# Lecture no. 7

## Transitive group actions on graphs

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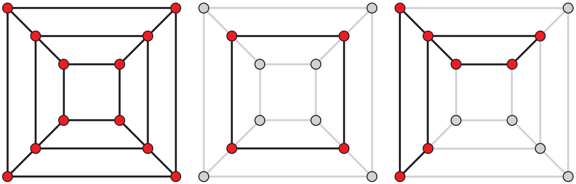
# The entire class of (simple) graphs is too big

- Graphs are used to represent various “complicated systems” in a compact way.
- But a graph is in fact just an irreflexive symmetric relation on a set.
- The entire class of graphs is thus too big to be investigated as a whole.
- We are forced to make some restrictions, that is, we focus on some special classes of graphs.
- One possibility is to require that a graph has a certain degree of symmetry.
- This is measured by the automorphism group of the graph in question.

# A quick review

- A *k*-path of a graph $X$ is a sequence of $k+1$ pairwise distinct vertices of $X$ such that each two consecutive vertices are adjacent.
- An *s*-arc of $X$ is a sequence of $s+1$ vertices of $X$ such that each two consecutive vertices are adjacent and any three consecutive vertices are pairwise distinct.
- A cycle of $X$ is a connected regular subgraph of $X$ of degree $2$.



# A quick review

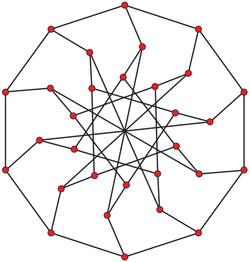
- An action of a group $G$ on a set $X$ is transitive if for any pair of points $x, y \in X$ there exists a $g \in G$ such that $y = xg$.
- The automorphism group $\mathrm{Aut}X$ of $X$ is defined as a certain permutation group acting on its vertex set.
- But it also acts naturally on the set of edges, arcs, $s$-arcs, $k$-paths, $k$-cycles, etc.
- This way we obtain some interesting classes of graphs.

# Classes of graphs

- Vertex-transitive graphs.
  - The group $\mathrm{Aut}X$ acts transitively on $V(X)$.
- Edge-transitive graphs.
  - $\mathrm{Aut}X$ acts transitively on $E(X)$.
  - The graph $X$ can be vertex-transitive or not.
  - In the latter case, $X$ is semisymmetric.
- Arc-transitive graphs.
  - $\mathrm{Aut}X$ acts transitively on the set $A(X)$ of $1$-arcs of $X$.
  - In this case $X$ is automatically edge-transitive. If it does not contain isolated vertices it is also vertex-transitive.
  - $s$-arc transitive graphs, where $s \geq 2$.
- Half-arc-transitive graphs.
  - $X$ is vertex-transitive, edge-transitive but not arc-transitive.

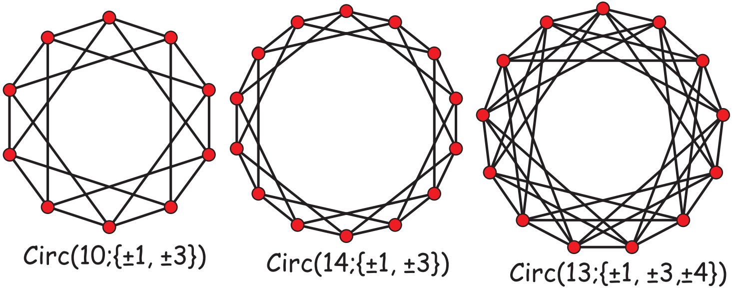
# $s$-arc-transitivity

- For which $s$ does there exist an $s$-arc-transitive graph?
- In 1947 Tutte showed that there exists no $s$-arc-transitive cubic graph for $s > 5$.
- In 1981 Weiss showed that there exists no $s$-arc-transitive graph for $s > 7$ (using the classification).
- The smallest 5-arc-transitive cubic graph: Tutte's 8-cage.
- An arc-transitive cubic graph $X$ of order $n$ is exactly $s$-arc-transitive iff $|\mathrm{Aut}X| = 3n \cdot 2^{s-1}$.



