Tutorial on Statistical Machine Learning

with Applications to Multimodal Processing

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Outline of the Tutorial

Outline

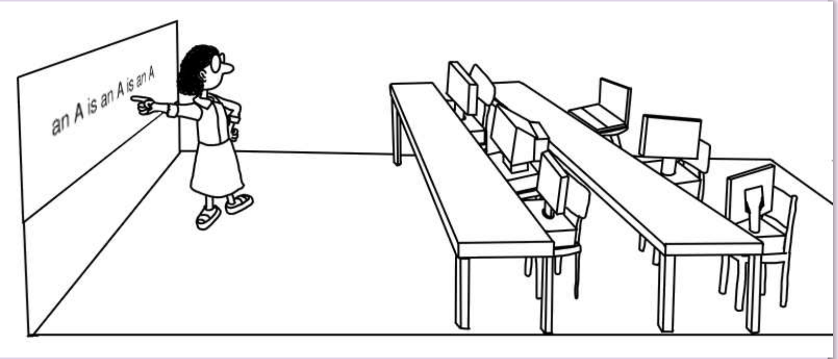
Updated slides

`http://www.idiap.ch/~bengio/icmi2005.pdf`

Part I

Introduction

What is Machine Learning? (Graphical View)

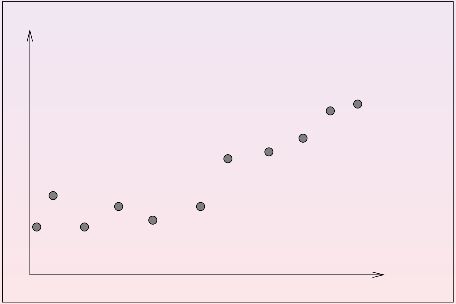


What is Machine Learning?

- Learning is an essential human property
- Learning means changing in order to be better (according to a given criterion) when a similar situation arrives
- Learning IS NOT learning by heart
- Any computer can learn by heart, the difficulty is to generalize a behavior to a novel situation

Why Learning is Difficult?

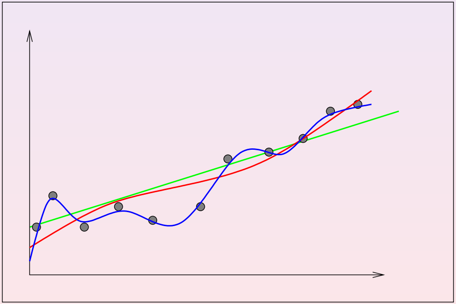
- Given a finite amount of training data, you have to derive a relation for an infinite domain
- In fact, there is an infinite number of such relations



- How should we draw the relation?

Why Learning is Difficult? (2)

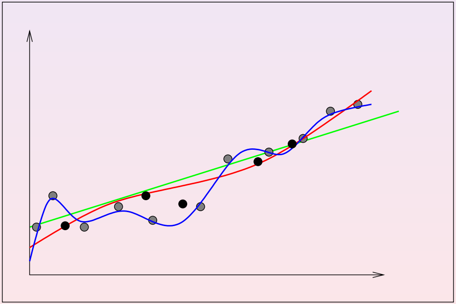
- Given a finite amount of training data, you have to derive a relation for an infinite domain
- In fact, there is an infinite number of such relations



- Which relation is the most appropriate?

Why Learning is Difficult? (3)

- Given a finite amount of training data, you have to derive a relation for an infinite domain
- In fact, there is an infinite number of such relations



- ... the hidden test points...

Occam's Razor's Principle

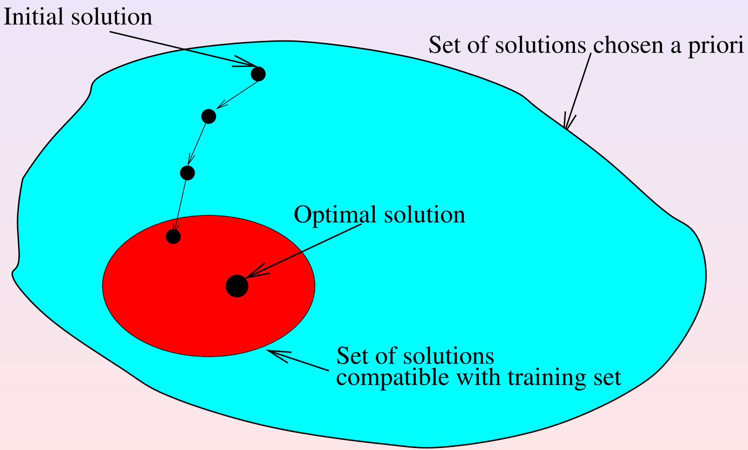
- William of Occam: Monk living in the 14th century
- Principle of Parcimony:

  One should not increase, beyond what is necessary, the number of entities required to explain anything

- When many solutions are available for a given problem, we should select the simplest one
- But what do we mean by simple?
- We will use prior knowledge of the problem to solve to define what is a simple solution

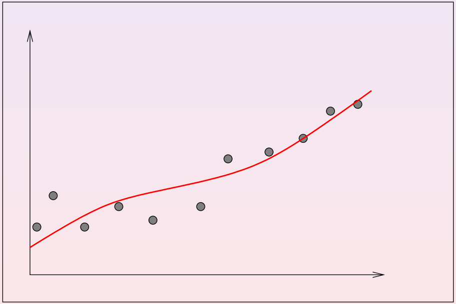
  Example of a prior: smoothness

Learning as a Search Problem



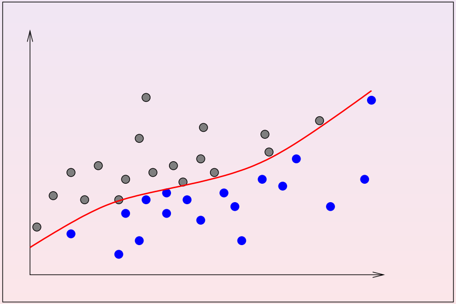
Types of Problems

- There are 3 kinds of problems:
  - regression



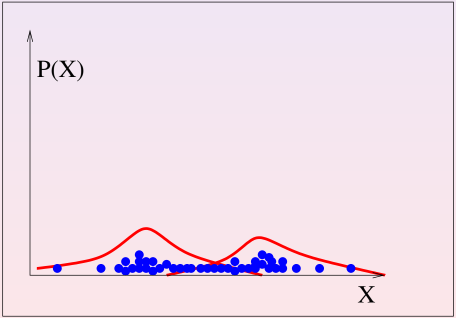
Types of Problems

- There are 3 kinds of problems:
  - regression, classification



Types of Problems

- There are 3 kinds of problems:
    - regression, classification, density estimation



Applications

- Vision Processing
  - Face detection/verification
  - Handwritten recognition
- Speech Processing
  - Phoneme/Word/Sentence/Person recognition
- Others
  - Finance: asset prediction, portfolio and risk management
  - Telecom: traffic prediction
  - Data mining: make use of huge datasets kept by large corporations
  - Games: Backgammon, go
  - Control: robots
- ... and plenty of others of course!

Part II

Statistical Learning Theory

The Data

Available training data

- Let $Z_1, Z_2, \cdots, Z_n$ be an n-tuple random sample of an unknown distribution of density $p(z)$.
- All Z_i are independently and identically distributed (iid).
- Let D_n be a particular instance $= \{z_1, z_2, \cdots, z_n\}$.

