ANU Logic Summer School
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Non-Classical Logic
Lecture 1: Natural Deduction

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1 Introducing Fitch-Style Natural Deduction

In this course, we will look at several non-classical logics in the framework of natural deduction. The idea behind natural deduction – and the reason that it is called "natural" – is that it is supposed to mirror the way in which mathematicians construct proofs. We will look at two styles of natural deduction proofs – Fitch-style proofs, and sequent-style proofs. This and the next two lectures will be on Fitch-style proofs.

Although this is a course on non-classical logic, we are going to start with classical propositional logic (PC), which is the logical system that is discussed in most of the other courses in the summer school.

A natural deduction proof is a list of statements of logic written vertically. Of course this is not a definition of the word 'proof'. Not every vertical list is a proof. Proofs can contain *subproofs*. Here is a simple proof that contains a subproof:

1.	$\mid (A \supset B)$	*hyp.*
2.	$\mid \; \mid A$	*hyp.*
3.	$\mid \; \mid (A \supset B)$	1, *reit*
4.	$\mid \; \mid B$	$2, 3, \supset E$

The first two lines are *hypotheses*. These are the premises of the proof. When we write a hypothesis, we also write a line down the left side of the proof or subproof which is begun by the hypothesis. This part

$\mid A$
$\mid A \supset B$
$\mid B$

is a subproof. Every proof or subproof starts with a hypothesis and ends when the hypothesis is *discharged*. We will get to discharging hypotheses later.

The final line is the conclusion of the proof – it is the statement that we are trying to prove. The third column tells us what rules we are using. The rule *hyp* ('hypothesis') tells us that we can assume a hypothesis. When can we do so? Anytime. But we will see that you do have to be careful when you assume hypotheses in proofs. The rule $\supset E$ ('implication elimination') is used to infer the final line from the first two. The rule *reit* ('reiteration') tells us that we can copy a line from a proof into any of its subproofs (note that we cannot copy lines from subproofs to proofs).

This proof tells us that the argument

$$\begin{array}{c} A \supset B \\ A \\ \hline \therefore\ B \end{array}$$

is valid. When we assume a hypothesis in a proof, it is the same thing as giving a premise in an argument. Thus, if we are trying to prove that an argument is valid, we begin by assuming all of its premises as hypothesis in a proof. Then we try to prove its conclusion.

2 Reiteration

Our first rule is one that you have already seen. This is the rule of reiteration. Stated precisely it is:

> If A occurs in a proof, it can be copied into any subproof of that proof or into a subproof of a proof, and so on.

Note that the notions of proof and subproof are relative. Consider the following proof:

1.	$A \supset B$	*hyp.*
2.	$\quad B \supset C$	*hyp.*
3.	$\quad\quad A$	*hyp.*
4.	$\quad\quad A \supset B$	$1,$ *reit.*
5.	$\quad\quad B$	$3, 4,\ \supset E$
6.	$\quad\quad B \supset C$	$2,$ *reit.*
7.	$\quad\quad C$	$5, 6,\ \supset E$
8.	$\quad A \supset C$	$2 - 7,\ \supset I$

The proof

$$\begin{array}{l} B \supset C \\ \quad A \\ \quad A \supset B \\ \quad B \\ \quad B \supset C \\ \quad C \\ A \supset C \end{array}$$

is a subproof of the whole proof and

$$\begin{array}{|l} A \\ A \supset B \\ B \\ B \supset C \\ C \end{array}$$

is a subproof of both the previous proof and the whole proof.

3 Introduction and Elimination Rules

For each logical connective, there are two rules. (Actually, for negation we will have four rules, but let's leave that aside for now.) One of these rules tells us how to add the connective, and is called an *introduction rule*. The other rule tells us how to get rid of the connective and is called an *elimination rule*.

4 Implication

The first connective that we will treat is perhaps the most interesting – implication. Its elimination rule is one that you have already seen. Stated precisely, it is:

If $A \supset B$ and A occur in the same proof, you may infer B.

We use '$\supset E$' to mean 'implication elimination'.

The introduction rule is a little more complicated. It tells us that, if we assume A as a hypothesis and then are able to infer B, we may *discharge* the hypothesis and infer that $A \supset B$. Here is an example:

$$\begin{array}{rlll}
1. & \mid\ A \supset B & & hyp. \\
2. & \mid\ \mid\ B \supset C & & hyp. \\
3. & \mid\ \mid\ \mid\ A & & hyp. \\
4. & \mid\ \mid\ \mid\ A \supset B & & 1,\ reit \\
5. & \mid\ \mid\ \mid\ B & & 3,4,\ \supset E \\
6. & \mid\ \mid\ \mid\ B \supset C & & 2,\ reit. \\
7. & \mid\ \mid\ \mid\ C & & 5,6,\ \supset E \\
8. & \mid\ \mid\ A \supset C & & 3-7,\ \supset I
\end{array}$$

The subproof

$$\begin{array}{|l} A \\ A \supset B \\ B \\ B \supset C \\ C \end{array}$$

proves C from A. Note that we don't have to do this proof "on its own". You can use material from the proof of which this is a subproof, using reiteration. After we have this proof, we can discharge the hypothesis A and conclude $A \supset B$. This means that we have ended the subproof beginning with A and the line to its left ends. We write '$A \supset B$' in the proof of which this is a subproof.

5 Conjunction

The easiest of the connectives to manipulate, and perhaps the least interesting, is conjunction. The introduction and elimination rules are perfectly symmetrical to one another. The introduction rule ($\wedge I$) says

If A and B occur in the same proof, you may infer $A \wedge B$.

