$) it follows

$$NvNw = NvN(v^{-1}v)w = N(vNv^{-1})vw = Nvw.$$

We have a well defined left action $(\mathcal{N}/N, F/N)$ (note the reversed order of the group and the set in notation for left actions).

Homework 7 *Using the notation as above, prove that $(\theta^{-1}, p) : (\mathrm{Aut}(M), Z) \to (\mathcal{N}/N, F/N)$ is an isomorphism of left actions.*

5.7 Representing $\mathrm{Aut}(M)$ with right actions

One can use right actions to represent $\mathrm{Aut}(M)$. Then for $M = (f, Z, G, \underline{\mathrm{id}})$, $\mathrm{Aut}(M) \leqslant \mathrm{Sym}_R(Z)$. Since for any $z \in Z$, $\alpha \in \mathrm{Aut}(M)$ and $w \in G$ (G also considered as a subgroup of $\mathrm{Sym}_R(Z)$) it follows

$$z\alpha w = zw\alpha,$$

it follows that $\alpha w \alpha^{-1} w^{-1} \in \mathrm{Stab}_G(z)$. But as this is true for any $z \in Z$ and (Z, G) is faithful, the intersection of all stabilizers (this is exactly $Z(G)$) is trivial and Therefore, $\alpha w = w\alpha$ or $\alpha^{-1} w \alpha = w$. Therefero $\mathrm{Aut}(M)$ is exactly the centralizer of G in $\mathrm{Sym}_R(Z)$. Note that in general, for G a group and K a subgroup, the centralizer is

$$\mathrm{Cent}_G(K) = \{g \in G \mid g^{-1}kg = k, \text{ for any } k \in K\}.$$

Example 13 *Continue from Example 11. Let us calculate the automorphism group of the tetrahedron.*

```
Norm := Normalizer(Mon, Stab);
Aut1 := quo<Norm| Stab>;
```

It is true that `Aut1` *is isomorphic to the automorphism group. However, for us is not very useful, since* `Aut1` *does not act on the same set of flags as* `Mon`*. To achieve this we use:*

```
Aut := Centralizer(S24, Mon);
```

To verify the theory check the following.

```
IsIsomorphic(Aut, Aut1);
```

5.8 About symmetries, regular maps

There are several interesting implications of Theorem 12, among them the following one.

Let n denote the number of flags. Since the monodromy group of a map M, denoted by $\mathcal{M}(M)$ is transitive it follows that

$$n \leqslant |\mathcal{M}(M)|.$$

Since the $\mathrm{Aut}(M)$ is semi-regular it follows

$$|\mathrm{Aut}(M)| \leqslant n.$$

Note that for $N = S(M)$ and $\mathcal{N} = \mathrm{Norm}_F(N)$, $\mathcal{M}(M) \cong F/\mathrm{Core}_F(N)$, while $\mathrm{Aut}(M) \cong \mathcal{N}/N$. The number of flags corresponds to the number of cosets. Equalities occur in the above two equations exactly at the same time: namely when $N = \mathrm{Core}_F(N)$, i.e. $\mathcal{N} = F$. In this special case we have the maximal symmetry, monodromy and automorphism group are of the same order equal to the number of flags. We have so called *regular maps.* An example of a regular map is the tetrahedron. For a regular map M, it follows $S(M) \lhd F$.

Clearly, parallel product of regular maps is a regular map (if $S(M)$ and $S(N)$ are normal in F, then $S(M) \cap S(N)$ is also normal in F).

Homework 8 *Let M and N be maps such that in both maps the automorphism α_w, $w \in F$, exists. Then α_w exists in $M \parallel N$.*

6 Map operations

A *map operation* is defined by a pair (φ, f), where $\varphi : F \to Q$ is an epimorphism and $h : F \to Q$ a homomorphism that takes subgroups of finite index to subgroups of finite index (not necessarily the same as the initial index). The operation is performed on stablizers. Let M be a rooted map. Then $\varphi^{-1}(h(S(M))$ is the stabilizer corresponding to the map obtained by the operation.

6.1 Dual, Petrie dual

The most common map operations arise from certain automorphisms of F. Two of the most interesting map operations are the Dual and the Petrie Dual of a map. They are defined by the corresponding automorphisms of F, defined on generators by:

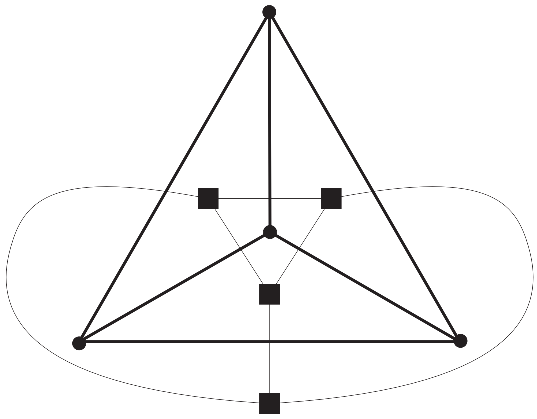
$$(s_0, s_1, s_2) \mapsto (s_2, s_1, s_0)$$

and

$$(s_0, s_1, s_2) \mapsto (s_0 s_2, s_1, s_2).$$

Homework 9 *Show that the operations Dual and Petrie dual generate a group of six operations which is isomorphic to S_3.*