Various forms of the data

- Classification: $Z = (X, Y) \in \mathbb{R}^d \times \{-1, 1\}$
  objective: given a new x, estimate $P(Y|X = x)$
- Regression: $Z = (X, Y) \in \mathbb{R}^d \times \mathbb{R}$
  objective: given a new x, estimate $E[Y|X = x]$
- Density estimation: $Z \in \mathbb{R}^d$
  objective: given a new z, estimate $p(z)$

The Function Space

Learning: search for a good function in a function space $\mathcal{F}$

Examples of functions $f(\cdot;\theta) \in \mathcal{F}$:

- Regression:

$$\hat{y} = f(x; a, b, c) = a \cdot x^2 + b \cdot x + c$$

- Classification:

$$\hat{y} = f(x; a, b, c) = \mathrm{sign}(a \cdot x^2 + b \cdot x + c)$$

- Density estimation

$$\hat{p}(z) = f(z; \mu, \Sigma) = \frac{1}{(2\pi)^{\frac{|z|}{2}} \sqrt{|\Sigma|}} \exp\left(-\frac{1}{2}(z-\mu)^T \Sigma^{-1} (z-\mu)\right)$$

The Loss Function

Learning: search for a good function in a function space $\mathcal{F}$

Examples of loss functions $L : \mathcal{Z} \times \mathcal{F}$

- Regression:

$$L(z, f) = L((x, y), f) = (f(x) - y)^2$$

- Classification:

$$L(z, f) = L((x, y), f) = \begin{cases} 0 & \text{if } f(x) = y \\ 1 & \text{otherwise} \end{cases}$$

- Density estimation:

$$L(z, f) = -\log p(z)$$

The Risk and the Empirical Risk

Learning: search for a good function in a function space $\mathcal{F}$

- Minimize the Expected Risk on $\mathcal{F}$, defined for a given f as

$$R(f) = E_Z[L(z, f)] = \int_Z L(z, f)p(z)dz$$

- Induction Principle:
  - select $f^* = \arg\min_{f \in \mathcal{F}} R(f)$
  - problem: $p(z)$ is unknown!!!
- Empirical Risk:

$$\hat{R}(f, D_n) = \frac{1}{n}\sum_{i=1}^{n} L(z_i, f)$$

The Risk and the Empirical Risk

- The empirical risk is an unbiased estimate of the risk:

$$E[\hat{R}(f, D)] = R(f)$$

- The principle of empirical risk minimization:

$$f^*(D_n) = \arg\min_{f \in \mathcal{F}} \hat{R}(f, D_n)$$

- Training error:

$$\hat{R}(f^*(D_n), D_n) = \min_{f \in \mathcal{F}} \hat{R}(f, D_n)$$

Is the training error a biased estimate of the risk?

The Training Error

Is the training error a biased estimate of the risk? yes.

$$E[R(f^*(D_n)) - \hat{R}(f^*(D_n), D_n)] \geq 0$$

- The solution $f^*(D_n)$ found by minimizing the training error is better on D_n than on any other set $D_n^{'}$ drawn from $p(Z)$.
- Can we bound the difference between the training error and the generalization error?

$$|R(f^*(D_n)) - \hat{R}(f^*(D_n), D_n)| \leq ?$$

- Answer: under certain conditions on $\mathcal{F}$, yes.
- These conditions depend on the notion of capacity h of $\mathcal{F}$.

The Capacity

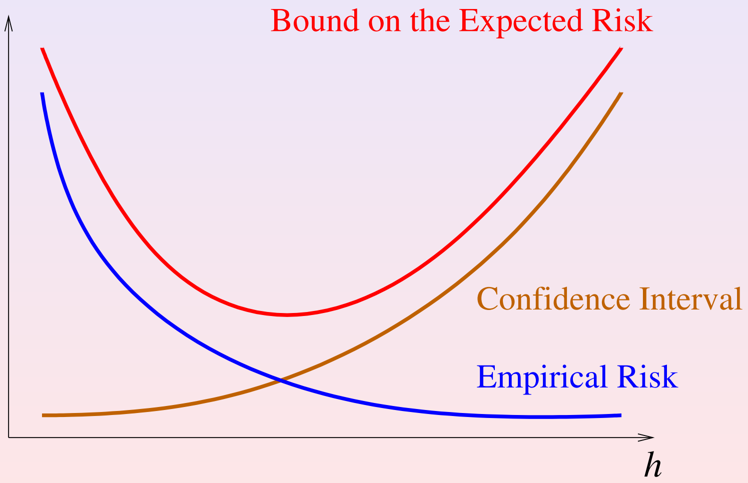
- The capacity $h(\mathcal{F})$ is a measure of its size, or complexity.
- Classification:

  The capacity $h(\mathcal{F})$ is the largest n such that there exist a set of examples D_n such that one can always find an $f \in \mathcal{F}$ which gives the correct answer for all examples in D_n, for any possible labeling.

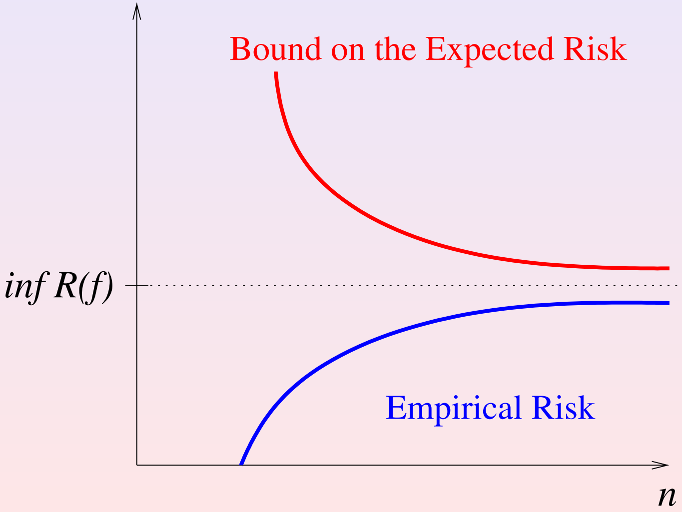
- Regression and density estimation: capacity exists also, but more complex to derive (for instance, we can always reduce a regression problem to a classification problem).
- Bound on the expected risk: let $\tau = \sup L - \inf L$. $\forall \eta$ we have

$$P\left(\sup_{f \in \mathcal{F}} |R(f) - \hat{R}(f, D_n)| \leq 2\tau \sqrt{\frac{h\left(\ln \frac{2n}{h} + 1\right) - \ln \frac{\eta}{9}}{n}}\right) \geq 1 - \eta$$

Theoretical Curves



Theoretical Curves



Methodology

- First: identify the goal! It could be
    1. to give the best model you can obtain given a training set?
    2. to give the expected performance of a model obtained by empirical risk minimization given a training set?
    3. to give the best model and its expected performance that you can obtain given a training set?
- If the goal is (1): use need to do model selection
- If the goal is (2), you need to estimate the risk
- If the goal is (3): use need to do both!
- There are various methods that can be used for either risk estimation or model selection:
    - simple validation
    - cross validation (k-fold, leave-one-out)

Model Selection - Validation

- Select a family of functions with hyper-parameter θ
- Divide your training set D_n into two parts
  - $D^{tr} = \{z_1, z_2, \cdots, z_{tr}\}$
  - $D^{va} = \{z_{tr+1}, z_{tr+2}, \cdots, z_{tr+va}\}$
  - $tr + va = n$
- For each value θ_m of the hyper-parameter θ
  - select $f^*_{\theta_m}(D^{tr}) = \arg\min_{f \in \mathcal{F}_{\theta_m}} \hat{R}(f, D^{tr})$
  - estimate $R(f^*_{\theta_m})$ with $\hat{R}(f^*_{\theta_m}, D^{va}) = \dfrac{1}{va} \sum_{z_i \in D^{va}} L(z_i, f^*_{\theta_m}(D^{tr}))$
- select $\theta^*_m = \arg\min_{\theta_m} R(f^*_{\theta_m})$
- return $f^*(D_n) = \arg\min_{f \in \mathcal{F}_{\theta^*_m}} \hat{R}(f, D_n)$

Model Selection - Cross-validation

- Select a family of functions with hyper-parameter θ
- Divide your training set D_n into K distinct and equal parts $D^1, \cdots, D^k, \cdots, D^K$
- For each value θ_m of the hyper-parameter θ
  - For each part D^k (and its counterpart $\bar{D}^k$)
    - select $f^*_{\theta_m}(\bar{D}^k) = \arg \min_{f \in \mathcal{F}_{\theta_m}} \hat{R}(f, \bar{D}^k)$
    - estimate $R(f^*_{\theta_m}(\bar{D}^k))$ with $\hat{R}(f^*_{\theta_m}(\bar{D}^k), D^k) = \frac{1}{|D^k|} \sum_{z_i \in D^k} L(z_i, f^*_{\theta_m}(\bar{D}^k))$
  - estimate $R(f^*_{\theta_m}(D_n))$ with $\frac{1}{K} \sum_k R(f^*_{\theta_m}(\bar{D}^k))$
- select $\theta^*_m = \arg \min_{\theta_m} R(f^*_{\theta_m}(D))$
- return $f^*(D_n) = \arg \min_{f \in \mathcal{F}_{\theta^*_m}} \hat{R}(f, D_n)$

