Identity Management on Homogeneous Spaces

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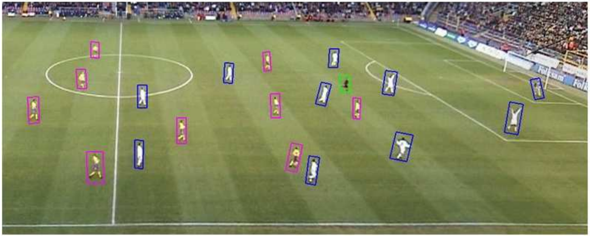
Ranking, Voting and Tracking



Problem in Identity Management

Problem in Identity Management

Markov Model for Identity Management

$$
\begin{array}{ccccccccc}
\sigma_0 & \xrightarrow{M_0} & \sigma_1 & \xrightarrow{M_1} & \sigma_2 & \xrightarrow{M_2} & \sigma_3 & \xrightarrow{M_3} & \cdots \\
\Big\downarrow{\scriptstyle L(z_0|\sigma_0)} & & \Big\downarrow{\scriptstyle L(z_1|\sigma_1)} & & \Big\downarrow{\scriptstyle L(z_2|\sigma_2)} & & \Big\downarrow{\scriptstyle L(z_3|\sigma_3)} & & \\
z_0 & & z_1 & & z_2 & & z_3 & &
\end{array}
$$

σ: true state; z: observations; M: markov matrix; $L(z|\sigma)$: likelihood function.

- Mixing Model: tracks swapped identities with some probability.
- Observation Model: identity on a particular track is observed.
- Problem: For each timestep, find posterior over σ_t conditioned on all past observations.
- Our Problem: Find posterior over **class characteristics** (red or blue) conditioned on all past observations.

Our Problem

- Define $\sigma^{(t)} \in S_n$ to be a mapping from identities $\{i_1, i_2, \cdots, i_{m+n}\}$ to tracks $T = \{t_1, t_2, \cdots, t_{m+n}\}$.
- After a random permutation among tracks $\tau^{(t)}$. The association of identities with tracks at time $t+1$ is $\sigma^{(t+1)} = \tau^{(t)}\sigma^{(t)}$.
- Assume n of the identities are red and the remaining m identities are blue.
- We care only about the **class characteristics** (red or blue) of identities.

Homogeneous Space

- **Homogeneous Space:** All $k-$subsets of $\{1, 2, \cdots, n\}$.
- Permutation groups act on homogeneous spaces.

Example

- Suppose $n = 3, k = 2$, homogeneous space X is all $2-$subset of $\{1, 2, 3\}$, i.e. $X = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$.
- Permutation group S_3 acts on X, e.g., if

$$\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

then $\tau(\{1, 2\}) = \{2, 3\}; \tau(\{2, 3\}) = \{1, 3\}; \tau(\{1, 3\}) = \{1, 2\}$.

Markov Process on Homogeneous Space

- A probability distribution Q on permutation groups induces a Markov process on the homogeneous space X with transition probability

$$P_x(y) = \sum_{\tau:\tau x = y} Q(\tau)$$

- **Naive Model:** Maintain beliefs on homogeneous space instead of full permutation group.

Running Example

Example (Markov Model on Homogeneous Space)

- Suppose $m = n = 3$ and we are sure that $\{t_1, t_2, t_3\}$ are red, then $f \in L(X)$

$$f(x) = \begin{cases} 1 & \text{if } x = \{t_1, t_2, t_3\} \\ 0 & \text{otherwise} \end{cases}$$

- If a mixing happened among tracks t_3 and t_4, then

$$Q(\tau) = \begin{cases} p & \tau = \text{id} \\ 1 - p & \tau = (t_3, t_4) \\ 0 & \text{otherwise} \end{cases}$$

- The Markov mixing matrix induced from Q would be

	$\{t_1, t_2, t_3\}$	$\{t_1, t_2, t_4\}$	$\{t_1, t_2, t_5\}$	$\cdots$	$\{t_3, t_5, t_6\}$	$\{t_4, t_5, t_6\}$
$\{t_1, t_2, t_3\}$	p	$1 - p$	0	$\cdots$	0	0
$\{t_1, t_2, t_4\}$	$1 - p$	p	0	$\cdots$	0	0
$\{t_1, t_2, t_5\}$	0	0	1	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\{t_3, t_5, t_6\}$	0	0	0	$\cdots$	p	$1 - p$
$\{t_4, t_5, t_6\}$	0	0	0	$\cdots$	$1 - p$	p