There are in fact two elimination rules (both called '$\wedge E$'):

If $A \wedge B$ occurs in a proof, you may infer A.

If $A \wedge B$ occurs in a proof, you may infer B.

6 Disjunction

The introduction rules for disjunction ($\vee I$) are very straightforward. They are

If A occurs in a proof, you may infer $A \vee B$

and

If B occurs in a proof, you may infer $A \vee B$.

Here is a proof that uses disjunction introduction:

1.	$(A \vee B) \supset C$	*hyp.*
2.	A	*hyp.*
3.	$A \vee B$	$2,\ \vee I$
4.	$(A \vee B) \supset C$	$1,$ *reit.*
5.	C	$3, 4,\ \supset E$
6.	$A \supset C$	$2-5,\ \supset I$

This proof shows that the argument

$$\frac{(A \vee B) \supset C}{\therefore\ A \supset C}$$

is valid.

The rule of disjunction elimination ($\vee E$) is somewhat stranger. It says

If $(A \vee B)$, $A \supset C$, and $B \supset C$ all occur in a proof, then you may infer C.

This rule is sort of a "double implication elimination", for it allows us to eliminate two implications at one fell swoop.

Here is a proof that uses disjunction elimination:

1.	$\mid A \supset C$	*hyp.*
2.	$\mid\ \mid\ \mid B \supset C$	*hyp.*
3.	$\mid\ \mid\ \mid A \vee B$	*hyp.*
4.	$\mid\ \mid\ \mid A \supset C$	1, *reit.*
5.	$\mid\ \mid\ \mid B \supset C$	2, *reit.*
6.	$\mid\ \mid\ \mid C$	$4, 5,\ \vee E$
7.	$\mid\ \mid (A \vee B) \supset C$	$2-6,\ \supset I$

This proves that the following argument is valid:

$$\begin{array}{c} A \supset C \\ B \supset C \\ \hline \therefore\ (A \vee B) \supset C \end{array}$$

7 Negation

There are two negation elimination rules. This is the first one:

($\neg E_1$) If A and $\neg A$ occur in the same proof, you may infer any formula in that proof.

This merely says that we can infer any formula from a contradiction. As we shall see, relevant logic rejects this rule and replaces it with a weaker rule.

Here is a proof that uses negation elimination ($\neg E_1$):

1.	$\mid A$	*hyp.*
2.	$\mid\ \mid \neg A$	*hyp.*
3.	$\mid\ \mid A$	1, *reit.*
4.	$\mid\ \mid C$	$2, 3,\ \neg E_1$
5.	$\mid \neg A \supset C$	$2-4,\ \supset I$

The introduction rule should remind you of the implication introduction rule. It says

> ($\neg I$) If we have a proof of $\neg A$ from the hypothesis A, then we can discharge this hypothesis and infer $\neg A$.

Here is a proof in which we use negation introduction:

1.	$\mid A$	*hyp.*
2.	$\mid\ \mid \neg A$	*hyp.*
3.	$\mid\ \mid A$	1, *reit*
4.	$\mid\ \mid \neg\neg A$	$2, 3,\ \neg E_1$
5.	$\mid \neg\neg A$	$2-4,\ \neg I$

8 Another Negation Elimination Rule

We said that there are two negation elimination rules, but we have only described one so far. The second one is often called the rule of reductio ad absurdum, but we will call it $\neg E_2$:

> ($\neg E_2$) If we have a proof of A from the hypothesis $\neg A$, then we can discharge this hypothesis and infer A.

We call this rule 'negation elimination 2' because it eliminates a negation that is used in a subproof. Here is a proof of $\neg\neg A \supset A$ that uses this rule:

1.	$\mid \neg\neg A$	*hyp*
2.	$\mid \mid \neg A$	*hyp*
3.	$\mid \mid \neg\neg A$	1, *reit*
4.	$\mid \mid A$	$2, 3,\ \neg E$
5.	$\mid A$	$2-4, \neg E_2$
6.	$\neg\neg A \supset A$	$1-5,\ \supset I$

9 Validity

Recall the truth-table definition of a valid argument. An argument is valid if and only if there is no row of a truth table in which all of its premises are true and its conclusion is false. We have said that a argument for which there is a corresponding proof in natural deduction. We won't prove that this correspondence really holds. This can be rather difficult and we should leave it to an upper-level course. Rather, we are going to look at another interesting connection.

Consider a one premise argument, such as

$$\frac{A}{\therefore\ B \supset A}.$$

I will let you prove that this is valid. What I want to consider instead is a closely related formula, that is, $(A \supset (B \supset A))$. This formula is also valid in the sense that it is true on every row of its truth table. In fact, if for any formulas A and B, the argument

$$\frac{A}{\therefore B}$$

is valid if and only if the formula

$$A \supset B$$

is also valid.

What about arguments with more than one premise? Consider implication elimination (i.e. modus ponens):

$$\begin{array}{c} A \supset B \\ A \\ \hline \therefore B \end{array}$$

This is clearly valid. So is the formula

$$(A \supset B) \supset (A \supset B).$$

In general, an argument of the form

$$\begin{array}{c} A_1 \\ \vdots \\ A_n \\ \hline \therefore\ B \end{array}$$

is valid if an only if the formula

$$A_1 \supset (... \supset (A_n \supset B) \ldots)$$

is also valid.

This connection holds true in natural deduction as well. In order to understand what that means, we need to understand what it is for a formula to be proved valid in our natural deduction system.