Estimation of the Risk - Validation

- Divide your training set D_n into two parts
  - $D^{tr} = \{z_1, z_2, \cdots, z_{tr}\}$
  - $D^{te} = \{z_{tr+1}, z_{tr+2}, \cdots, z_{tr+te}\}$
  - $tr + te = n$
- select $f^*(D^{tr}) = \arg\min_{f \in \mathcal{F}} \hat{R}(f, D^{tr})$

  (this optimization process could include model selection)

- estimate $R(f^*(D^{tr}))$ with

  $$\hat{R}(f^*(D^{tr}), D^{te}) = \frac{1}{te} \sum_{z_i \in D^{te}} L(z_i, f^*(D^{tr}))$$

Estimation of the Risk - Cross-validation

- Divide your training set D_n into K distinct and equal parts $D^1, \cdots, D^k, \cdots, D^K$
- For each part D^k
  - let $\bar{D}^k$ be the set of examples that are in D_n but not in D^k
  - select $f^*(\bar{D}^k) = \arg\min_{f \in \mathcal{F}} \hat{R}(f, \bar{D}^k)$

    (this process could include model selection)
  - estimate $R(f^*(\bar{D}^k))$ with
    $$\hat{R}(f^*(\bar{D}^k), D^k) = \frac{1}{|D^k|} \sum_{z_i \in D^k} L(z_i, f^*(\bar{D}^k))$$
- estimate $R(f^*(D_n))$ with $\displaystyle\frac{1}{K} \sum_k R(f^*(\bar{D}^k))$
- When $k = n$: leave-one-out cross-validation

Estimation of the Risk and Model Selection

- When you want both the best model and its expected risk.
- You then need to merge the methods already presented.
  For instance:
  - train-validation-test: 3 separate data sets are necessary
  - cross-validation + test: cross-validate on train set, then test on separate set
  - double-cross-validation: for each subset, need to do a second cross-validation with the $K-1$ other subsets
- Other important methodological aspects:
  - compare your results with other methods!!!!
  - use statistical tests to verify significance
  - verify your model on more than one datasets

Train - Validation - Test

- Select a family of functions with hyper-parameter θ
- Divide your training set D_n into three parts D^{tr}, D^{va}, and D^{te}
- For each value θ_m of the hyper-parameter θ
  - select $f^*_{\theta_m}(D^{tr}) = \arg \min_{f \in \mathcal{F}_{\theta_m}} \hat{R}(f, D^{tr})$
  - let $\hat{R}(f^*_{\theta_m}(D^{tr}), D^{va}) = \frac{1}{va} \sum_{z_i \in D^{va}} L(z_i, f^*_{\theta_m}(D^{tr}))$
- select $\theta^*_m = \arg \min_{\theta_m} \hat{R}(f^*_{\theta_m}(D^{tr}), D^{va})$
- select $f^*(D^{tr} \cup D^{va}) = \arg \min_{f \in \mathcal{F}_{\theta^*_m}} \hat{R}(f, D^{tr} \cup D^{va})$
- estimate $R(f^*(D^{tr} \cup D^{va}))$ with $\frac{1}{te} \sum_{z_i \in D^{te}} L(z_i, f^*(D^{tr} \cup D^{va}))$

Cross-validation + Test

- Select a family of functions with hyper-parameter θ
- Divide you dataset D_n into two parts:

  a training set D^{tr} and a test set D^{te}

- For each value θ_m of the hyper-parameter θ

  estimate $R(f^*_{\theta_m}(D^{tr}))$ with D^{tr} using cross-validation

- select $\theta^*_m = \arg\min_{\theta_m} R(f^*_{\theta_m}(D^{tr}))$
- retrain $f^*(D^{tr}) = \arg\min_{f \in \mathcal{F}_{\theta^*_m}} \hat{R}(f, D^{tr})$
- estimate $R(f^*(D^{tr}))$ with $\displaystyle\frac{1}{te} \sum_{z_i \in D^{te}} L(z_i, f^*(D^{tr}))$

Examples of Known Models

- Multi-Layer Perceptrons (regression, classification)
- Radial Basis Functions (regression, classification)
- Support Vector Machines (classification, regression)
- Gaussian Mixture Models (density estimation, classification)
- Hidden Markov Models (density estimation, classification)
- Graphical Models (density estimation, classification)
- AdaBoost and Bagging (classification, regression, density estimation)
- Decision Trees (classification, regression)

Part III

EM and Gaussian Mixture Models

Reminder: Basics on Probabilities

A few basic equalities that are often used:

1. (conditional probabilities)

$$P(A, B) = P(A|B) \cdot P(B)$$

2. (Bayes rule)

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

3. If $(\bigcup B_i = \Omega)$ and $\forall i, j \neq i \ (B_i \bigcap B_j = \emptyset)$ then

$$P(A) = \sum_i P(A, B_i)$$

What is a Gaussian Mixture Model

- A Gaussian Mixture Model (GMM) is a distribution
- The likelihood given a Gaussian distribution is

$$\mathcal{N}(x;\mu,\Sigma) = \frac{1}{(2\pi)^{\frac{|x|}{2}}\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$

  where μ is the mean and Σ is the covariance matrix of the Gaussian. Σ is often diagonal.
- The likelihood given a GMM is

$$p(x) = \sum_{i=1}^{N} w_i \cdot \mathcal{N}(x;\mu,\Sigma)$$

  where N is the number of Gaussians and w_i is the weight of Gaussian i, with

$$\sum w_i = 1 \text{ and } \forall i : w_i \geq 0$$

Characteristics of a GMM

- While Multi-Layer Perceptrons are universal approximators of functions,
- GMMs are universal approximators of densities.

  (as long as there are enough Gaussians of course)
- Even diagonal GMMs are universal approximators.
- Full rank GMMs are not easy to handle: number of parameters is the square of the number of dimensions.
- GMMs can be trained by maximum likelihood using an efficient algorithm: Expectation-Maximization.

Basics of Expectation-Maximization

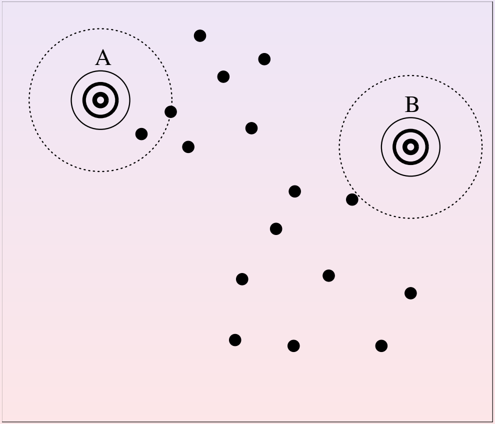
- Objective: maximize the likelihood $p(X;\theta)$ of the data X drawn from an unknown distribution, given the model parameterized by θ:

$$\theta^* = \arg\max_\theta p(X|\theta) = \arg\max_\theta \prod_{p=1}^{n} p(x_p|\theta)$$

- Basic ideas of EM:
  - Introduce a hidden variable such that *its knowledge would simplify the maximization of $p(X;\theta)$*
  - At each iteration of the algorithm:
    - E-Step: estimate the distribution of the hidden variable given the data and the current value of the parameters
    - M-Step: modify the parameters in order to maximize the joint distribution of the data and the hidden variable

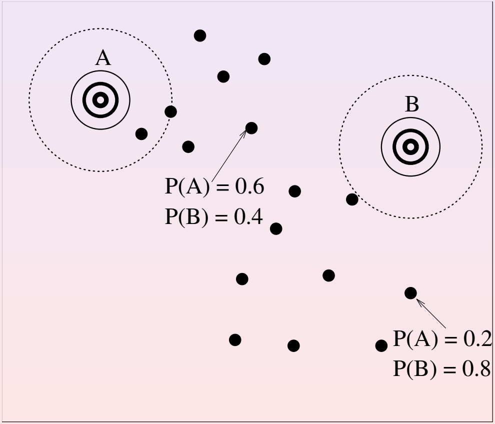
EM for GMM (Graphical View, 1)

Hidden variable: for each point, which Gaussian generated it?