# Examples

- Circulant graphs, that is Cayley graphs of cyclic groups.
  - $Circ(n; S)$, where $S \subset \mathbb{Z}_n$ with $0 \notin S$ and $-S = S$, is the graph with vertex set $\mathbb{Z}_n$ and where $i \sim j \iff j - i \in S$.
  - $Circ(10; \{\pm 1, \pm 3\})$ is exactly 2-arc-transitive.
  - $Circ(13; \{\pm 1, \pm 3, \pm 4\})$ is exactly 1-arc-transitive.
  - $Circ(14; \{\pm 1, \pm 3\})$ is just vertex-transitive.



# Homework 1

**H1:**

- Determine the largest integer $s$ for which there exists a connected $s$-arc-transitive circulant of valency at least three.
- Then classify all connected $s$-arc transitive circulants of valency at least three.

# Examples

- Generalized Petersen graphs $G(n,k)$, $k \leq \frac{n-1}{2}$.
- In 1971 Frucht, Graver and Watkins proved the following:
  - Let $A(n,k) = \mathrm{Aut}\, G(n,k)$ and let $B(n,k) \leq A(n,k)$ be the subgroup fixing the set of "spokes".
  - Let $\rho, \sigma, \tau$ be defined by:
    - $u_i\rho = u_{i+1}$, $v_i\rho = v_{i+1}$
    - $u_i\sigma = v_{ki}$, $v_i\sigma = u_{ki}$
    - $u_i\tau = u_{-i}$, $v_i\tau = v_{-i}$
  - Then
    - if $k^2 \not\equiv \pm 1 \pmod{n}$ then $B(n,k) = \langle \rho, \tau \rangle = D_{2\cdot n}$.
    - if $k^2 \equiv 1 \pmod{n}$ then $B(n,k) = \langle \rho, \sigma, \tau \rangle$, where $\sigma\tau = \tau\sigma$ and $\sigma^{-1}\rho\sigma = \rho^k$.
    - if $k^2 \equiv -1 \pmod{n}$ then $B(n,k) = \langle \rho, \sigma \rangle$, where $\sigma^{-1}\rho\sigma = \rho^k$.
    - $A(n,k) = B(n,k)$ unless $(n,k) \in \{(4,1),(5,2),(8,3),(10,2),(10,3),(12,5),(24,5)\}$.

# Examples

- It follows that

**Proposition**

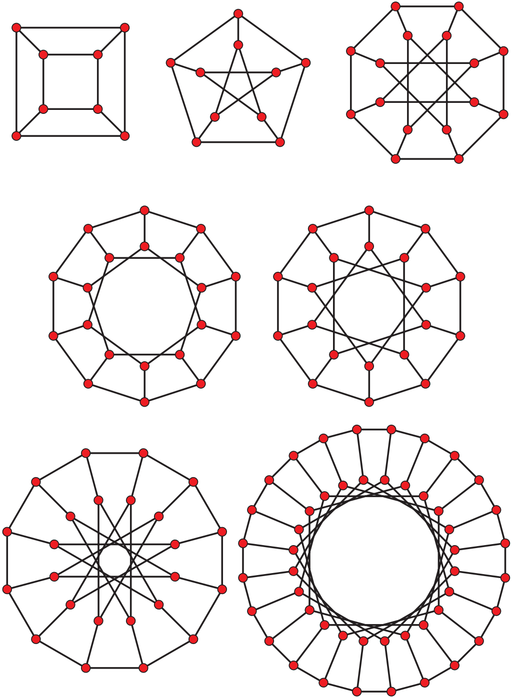
*The graph $G(n, k)$ is vertex-transitive iff $k^2 \equiv \pm 1 \pmod{n}$ or $n = 10$, $k = 2$.*

- It turns out that the above seven exceptional graphs are edge-transitive. Hence

**Proposition**

*The graph $G(n, k)$ is edge-transitive (and thus arc-transitive) iff $(n, k) \in \{(4, 1), (5, 2), (8, 3), (10, 2), (10, 3), (12, 5), (24, 5)\}$.*

- In fact, all of the above seven graphs are $2$-arc-transitive and the Petersen graph $G(5, 2)$ and the Desargues graph $G(10, 3)$ are also $3$-arc-transitive.