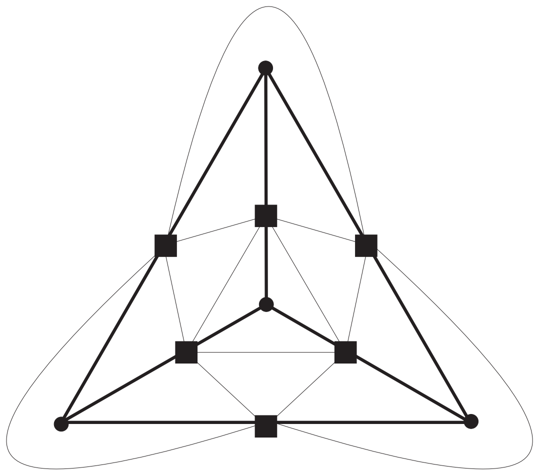
Both operations have the corresponding geometric interpretation. For instance, the change of roles in generators s_0 and s_2 in dual results in changing the role of vertices and faces. An edge separating two faces becomes in dual of the map an edge that connects the two faces (which are now in the role of vertices).



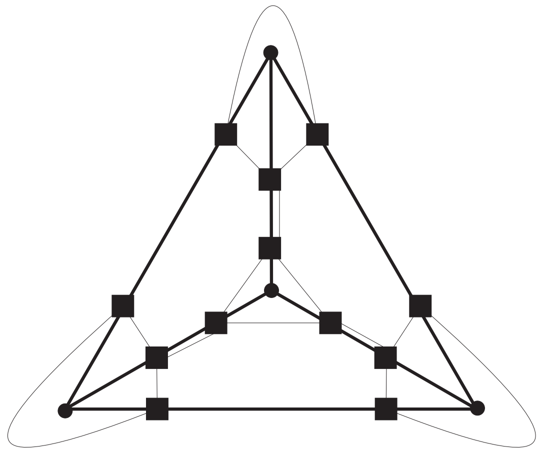
Homework 10 *Determine all the maps that can be obtained from the tetrahedron by applying the operations Dual and Petrie Dual. For each such map determine its underlying surface.*

There are also other interesting operations on maps on compact closed surfaces which preserve the surface. Among them the most important are the medial and the truncation.

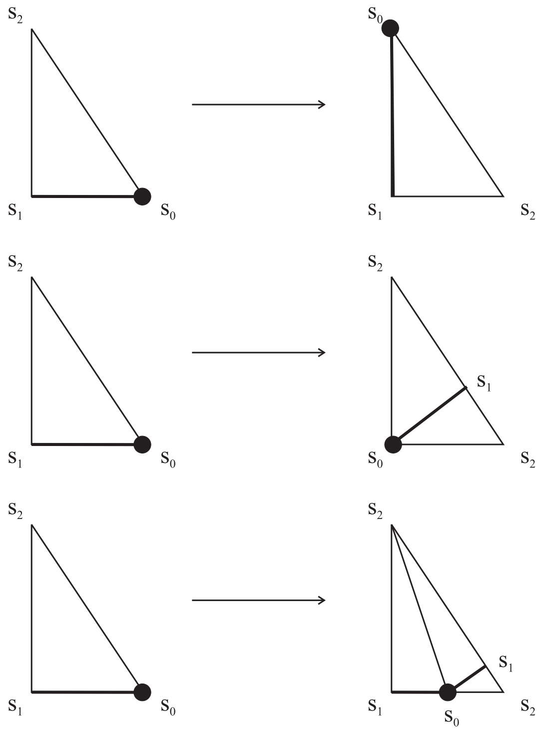
In the medial we subdivide each edge by adding edge-centers. The edge-centers become the new vertices of the medial of the map. Then we can imagine that we connect two edge-centers in the face where the edge-centers are incident. The example is shown in the figure below.



Another interesting operation is the truncation. What we basically do here is that we in a kind of a geometic way truncate the vertices. The figure below shows an example.



The dual, medial and the truncation are members of the family of the operations perserving the surface. This family is obtained by taking flags of an initial map M and considering the flags as triangles, an operation is defined by a subdivision of each triangle to smaller triangles together with the reinterpretation (or new labelling) of the new triangle vertices in a consistent way within triangle as well as between the (big) triangles representing the flags. The figures below represent such divisions for the dual, medial and the truncation.



Now let us just quickly see how algebraic definition for truncation goes.

Let $\mathcal{C}_3 = \langle s_0, s_1, s_2 \mid s_0^2, s_1^2, s_2^2, (s_0s_2)^2, (s_1s_2)^3\rangle$. The group $\mathcal{C}_3$ is a quotient of F and let $\varphi : \mathcal{C} \to \mathcal{C}_3$ be the epimorphism. Since the subgroup $K = \langle s_0, s_1s_0s_1, s_2\rangle$ is of index 3 in F, then

$\varphi(K)$ is also of index 3 in $\mathcal{C}_3$. Define:

$$\begin{aligned} r_0 &= \varphi(s_0), \\ r_1 &= \varphi(s_1 s_0 s_1), \\ r_2 &= \varphi(s_2). \end{aligned}$$

One can show that the mapping:

$$f : F \to f(K), \qquad f : s_i \to r_i$$

is a group epimorphism which extends to an homomorphism $f : F \to \mathcal{C}_3$. The truncation is defined by the pair (φ, f).

7 Generalizations of rooted maps

If instead of F we take some other group the theory still works. For instance, if

$$F = \langle s_0, s_1, s_2 \mid s_0^2, s_1^2, s_2^2 \rangle,$$

we obtain hypermaps.

If

$$F = \langle r, l \mid l^2 \rangle.$$

we get orientable maps (rotation and edge involution).