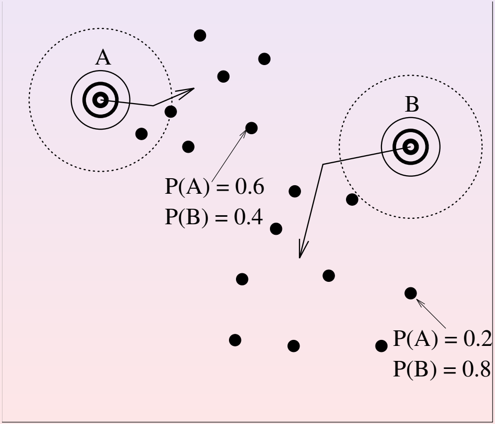
EM for GMM (Graphical View, 2)

E-Step: for each point, estimate the probability the each Gaussian generated it



EM for GMM (Graphical View, 3)

M-Step: modify the parameters according to the hidden variable to maximize the likelihood of the data (and the hidden variable)



EM: More Formally

- Let us call the hidden variable Q.
- Let us consider the following auxiliary function:

$$A(\theta, \theta^s) = E_Q[\log p(X, Q|\theta)|X, \theta^s]$$

- It can be shown that maximizing A

$$\theta^{s+1} = \arg\max_{\theta} A(\theta, \theta^s)$$

always increases the likelihood of the data $p(X|\theta^{s+1})$, and a maximum of A corresponds to a maximum of the likelihood.

EM for GMM: Hidden Variable

- For GMM, the hidden variable Q will describe which Gaussian generated each example.
- If Q was observed, then it would be simple to maximize the likelihood of the data: simply estimate the parameters Gaussian by Gaussian
- Moreover, we will see that we can easily estimate Q
- Let us first write the mixture of Gaussian model for one x_i:

$$p(x_i|\theta) = \sum_{j=1}^{N} P(j|\theta)p(x_i|j,\theta)$$

- Let us now introduce the following indicator variable:

$$q_{i,j} = \begin{cases} 1 & \text{if Gaussian } j \text{ emitted } x_i \\ 0 & \text{otherwise} \end{cases}$$

EM for GMM: Auxiliary Function

- We can now write the joint likelihood of all the X and Q:

$$p(X, Q|\theta) = \prod_{i=1}^{n} \prod_{j=1}^{N} P(j|\theta)^{q_{i,j}} p(x_i|j, \theta)^{q_{i,j}}$$

- which in log gives

$$\log p(X, Q|\theta) = \sum_{i=1}^{n} \sum_{j=1}^{N} q_{i,j} \log P(j|\theta) + q_{i,j} \log p(x_i|j, \theta)$$

EM for GMM: Auxiliary Function

Let us now write the corresponding auxiliary function:

$$
\begin{aligned}
A(\theta, \theta^s) &= E_Q[\log p(X, Q|\theta)|X, \theta^s] \\
&= E_Q\left[\sum_{i=1}^{n}\sum_{j=1}^{N} q_{i,j} \log P(j|\theta) + q_{i,j} \log p(x_i|j, \theta)|X, \theta^s\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{N} E_Q[q_{i,j}|X, \theta^s] \log P(j|\theta) + E_Q[q_{i,j}|X, \theta^s] \log p(x_i|j, \theta)
\end{aligned}
$$

EM for GMM: Update Rules

Means $$\hat{\mu}_j = \frac{\displaystyle\sum_{i=1}^{n} x_i \cdot P(j|x_i, \theta^s)}{\displaystyle\sum_{i=1}^{n} P(j|x_i, \theta^s)}$$

Variances $$(\hat{\sigma}_j)^2 = \frac{\displaystyle\sum_{i=1}^{n} (x_i - \mu_j)^2 \cdot P(j|x_i, \theta^s)}{\displaystyle\sum_{i=1}^{n} P(j|x_i, \theta^s)}$$

Weights: $$\hat{w}_j = \frac{1}{n} \sum_{i=1}^{n} P(j|x_i, \theta^s)$$

Initialization

- EM is an iterative procedure that is very sensitive to initial conditions!
- Start from trash $\rightarrow$ end up with trash.
- Hence, we need a good and fast initialization procedure.
- Often used: K-Means.
- Other options: hierarchical K-Means, Gaussian splitting.

Part IV

Hidden Markov Models

Markov Models

- Stochastic process of a temporal sequence: the probability distribution of the variable q at time t depends on the variable q at times $t-1$ to 1.

$$P(q_1, q_2, \ldots, q_T) = P(q_1^T) = P(q_1) \prod_{t=2}^{T} P(q_t | q_1^{t-1})$$

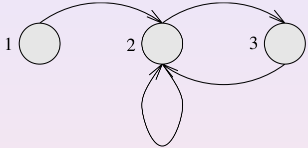
- First Order Markov Process:

$$P(q_t | q_1^{t-1}) = P(q_t | q_{t-1})$$

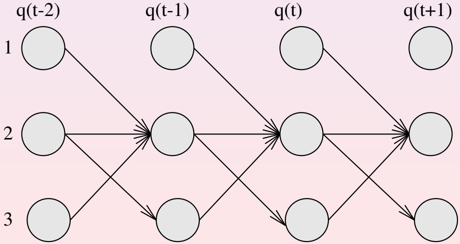
- Markov Model: model of a Markovian process with discrete states.
- Hidden Markov Model: Markov Model whose state is not observed, but of which one can observe a manifestation (a variable x_t which depends only on q_t).

Markov Models (Graphical View)

- A Markov model:

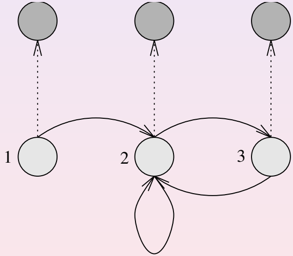


- A Markov model unfolded in time:

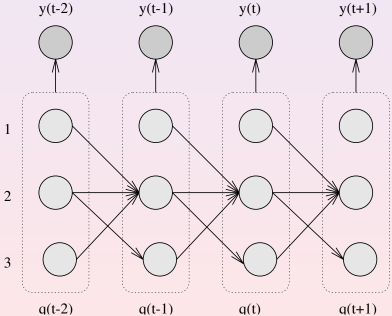


Hidden Markov Models

- A hidden Markov model:



- A hidden Markov model unfolded in time:



Elements of an HMM

- A finite number of states N.
- Transition probabilities between states, which depend only on previous state: $P(q_t{=}i|q_{t-1}{=}j, \theta)$.
- Emission probabilities, which depend only on the current state: $p(x_t|q_t{=}i, \theta)$ (where x_t is observed).
- Initial state probabilities: $P(q_0 = i|\theta)$.
- Each of these 3 sets of probabilities have parameters θ to estimate.

The 3 Problems of HMMs

- The HMM model gives rise to 3 different problems:
  - Given an HMM parameterized by θ, can we compute the likelihood of a sequence $X = x_1^T = \{x_1, x_2, \ldots, x_T\}$:

$$p(x_1^T|\theta)$$

  - Given an HMM parameterized by θ and a set of sequences D_n, can we select the parameters θ^* such that:

$$\theta^* = \arg\max_{\theta} \prod_{p=1}^{n} p(X(p)|\theta)$$

  - Given an HMM parameterized by θ, can we compute the optimal path Q through the state space given a sequence X:

$$Q^* = \arg\max_{Q} p(X, Q|\theta)$$

HMMs as Generative Processes

HMMs can be use to generate sequences:

- Let us define a set of starting states with initial probabilities $P(q_0 = i)$.
- Let us also define a set of final states.
- Then for each sequence to generate:
  1. Select an initial state j according to $P(q_0)$.
  2. Select the next state i according to $P(q_t = i | q_{t-1} = j)$.
  3. Emit an output according to the emission distribution $P(x_t | q_t = i)$.
  4. If i is a final state, then stop, otherwise loop to step 2.