# Semisymmetric graphs

## Proposition

*A semisymmetric graph is bipartite.*

## Proof.

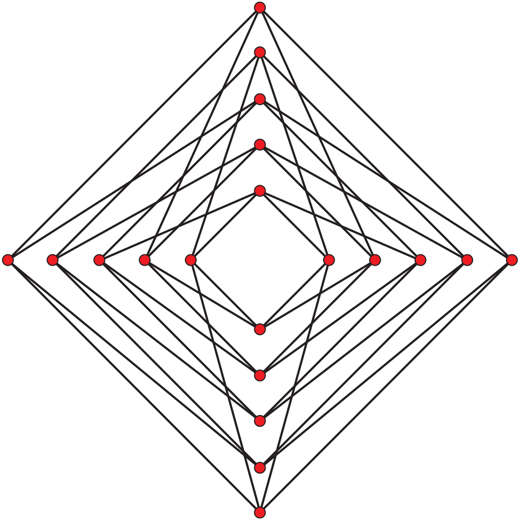
- We can assume that the graph $X$ is connected.
- Take any $uv \in E(X)$, let $\mathcal{U}$ and $\mathcal{V}$ be the $\mathrm{Aut}X$-orbits of $u$ and $v$, respectively.
- Let $w \in V(X)$. There exists $e \in E(X)$ incident with $w$.
- As $X$ is edge-transitive, some $\varphi \in \mathrm{Aut}X$ maps $uv$ to $e$.
- Hence, $w \in \mathcal{U} \cup \mathcal{V}$, and so, as $X$ is not vertex-transitive, $\mathrm{Aut}X$ has two orbits, namely $\mathcal{U}$ and $\mathcal{V}$, on $V(X)$.
- Clearly $\mathcal{U}$ and $\mathcal{V}$ are independent sets, and so $X$ is bipartite.

□

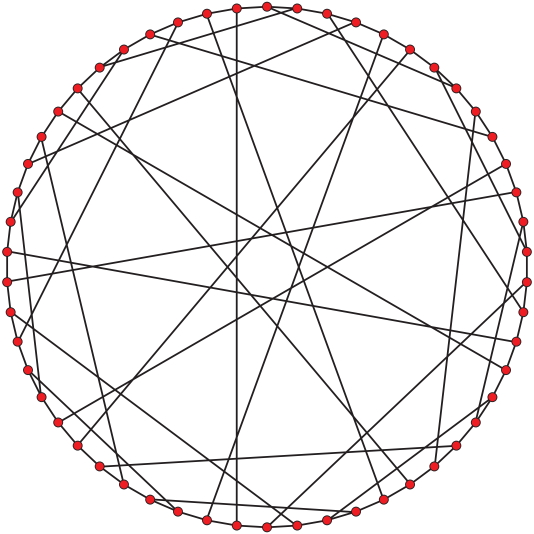
# Semisymmetric graphs

- It turns out that the smallest semisymmetric graph is the so-called Folkman graph. It is tetravalent and has $20$ vertices. The Folkman graph is of girth $4$. Its automorphism group is of order $3840$.
- It turns out that the smallest cubic semisymmetric graph is the so-called Gray graph having $54$ vertices. One of its LCF notations is $[7, -7, 13, -13, 25, -25]^9$. The Gray graph is of girth $8$. Its automorphism group is of order $1296$.

# The Folkman graph



# The Gray graph



# Homework 2

**H2:**

- Show that the Folkman graph is indeed semisymmetric.

# An open problem

- There are many interesting open problems about vertex-transitive graphs.
- Probably the most famous is the question of Lovász about the existence of Hamilton paths in vertex-transitive graphs.
- This question has been open for almost forty years.
- So far no example not having such a path is known.
- In fact, only four (excluding the trivial $K_2$) vertex-transitive graphs not possessing a Hamilton cycle are known. These are the Petersen graph, the Coxeter graph and the two graphs obtained from them by “replacing each vertex by a triangle”.
- None of these four graphs is a Cayley graph.

# An open problem

- Many papers on the subject.
- Complete answers only for graphs of certain order: $p$, $2p$, $3p$, $4p$, $5p$, $6p$, $p^2$, $p^3$, $p^4$, $2p^2$.
- The following result by Dobson, Gavlas, Morris[2] is also of interest:

**Theorem**

*Every connected vertex-transitive graph, other than the Petersen graph, whose automorphism group contains a transitive subgroup with a cyclic commutator subgroup of prime-power order, has a Hamilton cycle.*

# An open problem

- How do we tackle such a problem?
- Semiregular automorphisms.
- An automorphism of a graph of order $mn$ is $(m, n)$-semiregular if it has $m$ orbits of length $n$.