10 Validity of formulas in Natural Deduction

Everything will fall into place when we understand what it is for a formula to be valid in natural deduction. Then we will be able to prove the correspondence between valid formulas and valid arguments.

Consider the following proof of the validity of modus ponens in natural deduction:

1.	$A \supset B$	*hyp.*
2.	$\quad A$	*hyp.*
3.	$\quad A \supset B$	1, *reit*
4.	$\quad B$	$2, 3,\ \supset E$
5.	$A \supset B$	$2-4,\ \supset I$

We can do one more step here:

1.	$A \supset B$	*hyp.*
2.	$\quad A$	*hyp.*
3.	$\quad A \supset B$	1, *reit*
4.	$\quad B$	$2, 3,\ \supset E$
5.	$A \supset B$	$2-4,\ \supset I$
6.	$(A \supset B) \supset (A \supset B)$	$1-5,\ \supset I$

You might think that there are easier ways to do this proof, and there are, but that's not the point. The point is that we can discharge even the first hypothesis. Then we get a proof that has no undischarged hypotheses. And what we prove is a valid formula, or a *theorem* of the logic.

If we can show that

$$\begin{array}{c} A_1 \\ \vdots \\ A_n \\ \hline \therefore\ B \end{array}$$

is a valid argument, then we can also construct a natural deduction proof like:

$$\begin{array}{l|l} & A_1 \\ & \quad \ldots \quad \begin{array}{|l} A_n \\ \vdots \\ B \end{array} \\ & \qquad A_n \supset B \\ & \qquad\quad \vdots \\ \hline \multicolumn{2}{l}{A_1 \supset (\ldots(A_n \supset B)\ldots)} \end{array}$$

11 Exercises

Try to prove these valid:

1. $$\begin{array}{c} A \supset (A \supset B) \\ \hline \therefore\ A \supset B \end{array}$$

2. $$\begin{array}{c} A \\ \hline \therefore\ (A \supset B) \supset B \end{array}$$

3. $$\begin{array}{c} A \supset B \\ A \supset C \\ \hline \therefore\ A \supset (B \wedge C) \end{array}$$

4. $$\begin{array}{c} A \\ \hline \therefore\ B \supset (A \wedge B) \end{array}$$

5. $$\begin{array}{c} A \supset B \\ \hline \therefore\ \neg B \supset \neg A \end{array}$$

6. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

7. $((A \vee B) \supset C) \supset (A \supset C)$

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Lecture 2: Intuitionist Logic

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1 Intuitionist Logic: From Truth to Proof

The classical logician's view of validity is that an inference is valid if and only if its premises cannot all be true in the same circumstance in which its conclusion is false. This view is often called the *truth preservation theory of validity.* Thus, the notion of truth is central to classical logic. One of the first classical logicians, Gottlob Frege, says:

> Just as 'beautiful' points the way for aesthetics and 'good' for ethics, so do words like 'true' for logic. All sciences have truth as their goal; but logic is concerned with it in a quite different way: logic has much the same relation to truth as physics has to weight or heat. To discover truths is the task of all sciences; it falls to logic to discern the laws of truth. (Frege, "Thoughts" (1918) translated by P.T. Geach and R.H. Stoothoff in Frege, *Logical Investigations,* Oxford: Blackwell, 1977)

Intuitionist logic began as a way of formalizing *intuitionist mathematics.* Intuitionist mathematics was a form of mathematical practice that began in the early years of the 20th Century as a reaction to classical mathematics. Classical logic began (in the work of Frege, Bertrand Russell, and others) as a way of understanding the inferences made in classical mathematics. If we are to use the classical notion of validity to codify mathematical inference, then there must be a usable concept of mathematical truth. At the turn of the 20th Century, there were a few such notions available, but the one that concerns us here is the *Platonist* concept of mathematical truth. According to Platonism (a view held by Frege and the set theorist Georg Cantor among others), there are entities called "mathematical objects". A number is a mathematical object, so is a set, so is a function, and so on. Where are these mathematical objects? They are, according to Platonism, nowhere in space or time – they have their own "realm". Platonism has the virtue of giving a straightforward and rather standard theory of truth. A mathematical statement is true if and only if the things it talks about actually have the properties attributed to it by the statement. E.g., the statement '2+2=4' is true if and only if applying the function of addition to the pair $< 2, 2 >$ has the value 4.

Platonism, however, clearly also has important difficulties. First, it seems philosophically ad hoc to postulate a special realm of objects just to explain how certain sentences can be true. Second, if these objects are nowhere in space or time, then we cannot perceive them. If we cannot perceive them, how do can we know things about them. Surely there is mathematical knowledge, and this fact needs to be explained.

Intuitionism is a reaction against Platonism. We won't go over the original form of intuitionism, because although extremely interesting it is a complicated mix of 19th Century philosophy and mysticism. Rather, we will look at a more modern form due to Stephen Kleene and Michael Dummett.

According to this modern form of intuitionism, what is true in mathematics is what can be *constructibly proven*. The idea is that a mathematical statement is true if and only there is a step by step method that will prove it. In effect, what is true is what can (ideally) be proven by a computer. In this move from Platonist truth to constructive proof, we see an attempt to deal with the two problems we have stated above. First, the notion of proof is clearly central to mathematical practice – it is not ad hoc to make it central to a philosophy of mathematics. Second, the intuitionist view that takes truth to be what can be proven explains how we can know mathematical truths. Our proofs show that they are truth. The Platonist has to explain why we take proofs in classical logic to show that certain statements about Platonic objects are true. For the intuitionist, mathematical truth is just provability, so no further explanation is needed.