Mixing Model

- Suppose Q is a distribution on permutation group S_{m+n}, then the simplest mixing model is

$$Q(\tau) = \begin{cases} p & \tau = \mathrm{id} \\ 1-p & \tau = (t_i, t_j) \\ 0 & \text{otherwise} \end{cases}$$

- Q induces a Markov update of beliefs for $f \in L(X)$

$$f(y) \leftarrow \sum_x P_x(y) f(x)$$

where $P_x(y) = \sum_{\tau:\tau x = y} Q(\tau)$.

Observation Model

- The simplest model for observation consist of receiving information z that with some high probability, target on track t_i is red.
- Likelihood function have the form ($a \gg b$):

$$L(z|x) = \begin{cases} a & \text{if } t_i \in x \\ b & \text{if } t_i \notin x \end{cases}$$

- Posterior by Bayes rule

$$f(x|z) = \frac{L(z|x) \cdot f(x)}{\sum_x L(z|x) \cdot f(x)}$$

Decomposition of Homogeneous Space

- Function space of homogeneous space $M^{m,n}$ decomposes as

$$S^{m+n} \oplus S^{m+n-1,1} \oplus S^{m+n-2,2} \oplus \ldots \oplus S^{m,n}$$

- $S^{m+n-i,i}$ is invariant under actions by S_{m+n}.
- Hierarchical structures: Direct sum of the first j subspaces is a $\binom{m+n}{j}$ dimensional subspace, can be regarded as functions defined on all $j-$subsets (j^{th} order statistics).
-

$$\begin{aligned} M^{m,n} &= S^{m+n} \oplus S^{m+n-1,1} \oplus S^{m+n-2,2} \oplus \cdots \oplus S^{m,n} \\ &= M^{m+n-j,j} \oplus S^{m+n-j-1,j+1} \oplus \cdots \oplus S^{m,n} \end{aligned}$$

  - $M^{m,n}$: all $n-$subsets of $\{1, 2, \cdots, m+n\}$.
  - $M^{m+n-j,j}$: all $j-$subsets of $\{1, 2, \cdots, m+n\}$.

Radon Up Transformations

- For $1 \leq k \leq n$ define the *Radon up transform*

$$R^+ : M^{m+n-k,k} \to M^{m,n} \quad \text{by} \quad R^+ f(s) = \sum_{s \supset r} f(r)$$

where $r \in M^{m+n-k,k}$ is a $k-$subset and $s \in M^{m,n}$ is an $n-$subset.

Example

- Suppose $f^2 \in M^{4,2}$ is

$\{t_1, t_2\}$	$\{t_1, t_3\}$	$\{t_2, t_3\}$	$\{t_1, t_4\}$	$\cdots$	$\{t_4, t_6\}$	$\{t_5, t_6\}$
4	4	4	2	$\cdots$	0	0

- After Radon transformation, $f^3 = R^+_{2,3} f^2$ would be

$\{t_1, t_2, t_3\}$	$\{t_1, t_2, t_4\}$	$\{t_1, t_4, t_5\}$	$\cdots$	$\{t_4, t_5, t_6\}$
4+4+4	4+2+2	2+2+0	$\cdots$	0+0+0

Radon Down Transformations

- If $M^{m,n}$ and $M^{m+n-k,k}$ are given bases consisting of delta functions on $n-$subsets and $k-$subsets. For $1 \leq k \leq n$ define *Radon down transform* $R^- : M^{m,n} \to M^{m+n-k,k}$, the (r,s) element of R^- is

$$\frac{(-1)^{n-k}(n-k)}{(-1)^{|s-r|}|s-r|\binom{m+n-k}{|s-r|}}$$

  where $r \in M^{m+n-k,k}$ is a $k-$subset and $s \in M^{m,n}$ is an $n-$subset.
- Radon transform R^+ and R^- satisfy
  - $R^- R^+ = I$
  - $R^+ R^-$ is an orthogonal projection.