Markovian Assumptions

- Emissions: the probability to emit x_t at time t in state $q_t = i$ does not depend on anything else:

$$p(x_t | q_t = i, q_1^{t-1}, x_1^{t-1}) = p(x_t | q_t = i)$$

- Transitions: the probability to go from state j to state i at time t does not depend on anything else:

$$P(q_t = i | q_{t-1} = j, q_1^{t-2}, x_1^{t-1}) = P(q_t = i | q_{t-1} = j)$$

- Moreover, this probability does not depend on time t:

$$P(q_t = i | q_{t-1} = j) \text{ is the same for all } t$$

we say that such Markov models are homogeneous.

Derivation of the Forward Variable α

the probability of having generated the sequence x_1^t and being in state i at time t:

$$
\begin{aligned}
\alpha(i,t) &\stackrel{\text{def}}{=} p(x_1^t, q_t = i) \\
&= p(x_t | x_1^{t-1}, q_t = i) p(x_1^{t-1}, q_t = i) \\
&= p(x_t | q_t = i) \sum_j p(x_1^{t-1}, q_t = i, q_{t-1} = j) \\
&= p(x_t | q_t = i) \sum_j P(q_t = i | x_1^{t-1}, q_{t-1} = j) p(x_1^{t-1}, q_{t-1} = j) \\
&= p(x_t | q_t = i) \sum_j P(q_t = i | q_{t-1} = j) p(x_1^{t-1}, q_{t-1} = j) \\
&= p(x_t | q_t = i) \sum_j P(q_t = i | q_{t-1} = j) \alpha(j, t-1)
\end{aligned}
$$

From α to the Likelihood

- Reminder: $\alpha(i,t) \stackrel{\text{def}}{=} p(x_1^t, q_t = i)$
- Initial condition:

$$\alpha(i,0) = P(q_0 = i) \rightarrow \text{ prior probabilities of each state } i$$

- Then let us compute $\alpha(i,t)$ for each state i and each time t of a given sequence x_1^T
- Afterward, we can compute the likelihood as follows:

$$\begin{aligned} p(x_1^T) &= \sum_i p(x_1^T, q_T = i) \\ &= \sum_i \alpha(i, T) \end{aligned}$$

- Hence, to compute the likelihood $p(x_1^T)$, we need $\mathcal{O}(N^2 \cdot T)$ operations, where N is the number of states

EM Training for HMM

- For HMM, the hidden variable Q will describe in which state the HMM was for each observation x_t of a sequence X.
- The joint likelihood of all sequences $X(l)$ and the hidden variable Q is then:

$$p(X, Q|\theta) = \prod_{l=1}^{n} p(X(l), Q|\theta)$$

- Let us introduce the following indicator variable:

$$q_{i,t} = \begin{cases} 1 & \text{if } q_t = i \\ 0 & \text{otherwise} \end{cases}$$

Joint Likelihood

$$
\begin{aligned}
p(X, Q|\theta) &= \prod_{l=1}^{n} p(X(l), Q|\theta) \\
&= \prod_{l=1}^{n} \left(\prod_{i=1}^{N} P(q_0 = i)^{q_{i,0}} \right) \cdot \\
&\quad \prod_{t=1}^{T_l} \prod_{i=1}^{N} p(x_t(l)|q_t = i)^{q_{i,t}} \prod_{j=1}^{N} P(q_t = i|q_{t-1} = j)^{q_{i,t} \cdot q_{j,t-1}}
\end{aligned}
$$

Joint Log Likelihood

$$
\begin{aligned}
\log p(X, Q|\theta) \quad = \quad & \sum_{l=1}^{n} \sum_{i=1}^{N} q_{i,0} \log P(q_0 = i) + \\
& \sum_{l=1}^{n} \sum_{t=1}^{T_l} \sum_{i=1}^{N} q_{i,t} \log p(x_t(l)|q_t = i) + \\
& \sum_{l=1}^{n} \sum_{t=1}^{T_l} \sum_{i=1}^{N} \sum_{j=1}^{N} q_{i,t} \cdot q_{j,t-1} \log P(q_t = i|q_{t-1} = j)
\end{aligned}
$$

Auxiliary Function

Let us now write the corresponding auxiliary function:

$$
\begin{aligned}
A(\theta, \theta^s) &= E_Q[\log p(X, Q|\theta)|X, \theta^s] \\
&= \sum_{l=1}^{n} \sum_{i=1}^{N} E_Q[q_{i,0}|X, \theta^s] \log P(q_0 = i) + \\
&\quad \sum_{l=1}^{n} \sum_{t=1}^{T_l} \sum_{i=1}^{N} E_Q[q_{i,t}|X, \theta^s] \log p(x_t(l)|q_t = i) + \\
&\quad \sum_{l=1}^{n} \sum_{t=1}^{T_l} \sum_{i=1}^{N} \sum_{j=1}^{N} E_Q[q_{i,t} \cdot q_{j,t-1}|X, \theta^s] \log P(q_t = i|q_{t-1} = j)
\end{aligned}
$$

From now on, let us forget about index l for simplification.

Derivation of the Backward Variable β

the probability to generate the rest of the sequence x_{t+1}^T given that we are in state i at time t

$$\begin{aligned}
\beta(i,t) &\stackrel{\text{def}}{=} p(x_{t+1}^T | q_t{=}i) \\
&= \sum_j p(x_{t+1}^T, q_{t+1}{=}j | q_t{=}i) \\
&= \sum_j p(x_{t+1} | x_{t+2}^T, q_{t+1}{=}j, q_t{=}i) p(x_{t+2}^T, q_{t+1}{=}j | q_t{=}i) \\
&= \sum_j p(x_{t+1} | q_{t+1}{=}j) p(x_{t+2}^T | q_{t+1}{=}j, q_t{=}i) P(q_{t+1}{=}j | q_t{=}i) \\
&= \sum_j p(x_{t+1} | q_{t+1}{=}j) p(x_{t+2}^T | q_{t+1}{=}j) P(q_{t+1}{=}j | q_t{=}i) \\
&= \sum_j p(x_{t+1} | q_{t+1}{=}j) \beta(j, t+1) P(q_{t+1}{=}j | q_t{=}i)
\end{aligned}$$

Final Details About β

- Reminder: $\beta(i,t) = p(x_{t+1}^T | q_t{=}i)$
- Final condition:

$$\beta(i,T) = \begin{cases} 1 & \text{if } i \text{ is a final state} \\ 0 & \text{otherwise} \end{cases}$$

- Hence, to compute all the β variables, we need $\mathcal{O}(N^2 \cdot T)$ operations, where N is the number of states

E-Step for HMMs

- Posterior on emission distributions:

$$
\begin{aligned}
E_Q[q_{i,t}|X,\theta^s] &= P(q_t = i|x_1^T,\theta^s) = P(q_t = i|x_1^T) \\
&= \frac{p(x_1^T, q_t = i)}{p(x_1^T)} \\
&= \frac{p(x_{t+1}^T|q_t = i, x_1^t)p(x_1^t, q_t = i)}{p(x_1^T)} \\
&= \frac{p(x_{t+1}^T|q_t = i)p(x_1^t, q_t = i)}{p(x_1^T)} \\
&= \frac{\beta(i,t)\cdot\alpha(i,t)}{\sum_j \alpha(j,T)}
\end{aligned}
$$

E-Step for HMMs

- Posterior on transition distributions:

$$E_Q[q_{i,t} \cdot q_{j,t-1}|X, \theta^s] = P(q_t = i, q_{t-1} = j|x_1^T, \theta^s)$$

$$= \frac{p(x_1^T, q_t = i, q_{t-1} = j)}{p(x_1^T)}$$

$$= \frac{p(x_{t+1}^T|q_t{=}i)P(q_t{=}i|q_{t-1}{=}j)p(x_t|q_t{=}i)p(x_1^{t-1}, q_{t-1}{=}j)}{p(x_1^T)}$$

$$= \frac{\beta(i,t)P(q_t = i|q_{t-1} = j)p(x_t|q_t = i)\alpha(j, t-1)}{\sum_j \alpha(j, T)}$$