For the intuitionist, talk of mathematical objects is rather misleading. For them, there really isn't any*thing* that we should call the natural numbers, but instead there is counting. What intuitionists study, then, are mathematical processes, such as counting (in arithmetic), collecting things (in intuitionist set theory, sometimes called the "theory of species"), and so on. We will follow the intuitionists' practice of talking about mathematical objects, but note that this is really shorthand for talk of processes.

In classical mathematics, we talk about infinite sets. In fact, we talk about larger and larger infinite sets: the natural numbers, the real numbers, the set of functions over the real numbers, and so on. If we talk about the process of collecting things, rather than a complete collection itself, we get a rather different notion of infinity. Philosophers distinguish between a never-ending process (sometimes called a "potential infinity") and a completed infinity. Classical mathematics deals with completed infinities, whereas intuitionists accept only never-ending processes. Given that they reject the notion that there are completed infinities, intuitionists cannot accept the notion that there are different sizes of infinity. This leads also to problems regarding the real numbers (we usually think of irrational numbers in terms of infinitely long strings of digits), and the intuitionist theory of the reals is as a result extremely complicated, as is there treatment of calculus.

2 The BHK Interpretation of IL

The notion of truth, as we have seen, is central to the understanding of classical logic. The interpretation of the connectives of classical logic (conjunction, implication, ...) and the quantifiers is usually given in terms of *truth conditions*. Suppose that we have a language with a finite number of predicates. A model for this language is based on a domain of individuals. We begin the construction of a model for our language by stating the extensions of each of these predicates. For example, suppose Ox means 'x is an odd number', then the extension of O is the set $\{1, 3, 5, ...\}$. The extension of an n-place relation symbol is a set of n-place sequences. It might be that we cannot explicitly state the extensions of all the predicates, but we will leave that problem aside. We also assume a class of value assignments, which are functions v that takes each term (constant or variable) to an individual in the domain of our model.

After the extensions of the predicates are determined, we state a *schema* that gives us the truth condition for atomic formulas. Where P is an n-place predicate,

$\forall xA$ is true according to v
if and only if
A is true according to all x-variants of v.

$Pt_1...t_n$ is true according to v
if and only if
$< v(t_1), ..., v(t_n) >$ is in the extension of P.

Here are the schemas for the connectives. Where a formula is false (according to a value assignment) if and only if it fails to be true,

$A \wedge B$ is true according to v
if and only if
A is true according to v and B is true according to v.

$A \vee B$ is true according to v
if and only if
A is true according to v or B is true according to v.

$A \supset B$ is true according to v
if and only if
A is false according to v or B is true according to v.

$$\neg A \text{ is true according to } v$$
$$\text{if and only if}$$
$$A \text{ is false according to } v\ .$$

$$\forall x A \text{ is true according to } v$$
$$\text{if and only if}$$
$$A \text{ is true according to all } x\text{-variants of } v.$$

$$\exists x A \text{ is true according to } v$$
$$\text{if and only if}$$
$$A \text{ is true according to some } x\text{-variant of } v.$$

An x-variant of v is a function that has the same value for each constant and variable as does v except perhaps x.

From a philosophical point of view, the main advantage of this semantics is that it is *compositional*. According to the standard view of meaning (called *truth conditional semantics*), we understand a sentence when we understand the conditions under which it is true. This construction of models for classical logic shows us that if we understand the meaning of the atomic formulas of the language, then we can use these schemas to determine the truth conditions of any formula of our language. This tells us how we can understand new sentences of a language that we have never heard before.

Intuitionist logic does not take truth to be central. Rather, as we have seen, proof plays a very similar role. To be a real alternative to classical logic, intuitionists need to provide us with a compositional interpretation based on the notion of proof. They have done so. Such an interpretation is the Brouwer-Heyting-Kolmogorov (BHK) interpretation, named after Jan Brouwer (the father of intuitionist mathematics), Arend Heyting (the person who first formulated intuitionist logic in its current form), and Andrey Kolmogorov (the great Russian mathematician).

This presentation of the BHK interpretation is taken (with minor alterations) from Iemhoff 2008.

$$\text{A proof of } A \wedge B \text{ is a proof of both } A \text{ and } B$$
$$\text{A proof of } A \vee B \text{ is a proof of either } A \text{ or } B$$

$$\text{A proof of } A \supset B$$
$$\text{is a proof that any proof of } A \text{ can be transformed into a proof of } B$$

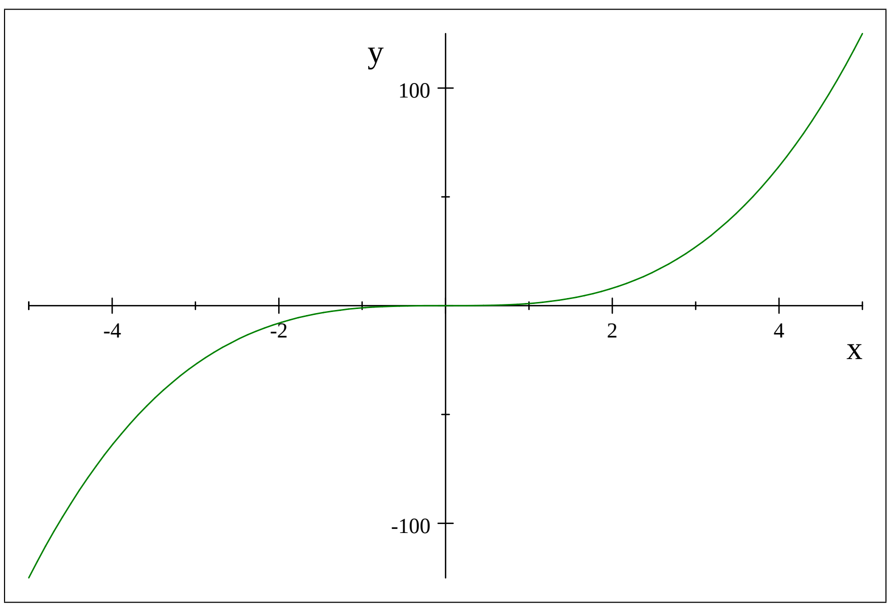
$$\text{A proof of } \neg A$$
$$\text{is a proof that any proof of } A \text{ can be transformed into a proof of a contradiction.}$$