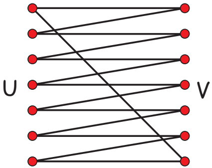
**Proposition (Marušič, 1981)**

*A vertex transitive graph of order $mp$, where $m \leq p$, $p$ a prime, admits an $(m, p)$-semiregular automorphism.*

- Why semiregular automorphisms?

# Using semiregular automorphisms

- We show that, except for the Petersen graph, every vertex-transitive graph of order $2p$ has a Hamilton cycle.
- We can assume that $p \geq 3$.
- By the above proposition $X$ has a $(2, p)$-semiregular automorphism. Let $U$ and $V$ be its two orbits.
- If the bipartite subgraph $[U, V]$ is of valency greater than $1$, then $X$ clearly has a Hamilton cycle.
- We can thus assume that $[U, V]$ is a matching and hence each of the circulants $[U]$ and $[V]$ is connected.



# Using semiregular automorphisms

- It is easy to see that $[U]$ is Hamilton-connected if it is not a cycle. (Chen-Quimpo)
- We can thus assume that $X \cong G(p, k)$ is a generalized Petersen graph.
- Thus $k^2 \equiv \pm 1 \pmod{n}$.
- K. Bannai (1978) showed that $G(n, k)$ has a Hamilton cycle whenever $gcd(n, k) = 1$ except when $n \equiv 5 \pmod{6}$ and $k \in \{2, \frac{n-1}{2}\}$.
- Thus, if $X$ is not the Petersen graph, it contains a Hamilton cycle.

# Constructing vertex-transitive graphs

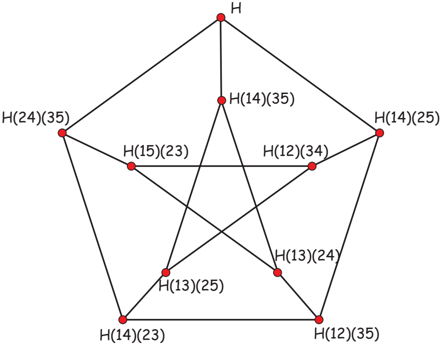
- Let $G$ be a transitive permutation group acting on the set $X$.
- Let $\mathcal{O}$ be a union of orbitals of $G$ on $X$ (orbits of $G$ on $X \times X$).
- The generalized orbital (di)graph $GenOrb(G, X, \mathcal{O})$ relative to $G$, $X$ and $\mathcal{O}$ is then the (di)graph with vertex set $X$ and edge set $\mathcal{O}$.
- The digraph $GenOrb(G, X, \mathcal{O})$ is a graph iff $\mathcal{O}$ coincides with $\mathcal{O}^* = \{(y, x) \mid (x, y) \in \mathcal{O}\}$.
- The digraph $GenOrb(G, X, \mathcal{O})$ is of course vertex-transitive.

# Constructing vertex-transitive graphs

- Let $G$ be an arbitrary group and let $H$ be its subgroup.
- Then $G$ acts on $G/H = \{Hg \mid g \in G\}$ by right multiplication.
- Let $\mathcal{O}$ be a union of orbits of $H$ on $G/H$, that is, $\mathcal{O}$ is a union of double cosets $HgH$.
- The generalized orbital (di)graph $GenOrb(G, H, \mathcal{O})$ relative to $G$, $H$ and $\mathcal{O}$ is then the (di)graph with vertex set $G/H$ and where $Hg \to Hg' \iff g'g^{-1} \in \mathcal{O}$.
- Every vertex-transitive graph is a generalized orbital graph of some group - its automorphism group.

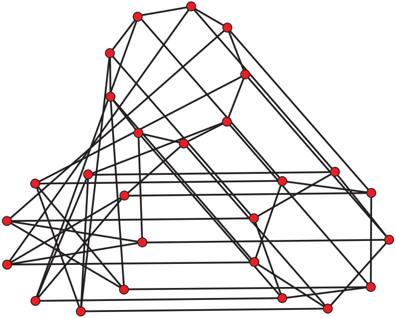
# For example:

- Let $G = Alt(5)$ be the alternating group of order 60.
- Let $H = \langle (1\,2\,3), (1\,2)(4\,5) \rangle = \{id, (1\,2\,3), (1\,3\,2), (1\,2)(4\,5), (1\,3)(4\,5), (2\,3)(4\,5)\}$.
- Let $\mathcal{O} = H(2\,4)(3\,5)H$.
- The graph $GenOrb(G, H, \mathcal{O})$ is isomorphic to $G(5, 2)$.