E-Step for HMMs

- Posterior on initial state distribution:

$$
\begin{aligned}
E_Q[q_{i,0}|X,\theta^p] &= P(q_0 = i|x_1^T,\theta^s) = P(q_0 = i|x_1^T) \\
&= \frac{p(x_1^T, q_0 = i)}{p(x_1^T)} \\
&= \frac{p(x_1^T|q_0 = i)P(q_0 = i)}{p(x_1^T)} \\
&= \frac{\beta(i,0)\cdot P(q_0 = i)}{\sum_j \alpha(j,T)}
\end{aligned}
$$

M-Step for HMMs

- Find the parameters θ that maximizes A, hence search for

$$\frac{\partial A}{\partial \theta} = 0$$

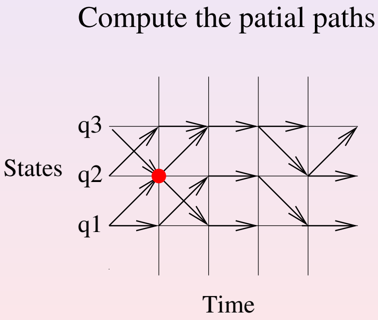
- When transition distributions are represented as tables, using a Lagrange multiplier, we obtain:

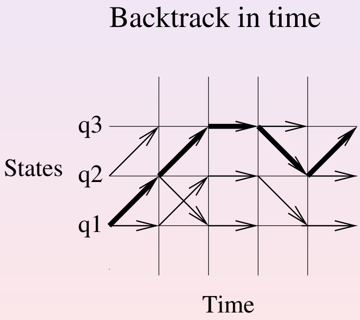
$$P(q_t = i | q_{t-1} = j) = \frac{\displaystyle\sum_{t=1}^{T} P(q_t = i, q_{t-1} = j | x_1^T, \theta^s)}{\displaystyle\sum_{t=1}^{T} P(q_t = i | x_1^T, \theta^s)}$$

- When emission distributions are implemented as GMMs, use already given equations, weighted by the posterior on emissions $P(q_t = i | x_1^T, \theta^s)$.

The Most Likely Path (Graphical View)

- The Viterbi algorithm finds the best state sequence.





The Viterbi Algorithm for HMMs

The Viterbi algorithm finds the best state sequence.

$$
\begin{aligned}
V(i,t) &\stackrel{\text{def}}{=} \max_{q_1^{t-1}} p(x_1^t, q_1^{t-1}, q_t{=}i) \\
&= \max_{q_1^{t-1}} p(x_t | x_1^{t-1}, q_1^{t-1}, q_t{=}i) p(x_1^{t-1}, q_1^{t-1}, q_t{=}i) \\
&= p(x_t | q_t{=}i) \max_{q_1^{t-2}} \max_j p(x_1^{t-1}, q_1^{t-2}, q_t{=}i, q_{t-1}{=}j) \\
&= p(x_t | q_t{=}i) \max_{q_1^{t-2}} \max_j p(q_t{=}i | q_{t-1}{=}j) p(x_1^{t-1}, q_1^{t-2}, q_{t-1}{=}j) \\
&= p(x_t | q_t{=}i) \max_j p(q_t{=}i | q_{t-1}{=}j) \max_{q_1^{t-2}} p(x_1^{t-1}, q_1^{t-2}, q_{t-1}{=}j) \\
&= p(x_t | q_t{=}i) \max_j p(q_t{=}i | q_{t-1}{=}j) V(j, t-1)
\end{aligned}
$$

From Viterbi to the State Sequence

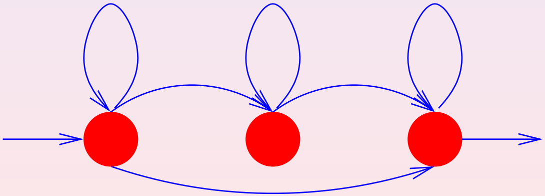
- Reminder: $V(i,t) = \max_{q_1^{t-1}} p(x_1^t, q_1^{t-1}, q_t{=}i)$
- Let us compute $V(i,t)$ for each state i and each time t of a given sequence x_1^T
- Moreover, let us also keep for each $V(i,t)$ the associated argmax previous state j
- Then, starting from the state $i = \arg\max_j V(j,T)$ backtrack to decode the most probable state sequence.
- Hence, to compute all the $V(i,t)$ variables, we need $\mathcal{O}(N^2 \cdot T)$ operations, where N is the number of states

HMMs for Speech Recognition

- Application: continuous speech recognition:

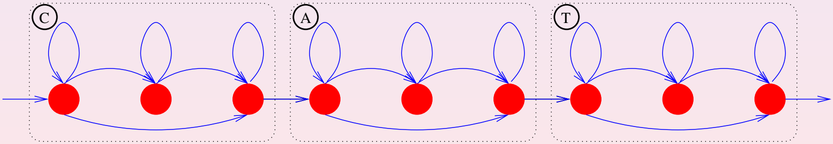
  Find a sequence of phonemes (or words) given an acoustic sequence

- Idea: use a phoneme model



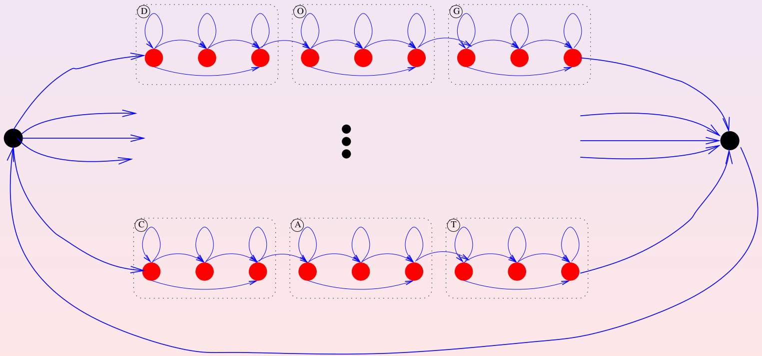
Embeded Training of HMMs

- For each acoustic sequence in the training set, create a new HMM as the concatenation of the HMMs representing the underlying sequence of phonemes.
- Maximize the likelihood of the training sentences.



HMMs: Decoding a Sentence

- Decide what is the accepted vocabulary.
- Optionally add a language model: $P(\text{word sequence})$
- Efficient algorithm to find the optimal path in the decoding HMM:



Measuring Error

- How do we measure the quality of a speech recognizer?
- Problem: the target solution is a sentence, the obtained solution is also a sentence, but they might have different size!
- Proposed solution: the Edit Distance:
    - assume you have access to the operators insert, delete, and substitute,
    - what is the smallest number of such operators we need to go from the obtained to the desired sentence?
    - An efficient algorithm exists to compute this.
- At the end, we measure the error as follows:

$$\text{WER} = \frac{\#\text{ins} + \#\text{del} + \#\text{subst}}{\#\text{words}}$$

- Note that the word error rate (WER) can be greater than 1...

Part V

Advanced Models for Multimodal Processing

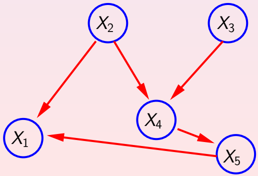
Graphical Models

- A tool to efficiently model complex joint distributions:

$$p(X_1, X_2, \cdots, X_n) = \prod_{i=1}^{n} p(X_i | \text{parents}(X_i))$$

- Introduces and makes use of known conditional independences.