$$\text{A proof of } \forall x A(x) \text{ is a proof}$$
$$\text{that given any object } i \text{ of the domain we can construct a proof of } A(i)$$

$$\text{A proof of } \exists x A(x) \text{ is a proof}$$
$$\text{that } A(i) \text{ for some object } i \text{ in the domain}$$

Note that there is no general procedure given for proving atomic formulas. Our knowledge of such proofs is determined by the contents of the atomic formulas themselves. But we still have a method for understanding complex statements on the basis of our understanding of simple ones, just as in the semantics for classical logic.

The treatment of the quantifiers has very interesting consequences for intuitionist mathematics. Consider a continuous function that has at least one value below 0 and at least one value above 0, e.g., $y = x^3$



In this case we can see that the value of x for which $y = 0$ is itself 0, but consider the more general statement 'for any function f, if there is an x for which $f(x) \leq 0$ and an x' for which $f(x') > 0$ then there is some x'' such that $f(x'') = 0$'. According to the BHK interpretation, to prove this expression we need to show that there is a proof that, given *any* function f there is a way of finding (i.e. computing) a number x'' at which $f(x'') = 0$'. In other words, we need a general method of finding the zeroing value for any function. Unfortunately, there is no such general method and so this (the intermediate value theorem) is not a theorem of intuitionist logic.

3 The Falsum and Negation

The treatment of negation in intuitionist logic is particularly interesting for us here. According to intuitionist logic, all contradictions are equivalent to one another. This is true in classical logic, but (as we shall see) not in relevant logic. It is standard in formulations of intuitionist logic to add a propositional constant f (sometimes $\bot$) that represents any contradiction. This constant is called the "falsum". Its proof condition is simple: there is no proof of f. Using f, we can define negation:

$$\neg A \ =_{df} \ A \supset f.$$

If we define negation, we remove it from the primitive vocabulary of the language.

One of the central truths of ancient logic, which Aristotle called the *eternal truths*, is the law of excluded middle, i.e.

$$A \vee \neg A.$$

The rewriting this with the falsum, we get

$$A \vee (A \supset f).$$

This schema is read, according to the BHK interpretation as 'for any formula A, we can either prove A or find a proof that a proof of A can be transformed into a proof of a contradiction'. Clearly, we cannot prove this statement. Thus, the law of excluded middle is not a theorem of intuitionist logic.

Note that we cannot show that $\neg(A \vee \neg A)$ in intuitionist logic. For every classical contradiction is also an intuitionist contradiction. But what we can show is that $\neg \vdash A \vee \neg A$, where the turnstile means 'is provable in intuitionist logic'.

There are other familiar theorems of classical logic that fail in intuitionist logic. Perhaps the most famous is double negation elimination, viz.,

$$\neg\neg A \supset A.$$

Convince yourself that this can't be proven by reading it using the BHK interpretation. On the other hand, the principle of double negation introduction is provable:

$$A \supset \neg\neg A.$$

This principle is just an instance of $A \supset ((A \supset B) \supset B)$, which is also provable.

4 Natural Deduction for Intuitionist Logic

For the present let us include in our language both the falsum and negation as primitive. The natural deduction system for intuitionist logic is exactly like the classical system except for the negation rules and the rule for f. The negation introduction rule is

If there is a proof of f from the hypothesis that A,
then we can discharge the hypothesis and infer $\neg A$.

The negation elimination rule is the following:

From A and $\neg A$, we may infer f.

There is no introduction rule for f. The elimination rule for f is similar to the negation elimination rule in classical logic:

From f we may infer B.

That is, from a contradiction we may infer any formula.

In intuitionist logic $\neg\neg A$ is a theorem if A is a theorem of classical logic. Here are proofs of $\neg\neg(A \lor \neg A)$ and $\neg\neg(\neg\neg A \supset A)$:

$$\begin{array}{lll}
1. & \mid\ \neg(A \lor \neg A) & hyp \\
2. & \mid\ \mid\ A & hyp \\
3. & \mid\ \mid\ A \lor \neg A & 2,\ \lor I \\
4. & \mid\ \mid\ \neg(A \lor \neg A) & 1,\ reit \\
5. & \mid\ \mid\ f & 3,4,\ \neg E \\
6. & \mid\ \neg A & 2-5,\ \neg I \\
7. & \mid\ A \lor \neg A & 6,\ \lor I \\
8. & \mid\ f & 1,7,\ \neg E \\
9. & \neg\neg(A \lor \neg A) & 1-8,\ \neg I
\end{array}$$