Bandlimited Mixing Model

- Bandlimiting: Maintain k^{th} order statistics $f^k \in M^{m+n-k,k}$, which can be interpreted as the likelihood of a particular $k-$subset being all red.
- Induce mixing model Q to $M^{m+n-k,k}$ and update f^k by

$$f^k(y) \leftarrow \sum_x P_x(y) f^k(x)$$

$$\begin{array}{ccccccccc}
f_0^n & \xrightarrow{\mathrm{M}_0^\mathrm{n}} & f_1^n & \xrightarrow{\mathrm{M}_1^\mathrm{n}} & f_2^n & \xrightarrow{\mathrm{M}_2^\mathrm{n}} & f_3^n & \xrightarrow{\mathrm{M}_3^\mathrm{n}} & \cdots \\
R_{n,k}^- \downarrow & & R_{n,k}^- \downarrow & & R_{n,k}^- \downarrow & & R_{n,k}^- \downarrow & & \\
f_0^k & \xrightarrow{\mathrm{M}_0^\mathrm{k}} & f_1^k & \xrightarrow{\mathrm{M}_1^\mathrm{k}} & f_2^k & \xrightarrow{\mathrm{M}_2^\mathrm{k}} & f_3^k & \xrightarrow{\mathrm{M}_3^\mathrm{k}} & \cdots
\end{array}$$

Theorem

Both R^+ and R^- commute with the Markov mixing matrices induced from probability Q on permutation group S_{m+n}.

Bandlimited Observation Model

- Observation consists of first order statistics (observing the identity on track t_i is red with high probability)
- Lift first order statistics to k^{th} order statistics by Radon up transform.
- Use Bayes update to get posterior.

Classification Criteria

- We project k^{th} order statistics to first order statistics using Radon down transform.
- Predict the tracks with highest n scores as red members.

Real Camera Data

- Real Network with 8 Cameras
- 11 People (5 red, 6 blue)
- Experiments with different number of mixing events and observation events

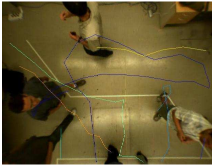


Figure: Sample Image.

Table: Experiments Data Summary

Experiment	#Mixings	#Observations	Explanations
1	8	76	few mix, lots of obs
2	169	184	
3	226	116	
4	261	64	lots of mix, few obs

Energy Distributions

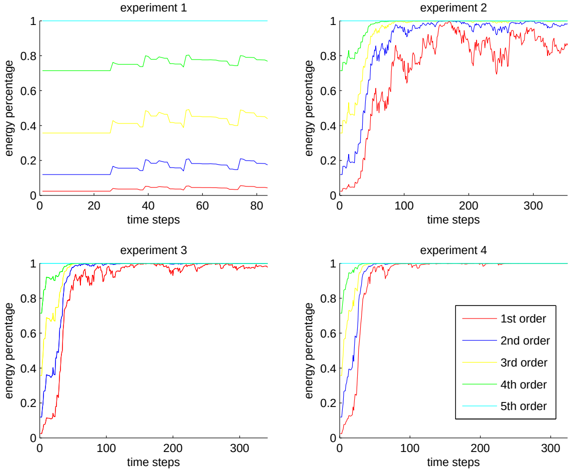


Figure: Energy distributions for four experiments

Classification Accuracy

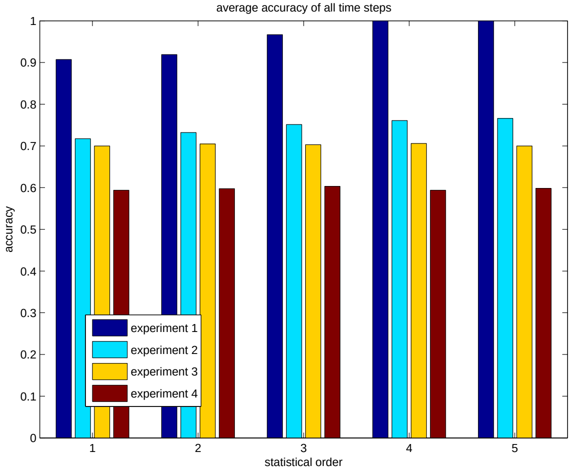


Figure: Classification accuracy of implementation with different statistical order.

Conclusions and Future Work

Conclusions

- Distributions on homogeneous spaces can be compactly summarized.
- Radon transforms useful for mapping distributions between different statistical orders.
- Evaluation of our model on a real camera network.

Future

- Use similar ideas to study other machine learning problems arising from ranking and voting.
- Smarter ways of projecting data on homogeneous spaces to low order statistics.