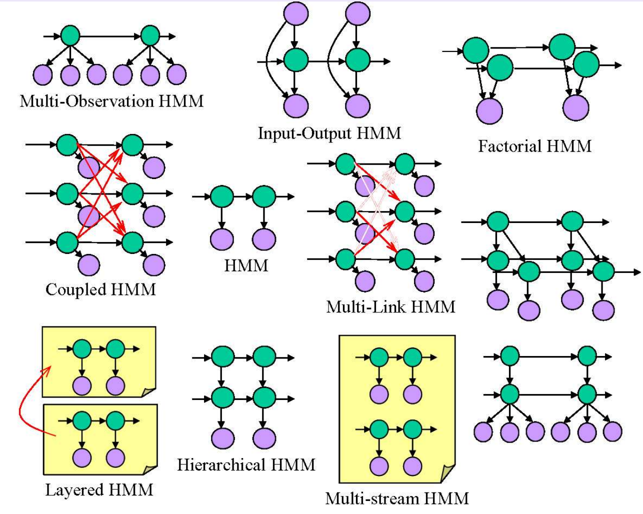
$$p(X_1, X_2, \cdots, X_n) = \prod \begin{cases} p(X_1|X_2, X_5) \\ p(X_2) \\ p(X_3) \\ p(X_4|X_2, X_3) \\ p(X_5|X_4) \end{cases}$$



Graphical Models

- Can handle an arbitrary number of random variables
- Junction Tree Algorithm (JTA): used to estimate joint probability (inference)
- Expectation-Maximization (EM): used to train
- Depending on the graph, EM and JTA are tractable or not
- Dynamic Bayes Nets: temporal extension of graphical models

A Zoo of Graphical Models for Multi-Channel Processing



Notation for Multimodal Data

Notation for One Stream

- Let us have a training set of L observations sequences.
- The observation sequence l: $\mathbf{O}^l = \left(\mathbf{o}_1^l, \mathbf{o}_2^l, \ldots, \mathbf{o}_{T_l}^l\right)$ with $\mathbf{o}_t^l$ the vector of multimodal features at time t for sequence l, of length T_l.

Notation for Several Streams

- Let us consider N streams for each observation sequence.
- The stream n of observation sequence l: $\mathbf{O}^{l:n} = \left(\mathbf{o}_1^{l:n}, \mathbf{o}_2^{l:n}, \ldots, \mathbf{o}_{T_l}^{l:n}\right)$ with $\mathbf{o}_t^{l:n}$ the vector of features of stream n at time t for sequence l, of length T_l.

Goals for Multimodal Data

Inference

$$\text{Likelihood: } \prod_{l=1}^{L} p\left(\left\{\mathbf{O}^{l:n}\right\}_{n=1}^{N};\theta\right)$$

Training

$$\theta^* = \arg\max_{\theta} \prod_{l=1}^{L} p\left(\left\{\mathbf{O}^{l:n}\right\}_{n=1}^{N};\theta\right)$$

Early-Integration HMMs

- Sample all streams at the same frame rate
- Emission distributions

$$p\left(\left\{\mathbf{o}_t^{l:n}\right\}_{n=1}^{N} | q_t = i\right)$$

- Transition distributions

$$p(q_t = i | q_{t-1} = j)$$

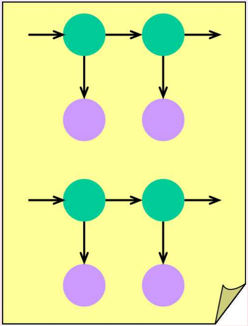
Multi-Stream HMMs

Train a separate model for each stream n

$$\theta_n^* = \arg\max_\theta \prod_{l=1}^{L} p\left(\mathbf{O}^{l:n}; \theta_n\right)$$

Inference, for each state $q_t = i$

$$p\left(\left\{\mathbf{o}_t^{l:n}\right\}_{n=1}^{N} | q_t = i\right) = \prod_{n=1}^{N} p\left(\mathbf{o}_t^{l:n} | q_t = i; \theta_n\right)$$



Multi-Stream HMMs

Training and Inference Complexity

- Per iteration, per observation sequence l:

$$\mathcal{O}(N \cdot S^2 \cdot T_l)$$

with S the number of states of each HMM stream, N the number of streams, T_l the length of the observation sequence

Additional Notes

- All stream HMMs need to be of the same topology.
- One can add stream weights ω_n as follows:

$$p\left(\left\{\mathbf{o}_t^{l:n}\right\}_{n=1}^{N} | q_t = i\right) = \prod_{n=1}^{N} p\left(\mathbf{o}_t^{l:n} | q_t = i; \theta_n\right)^{\omega_n}$$

Multi-Stream HMMs

Pros

- Efficient training (linear in the number of streams)
- Robust to stream-dependent noise
- Easy to implement

Cons

- Does not maximize the joint probability of the data
- Each stream needs to use the same HMM topology
- Assumes complete stream independence during training, and stream independence given the state during decoding.

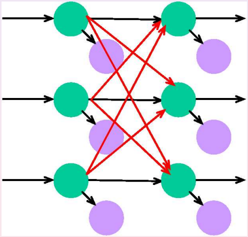
Coupled HMMs

- Let $q_t^n = i$ be the state at time t for stream n
- Stream emission distributions:

$$p\left(\mathbf{o}_t^{l:n} | q_t^n = i; \theta_n\right)$$

- Coupled transition distributions:

$$p\left(q_t^n | \left\{q_{t-1}^m\right\}_{m=1}^N; \theta_n\right)$$



Coupled HMMs

Complexity

- Per iteration, per observation sequence l:

$$\mathcal{O}(S^{2 \cdot N} \cdot T_l)$$

  with S the number of states of each HMM stream, N the number of streams, T_l the length of the observation sequence
- N-Heads approximation:

$$\mathcal{O}(N^2 \cdot S^2 \cdot T_l)$$

Coupled HMMs

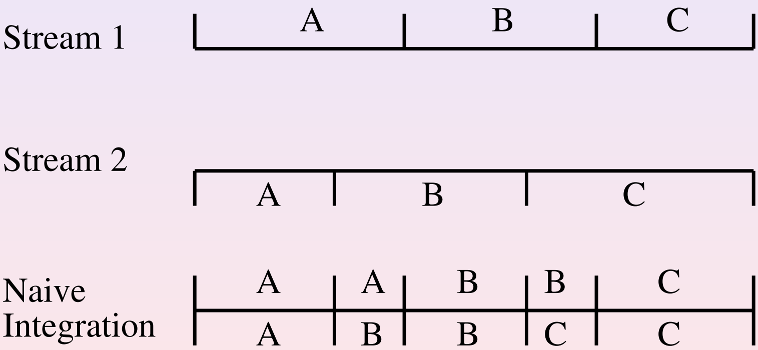
Pros

- Considers correlations between streams during training and decoding.

Cons

- The algorithm is intractable (unless using the approximation).
- Assumes synchrony between the streams.

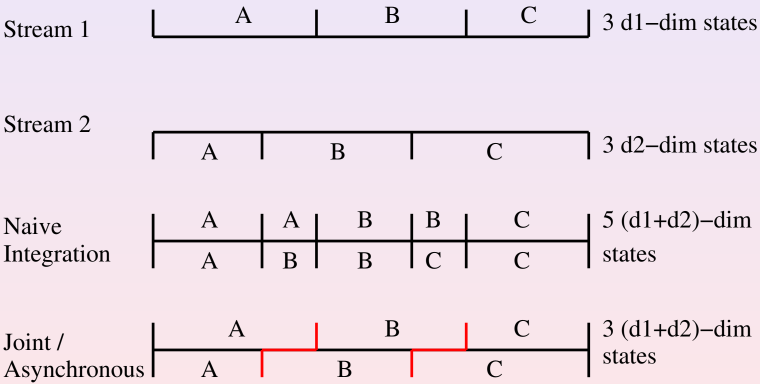
Challenges in Multi Channel Integration: Asynchrony



Some Evidence of Stream Asynchrony

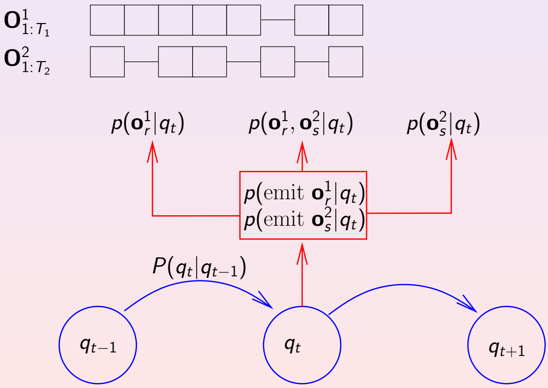
- Audio-visual speech recognition with lip movements: lips do not move at the same time as we hear the corresponding sound.
- Speaking and pointing: pointing to a map and saying "I want to go there".
- Gesticulating, looking at, and talking to someone during a conversation.
- In a news video, the delay between the moment when the newscaster says "Bush" and the moment when Bush's picture appears.
- ...