$$\begin{array}{lll}
1. & \mid\ \neg(\neg\neg A \supset A) & hyp \\
2. & \mid\ \mid\ \neg\neg A & hyp \\
3. & \mid\ \mid\ \mid\ A & hyp \\
4. & \mid\ \mid\ \mid\ \mid\ \neg\neg A & hyp \\
5. & \mid\ \mid\ \mid\ \mid\ A & 3,\ reit \\
6. & \mid\ \mid\ \mid\ \neg\neg A \supset A & 4-5,\ \supset I \\
7. & \mid\ \mid\ \mid\ \neg(\neg\neg A \supset A) & 1,\ reit \\
8. & \mid\ \mid\ \mid\ f & 6,7,\ \neg E \\
9. & \mid\ \mid\ \neg A & 3-8,\ \neg I \\
10. & \mid\ \mid\ f & 2,9,\ \neg E \\
11. & \mid\ \mid\ A & 10,\ fE \\
12. & \mid\ \neg\neg A \supset A & 2-11,\ \supset I \\
13. & \mid\ f & 1,12,\ \neg E \\
14. & \neg\neg(\neg\neg A \supset A) & 1-13,\ \neg I
\end{array}$$

5 Further Reading

- E.W. Beth, *The Foundations of Mathematics*, Amsterdam: North Holland, 1959

- Chapter 9 of Stuart Brock and Edwin Mares, *Realism and Anti-Realism*, Stocksfield: Acumen, 2007 is about the philosophy of mathematics and discusses intuitionism.
- Michael Dummett, *Elements of Intuitionism*, Oxford: Oxford University Press, 1977
- Arend Heyting, *Intuitionism: An Introduction*, Amsterdam: North Holland, 1972
- Rosalie Iemhoff "Intuitionism in the Philosophy of Mathematics" in the Stanford Encyclopedia, http://plato.stanford.edu/entries/intuitionism/ , 2008

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Non-Classical Logic
Lecture 3: Relevant Logic

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1 Implication

Here we will look at a natural deduction system for the relevant logic, R. "R" stands for 'relevant implication'. The key notion for understanding this natural deduction system is that of a real use of an hypothesis. In order to make sure that an hypothesis is really used in an inference, we label each hypothesis with a number and then we put a subscript on each line of the proof that indicates which hypotheses were used to infer that line. For example:

1.	$A \to B_{\{1\}}$	*hyp.*
2.	$\quad A_{\{2\}}$	*hyp.*
3.	$\quad A \to B_{\{1\}}$	$1, reit$
4.	$\quad B_{\{1,2\}}$	$3, 4, \to E$

Here the rule for $\to E$ is: *From* $A \to B_{\alpha}$ and A_{β} we can infer $B_{\alpha\cup\beta}$. $\alpha \cup \beta$ is the set of numbers that belong to either α or β (it collects all of the numbers in the two sets together).

This proof shows that we can validly and relevantly infer B from $A \to B$ and A. The hypotheses that $A \to B$ and A are really used to infer B. We can see this because the hypotheses numbers for these premises show up in the subscript for the conclusion B.

Now let's look at another inference. This time we will infer from the single premise A the conclusion that $(A \to B) \to B$.

1.	$A_{\{1\}}$	*hyp.*
2.	$\quad A \to B_{\{2\}}$	*hyp.*
3.	$\quad A_{\{1\}}$	$1,$ *reit.*
4.	$\quad B_{\{1,2\}}$	$2, 3, \to E$
5.	$(A \to B) \to B_{\{1\}}$	$2-4, \to I$

What happened to the number 2 in the subscript to line 5? Surely the second hypothesis was really used in its derivation. When a hypothesis is discharged, its number is removed from the subscript of the line that is produced by the rule of implication introduction. This is the difference between a hypothesis and a premise in an inference. A premise is an hypothesis that never gets discharged.

The rule for implication introduction is: From a proof that B_α from the hypothesis $A_{\{k\}}$ (where k is a number), we can infer $B_{\alpha-\{k\}}$, *where k really is in α*. ($\alpha - \{k\}$ is just the set α with k removed from it.)

A valid formula in this system is just one that can be proven with the subscipt $\emptyset$ (the empty set). Consider the following proof:

1.	$A \rightarrow B_{\{1\}}$	*hyp*
2.	$B \rightarrow C_{\{2\}}$	*hyp*
3.	$A_{\{3\}}$	*hyp*
4.	$A \rightarrow B_{\{1\}}$	1, *reit*
5.	$B_{\{1,3\}}$	$3, 4, \rightarrow E$
6.	$B \rightarrow C_{\{2\}}$	2, *reit*
7.	$C_{\{1,2,3\}}$	$5, 6, \rightarrow E$
8.	$A \rightarrow C_{\{1,2\}}$	$3-7, \rightarrow I$
9.	$(B \rightarrow C) \rightarrow (A \rightarrow C)_{\{1\}}$	$2-8, \rightarrow I$
10.	$(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))_{\emptyset}$	$1-9, \rightarrow I$

(Write the justifications in yourself. I didn't have room on the page!) Here we have proven that $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$ is valid. It is said to be a *theorem of the logic R*.

2 From Truth to Proof to Information

I have my own line on how to understand the models for relevant logic, so I will give that you that here. Consider for a moment again the natural deduction system for R. A hypothesis in this system isn't really just a formula, but a formula subscripted with a number, say, $A_{\{1\}}$. Let's now think about what this means.