# Half-arc-transitive graphs

- Tutte (1966): half-arc-transitive $\Rightarrow$ even valency.
- The question of the existence of half-arc-transitive graphs of prescribed even valency.
- Bower (1970): they exist.
- Doyle (1976) and Holt (1981): found one of order 27.
- This is in fact the smallest half-arc-transitive graph.



# Tetravalent half-arc-transitive graphs

- Smallest possible valency is 4.
- Many papers dealing with these graphs.
- Even with this restriction the classification is presently beyond our reach.
- There has been some progress.

# We cannot flip an edge

**Proposition (Proposition 2.1.(Marušič 1998))**

*Let $X$ be a $G$-half-arc-transitive graph for some $G \leq \mathrm{Aut} X$. Then no element of $G$ interchanges a pair of adjacent vertices of $X$.*

- Two (paired) oriented graphs $D_G(X)$ correspond to a $G$-half-arc-transitive graph.
  - Fix an edge $uv$ and choose one of the two orientations.
  - As $X$ is $G$-edge transitive, we can map any edge $xy$ to $uv$.
  - By the above proposition always in “the same way”.

# Alternating cycles and attachment number

- In tetravalent half-arc-transitive graphs we thus have alternating cycles.
- Half of their length is called the radius of the graph in question.
- Any two nondisjoint alternating cycles meet in the same number of vertices.
- This number is called the attachment number of the graph in question.
- The relation between these two numbers is very important.

# The approach to classification

**Theorem (Marušič, Praeger, 1999)**

*If $X$ is a connected tetravalent $G$-half-arc-transitive graph, then either $X$ is tightly $G$-attached or it is a cover over a loosely or antipodally $G$-attached graph.*

- We thus need to classify these three special families.
- Then investigate these covers.
- So far the first step has been done: the classification of tightly attached graphs has been completed.

# The two theorems

**Theorem (Marušič, 1998)**

*A connected tetravalent graph $X$ is a tightly attached half-arc-transitive graph of odd radius $n$ if and only if $X \cong \mathcal{X}_o(m, n; r)$, where $m \geq 3$ and $r \in \mathbb{Z}_n^*$ satisfies $r^m = \pm 1$, and moreover none of the following conditions is fulfilled:*

(i) $r^2 = \pm 1$;

(ii) $(m, n; r) \in \{(3, 7; 2), (3, 7; 4)\}$;

(iii) $(m, n; r) = (6, 7n_1; r)$, *where* $n_1 \geq 1$ *is odd and coprime to* $7$, $r^6 = 1$, *and there exists a unique solution* $r' \in \{r, -r, r^{-1}, -r^{-1}\}$ *of the equation* $2 - x - x^2 = 0$ *such that* $7(r' - 1) = 0$ *and* $r' \equiv 5 \pmod{7}$.

# The two theorems

## Theorem (Šparl, 200?)

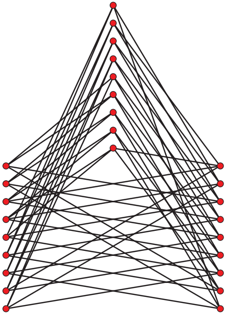
*A connected tetravalent graph* $X$ *is a tightly attached half-arc-transitive graph of even radius* $n$ *if and only if* $X \cong \mathcal{X}_e(m, n; r, t)$, *where* $m \geq 4$ *is even,* $r \in \mathbb{Z}_n^*$, $t \in \mathbb{Z}_n$ *are such that* $r^m = 1$, $t(r-1) = 0$ *and* $1 + r + \cdots + r^{m-1} + 2t = 0$, *and none of the following two conditions is fulfilled:*

(i) $r^2 = \pm 1$;

(ii) $m = 6$, $n = 14n_1$, *where* $n_1$ *is coprime to* $7$, *and there exists a unique solution* $r' \in \{r, -r, r^{-1}, -r^{-1}\}$ *of the equation* $2 - x - x^2 = 0$ *such that* $r' \equiv 5 \pmod 7$ *and* $2 + r' + t' = 0$, *where* $t' = t$ *in case* $r' \in \{r, r^{-1}\}$ *and* $t' = t + r + r^3 + \cdots + r^{m-1}$ *in case* $r' \in \{-r, -r^{-1}\}$.