Asynchrony Revisited



Asynchronous HMMs

- Enables re-synchronization of streams.
- One HMM: maximizes the likelihood of all streams jointly.



Training and Complexity

Training

EM algorithm maximizes the joint probability $p(\mathbf{x}_1^{T_1}, \mathbf{y}_1^{T_2}, \cdots)$.

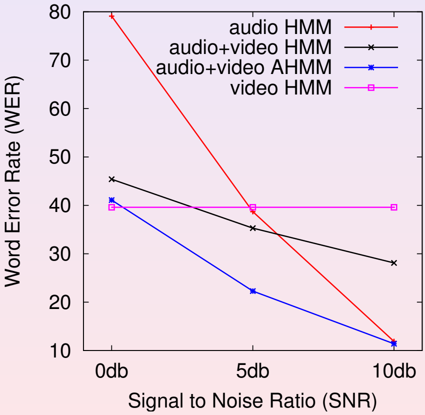
Complexity grows with number of streams, can be controlled

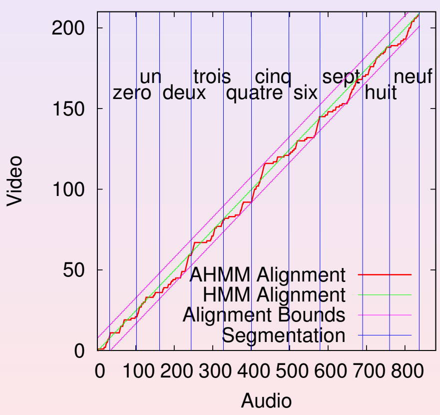
- Exact complexity: $\mathcal{O}(S^2 \cdot \prod_{i=1}^{N} T_i)$
- Introduction of temporal constraints: $\mathcal{O}(S^2 \cdot d \cdot T)$ with d the maximum delay allowed between streams.

Applications

- Easily adaptable to complex tasks such as speech recognition.
- Significant performance improvements in
  - audio-visual speech and speaker recognition
  - meeting analysis.

Alignments with Asynchronous HMMs

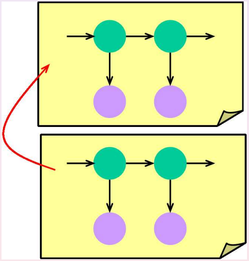




Layered HMMs

Philosophy

- Divide and Conquer for complex tasks
- Idea: transform the data from a raw representation into a higher level abstraction
- Then, transform again the result into yet another and higher level of abstraction, and continue as needed
- For temporal sequences: go from a fine time granularity to a coarser one.



General Algorithm

Layered Algorithm

For $i = 0$ to M layers, do:

1. Let $\mathbf{O}^i_{1:T}$ be a sequence of observations of HMM model m_i.
2. Train m_i using EM in order to maximise $p(\mathbf{O}^i_{1:T}|m_i)$.
3. Compute, for each state s^i_j for m_i the data posterior: $p(s^i_j|\mathbf{O}^i_{1:T}, m_i)$
4. Define $\mathbf{O}^{i+1}_{1:T}$ as the sequence of vectors of $p(s^i_j|\mathbf{O}^i_{1:T}, m_i)$

Complexity and Additional Notes

Complexity

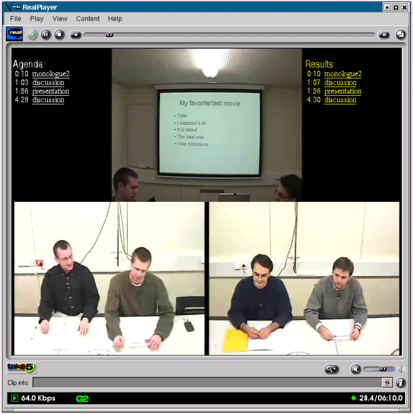
Training complexity: $\mathcal{O}(N \cdot S^2 \cdot M)$ for M layers.

A Speech Example

- A phoneme layer (with phoneme constraints)
- A word layer (with word constraints)
- A sentence layer (with language constraints)

Complexity of the Meeting Scenario

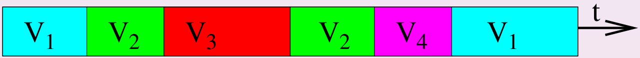
- Modeling multimodal group-level human interactions in meetings.
- Multimodal nature: data collected from multiple sensors (cameras, microphones, projectors, white-board, etc).
- Group nature: involves multiple interacting persons at the same time.



Meeting Analysis

- Structure a meeting as a sequence of group actions taken from an exhaustive set V of N possible actions:

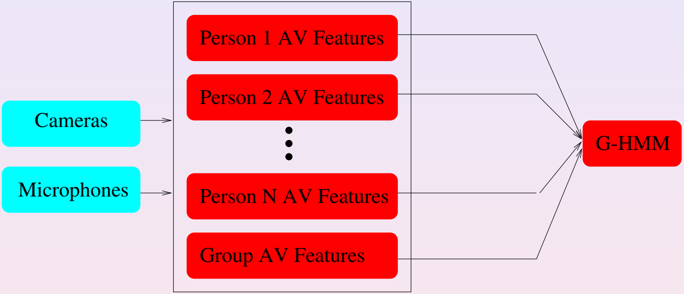
$$V = \{v_1, v_2, v_3, \cdots, v_N\}$$



- Recorded and annotated 30 training a 30 test meetings.
- Extract high level audio and visual features.
- Try to recover the target action sequence of unseen meetings.

I. McCowan, D. Gatica-Perez, S. Bengio, G. Lathoud, M. Barnard, and D. Zhang. Automatic Analysis of Multimodal Group Actions in Meetings. *IEEE Transactions on PAMI*, 27(3), 2005.

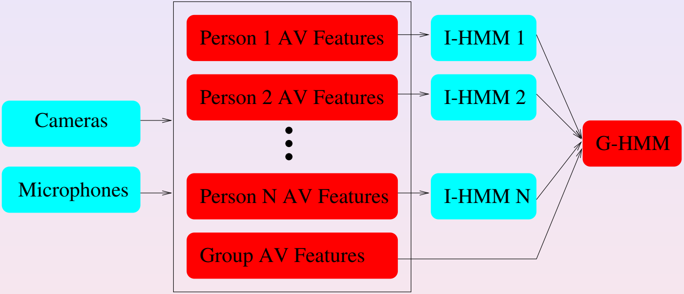
A One-Layer Approach



Classical Approach

- A large vector of audio-visual features from each participant and group-level features are concatenated to define the observation space.
- A general HMM is trained using a set of labeled meetings.

A Two-Layer Approach



Advantages

- Smaller observation space.
- I-HMMs share parameters, person independent, trained on simple task (write, talk, passive).
- Last layer less sensitive to variations of low-level data.

Experiments

Group Actions: Turn-Taking

- Discussion
- Monologue
- Monologue/Note-Taking
- Presentation
- Presentation/Note-Taking
- White-board
- White-board/Note-Taking

Individual Actions

Speaking - Writing - Passive

Results

Method	Features	AER
One-Layer	Visual Only	48.20
	Audio Only	36.70
	Audio Visual	23.74
Two-Layer	Visual Only	42.45
	Audio Only	32.37
	Audio Visual	16.55
	Async HMM	15.11

D. Zhang, D. Gatica-Perez, S. Bengio, I. McCowan, and G. Lathoud. Modeling Individual and Group Actions in Meetings: a Two-Layer HMM Framework. In *CVPR*, 2004.

Part VI

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   NOTE: The theory is explained here with all the equations.
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2. B. D. Ripley. *Pattern recognition and Neural networks*. Cambridge University Press, Cambridge, UK, 1996.
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2. L. R. Rabiner. A tutorial on hidden markov models and selected applications in speech recognition. *Proceedings of the IEEE*, 77(2):257–286, 1989.
   NOTE: A good introduction to HMMs and speech recognition.
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