The idea behind the theory that I am going to present comes from *situation semantics*, which was developed by Jon Barwise and John Perry in the 1980s. On their view, there are not just possible worlds, but also *situations*. A situation is a part of a world. For example, consider the room that you are in right now. There is certain information available to you in that room. If it is our lecture room, then the information is available to you about whether the projector is on or off and about what the lecturer is saying right now. But there is other information not available to you that is available to people in other situations, for example, someone in Singapore will have the information available to her about whether or not it is raining there, but won't have the information about whether the projector in our lecture room is on. So, in a single possible world, there are many different situations, each containing different information. We say that each situation contains *partial information*, because it does not (necessarily) tell us about the whole world. (On some views about situation semantics, we can consider the whole universe also to be one big situation, but on other views this is not the case. We will get back to that topic later.)

We can situations in this sense to think about relevant logic. Consider again the hypothesis $A_{\{1\}}$. If this is hypothesized in a proof, what it means is "suppose that there is a situation (call it s_1) in a world which contains the information that A". Now, suppose that we make further hypotheses in the same proof, for example, $B_{\{2\}}$. We are now saying "suppose that there is also a situation (call it s_2) in the same world which contains the information that B".

Consider the following proof:

1.	$A_{\{1\}}$	*hyp*
2.	$\quad A \to B_{\{2\}}$	*hyp*
3.	$\quad A_{\{1\}}$	1, *reit*
4.	$\quad B_{\{1,2\}}$	$2, 3, \to E$
5.	$(A \to B) \to B_{\{1\}}$	$2-4, \to I$
6.	$A \to ((A \to B) \to B)_{\emptyset}$	$1-5, \to I$

Let's forget about the last line for a moment. The first line says "suppose that there is a situation s_1 in a world in which A". The second line says "suppose there is a situation s_2 in the same world in which $A \to B$". The third line just reiterates the first line, but the fourth line is interesting. It says that there is a situation s' in the same world in which B, and we know that there is this situation because we have derived that it is so by *really using* the information in s_1 and s_2.

The fifth line tells of course that we know (from the discharged subproof in steps 2-4) that in s_1 there is the information that $(A \to B) \to B$.

Now we turn to the final line of the proof. What does "$A \to ((A \to B) \to B)_{\emptyset}$" mean? As we know, it means that this formula is valid. But what does "valid" mean here? It means that $A \to ((A \to B) \to B)$ is true in every *normal* situation. A normal situation (on my view) is just a situation that captures all and only the information that is in a possible world. Or, rather, a normal situation is just a possible world (that is, a complete possible universe). Thus, a formula is valid if and only if it is true in every possible world.

3 Conjunction

Now we will add conjunction. Here's a proof using conjunction:

1.	$(A \to B) \wedge A_{\{1\}}$	*hyp.*
2.	$A \to B_{\{1\}}$	$1, \wedge E$
3.	$A_{\{1\}}$	$1, \wedge E$
4.	$B_{\{1\}}$	$2, 3, \to E$
5.	$((A \to B) \wedge A) \to B_{\emptyset}$	$1-4, \to I$

The conjunction elimination rule ($\wedge E$) is: From $A \wedge B_{\alpha}$ we can infer A_{α} and B_{α}.

The conjunction introduction rule is just the reverse. It says that from A_α and B_α we can infer $A \wedge B_\alpha$. Note that in order to do a conjunction introduction, the two formulas that you want to conjoin have to have the exact same numbers in their subscript. Here is a proof using conjunction introduction:

1.	$(A \rightarrow B) \wedge (A \rightarrow C)_{\{1\}}$	*hyp.*
2.	$\quad A_{\{2)}$	*hyp.*
3.	$\quad (A \rightarrow B) \wedge (A \rightarrow C)_{\{1\}}$	1, *reit.*
4.	$\quad A \rightarrow B_{\{1\}}$	3, $\wedge E$
5.	$\quad A \rightarrow C_{\{1\}}$	3, $\wedge E$
6.	$\quad B_{\{1,2\}}$	$2, 4, \rightarrow E$
7.	$\quad C_{\{1,2\}}$	$2, 5, \rightarrow E$
8.	$\quad B \wedge C_{\{1,2\}}$	$6, 7, \wedge I$
9.	$A \rightarrow (B \wedge C)_{\{1,2\}}$	$2-8, \rightarrow I$
10.	$((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))_\emptyset$	$1-9, \rightarrow I$

4 Disjunction

The disjunction rules are a lot like the rules for classical logic, except that you have to be careful about how you use subscripts. The disjunction introduction rule ($\vee I$) is easy:

From A_α in a proof, you may infer $A \vee B_\alpha$ and/or $B \vee A_\alpha$.

Here the rule is just the same as it was for PC except that you have to carry along the same subscript to $A \vee B$ (or $B \vee A$) that was on A.

As before, the disjunction elimination rule ($\vee E$) is a lot more difficult. The easiest way to state it for relevant logic is the following:

From $A \vee B_\alpha$ and $A \rightarrow C_\beta$ and $B \rightarrow C_\beta$, you may infer $C_{\alpha \cup \beta}$.

This might not look a lot like the rule for PC, but it is a lot like it. For it tells us that a proof of the following form is valid:

$$\begin{array}{l}
A \vee B_\alpha \\
\quad A_{\{j\}} \\
\quad \vdots \\
\quad C_{\beta \cup \{j\}} \\
A \rightarrow C_\beta \\
\quad B \\
\quad \vdots \\
\quad C_{\beta \cup \{k\}} \\
B \rightarrow C_\beta \\
C_{\alpha \cup \beta}
\end{array}$$

Note that the subscript on $A \rightarrow C$ and $B \rightarrow C$ has to be the same to use this rule.