# The odd radius graphs

- For every pair of integers $m, n \geq 3$, $n$ odd, and for any $r \in \mathbb{Z}_n^*$ for which $r^m = \pm 1$ let $\mathcal{X}_o(m, n; r)$ be the graph with
  - vertex set $V = \{u_i^j \mid i \in \mathbb{Z}_m,\ j \in \mathbb{Z}_n\}$
  - edges defined by $u_i^j \sim u_{i+1}^{j \pm r^i}$.
- These graphs admit a half-arc-transitive subgroup of automorphisms.



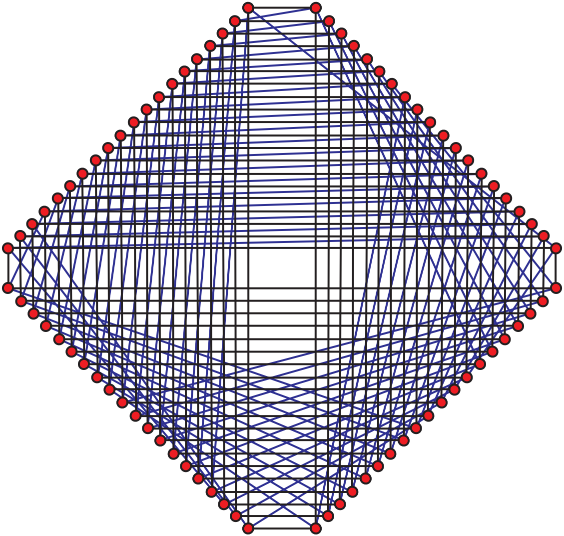
# The even radius graphs

- For every pair of even integers $m, n \geq 4$ and for every $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$, satisfying $r^m = 1$, $t(r-1) = 0$ and $1 + r + \cdots + r^{m-1} + 2t = 0$ let $\mathcal{X}_e(m, n; r, t)$ be the graph with
  - vertex set $V = \{u_i^j \mid i \in \mathbb{Z}_m,\ j \in \mathbb{Z}_n\}$
  - edges given by the following adjacencies:

$$u_i^j \sim \begin{cases} u_{i+1}^j,\ u_{i+1}^{j+r^i} & ; \quad i \in \mathbb{Z}_m \backslash \{m-1\},\ j \in \mathbb{Z}_n \\ u_0^{j+t},\ u_0^{j+r^{m-1}+t} & ; \quad i = m-1,\ j \in \mathbb{Z}_n. \end{cases}$$

- These graphs admit a half-arc-transitive subgroup of automorphisms.

# The smallest even radius graph



# Some tightly attached graphs with small girth

- Girth 3:
  - $\mathcal{X}_o(3, 13; 3)$
  - $\mathcal{X}_o(3, 19; 7)$
  - $\mathcal{X}_o(3, 21; 4)$
- Girth 4:
  - $\mathcal{X}_o(4, 15; 2)$
  - $\mathcal{X}_o(4, 35; 8)$
  - $\mathcal{X}_e(4, 20; 3, 0)$
  - $\mathcal{X}_e(4, 20; 7, 10)$
  - $\mathcal{X}_e(4, 30; 23, 0)$
- Girth 5:
  - $\mathcal{X}_o(3, 9; 2)$
  - $\mathcal{X}_o(5, 11; 2)$
  - $\mathcal{X}_o(5, 11; 3)$
- Girth 6:
  - $\mathcal{X}_o(6, 9; 2)$
  - $\mathcal{X}_o(9, 7; 2)$
  - $\mathcal{X}_o(4, 17; 2)$
  - $\mathcal{X}_e(6, 14; 5, 0)$
  - $\mathcal{X}_e(6, 18; 7, 15)$
  - $\mathcal{X}_e(6, 26; 9, 0)$
  - $\mathcal{X}_e(6, 26; 23, 0)$
- Girth 8:
  - $\mathcal{X}_o(8, 17; 3)$
  - $\mathcal{X}_o(8, 17; 5)$
  - $\mathcal{X}_e(6, 18; 7, 6)$
  - $\mathcal{X}_e(4, 30; 13, 25)$
  - $\mathcal{X}_e(8, 16; 11, 0)$
  - $\mathcal{X}_e(8, 16; 11, 8)$

# Homework 3

**H3:**

- Show that there exists no half-arc-transitive Cayley graph of an Abelian group.