Here is a proof that uses disjunction elimination:

1.	$(A \to C) \wedge (B \to C)_{\{1\}}$	*hyp*
2.	$\quad A \vee B_{\{2\}}$	*hyp*
3.	$\quad (A \to C) \wedge (B \to C)_{\{1\}}$	1, *reit*
4.	$\quad A \to C_{\{1\}}$	3, $\wedge E$
5.	$\quad B \to C_{\{1\}}$	3, $\wedge E$
6.	$\quad C_{\{1,2\}}$	$2, 4, 5,\ \vee E$
7.	$(A \vee B) \to C_{\{1\}}$	$2 - 6,\ \to I$
8.	$((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)_{\emptyset}$	$1 - 7,\ \to I$

5 Distribution

One thing that is rather odd about the natural deduction system for relevant logic is that it does not give us a clean proof of the distribution of conjunction over disjunction. The proof in PC looks like this:

1.	$A \wedge (B \vee C)$	*hyp*
2.	A	1, $\wedge E$
3.	$B \vee C$	1, $\wedge E$
4.	$\quad B$	*hyp*
5.	$\quad A$	2, *reit*
6.	$\quad A \wedge B$	$4, 5,\ \wedge I$
7.	$\quad (A \wedge B) \vee (A \wedge C)$	6, $\vee I$
8.	$\quad C$	*hyp*
9.	$\quad A$	2, *reit*
10.	$\quad A \wedge C$	$8, 9,\ \wedge I$
11.	$\quad (A \wedge B) \vee (A \wedge C)$	10, $\vee I$
12.	$(A \wedge B) \vee (A \wedge C)$	$3, 4 - 7, 8 - 11,\ \vee E$
13.	$(A \wedge (B \vee C)) \supset ((A \wedge B) \vee (A \wedge C))$	$1 - 12,\ \to I$

The problem with this proof, from a relevant point of view, is that once we add subscripts we can't do the conjunction introduction line 6 (or again at line 10):

1.	$A \wedge (B \vee C)_{\{1\}}$	*hyp*
2.	$A_{\{1\}}$	1, $\wedge E$
3.	$B \vee C_{\{1\}}$	1, $\wedge E$
4.	$\quad B_{\{2\}}$	*hyp*
5.	$\quad A_{\{1\}}$	2, *reit*
6.	$\quad A \wedge B_{\{???\}}$	$4, 5,\ \wedge I$
$\vdots$	$\quad \vdots$	$\vdots$

We require that the subscripts at lines 4 and 5 to be the same to do the conjunction introduction in line 6. But they aren't and there is no other way in the natural deduction system with the rules we have so far to get around this.

So what we do is add distribution as its own rule:

(Dist) From $A \wedge (B \vee C)_\alpha$ you may infer $(A \wedge B) \vee (A \wedge C)_\alpha$.

This is a bit ugly. It would be nice to be able to derive distribution from other ("deeper") features of conjunction and disjunction. [There is a slightly modified natural deduction system due to Ross Brady that does allow for the derivation of distribution, but we will not examine it here.]

6 Negation

Like intuitionist logic, to treat negation in relevant logic we add a falsum, f. Here f means 'a contradiction occurs'. Unlike intuitionist logic, relevant logic does not treat every contradiction as equivalent. Rather, the falsum can be understood as the (infinite) disjunction of all of the contradictions. In algebraic terms, it is the least upper bound of all the contradictions. The key is the introduction of a new constant, f. This is a proposition which means 'a contradiction occurs'. Thus, the formula '$A \rightarrow f$' means 'A implies that there is a contradiction. We take $A \rightarrow f$ to mean the same thing as $\neg A$. Thus, to say that it is not the case that A is to say the same thing as A implies that there is a contradiction.

Thus, we start with the following rule of negation introduction:

($\neg I$) From a proof of f from the hypothesis that A,
you may discharge the hypothesis and infer $\neg A$

Or, in more graphically:

$$\begin{array}{l} \left| \begin{array}{l} A_{\{k\}} \\ \quad\vdots \\ \quad f_\alpha \end{array} \right. \\ \neg A_{\alpha - \{k\}} \end{array}$$

where k really is in α. We also have the following version of negation elimination:

($\neg E_1$) From A_α and $\neg A_\alpha$ you may infer $f_{\alpha \cup \beta}$.

Note that in relevant logic we cannot infer just anything from a contradiction. But we are allowed to infer f from a contradiction, since f means that there is a contradiction.

We need one more rule. This is a rule that ensures that we can eliminate double negations. This rule looks a little like one that we met in the natural deduction system for PC:

($\neg E_2$) From a proof of f_α on the hypothesis that $\neg A_{\{k\}}$,
you may discharge the hypothesis and infer $A_{\alpha - \{k\}}$

where k really is in α.

Here is a proof of $\neg\neg A \to A$:

$$\begin{array}{lll}
1. & \quad \neg\neg A_{\{1\}} & hyp \\
2. & \qquad \neg A_{\{2\}} & hyp \\
3. & \qquad \neg\neg A_{\{1\}} & 1,\ reit \\
4. & \qquad f_{\{1,2\}} & 2,3,\ \neg E_1 \\
5. & \quad A_{\{1\}} & 2-4,\ \neg E_2 \\
6. & \neg\neg A \to A_{\emptyset} & 1-5,\ \to I
\end{array}$$

Here is a proof of $A \to \neg\neg A$:

$$\begin{array}{lll}
1. & \quad A_{\{1\}} & hyp \\
2. & \qquad \neg A_{\{2\}} & hyp \\
3. & \qquad A_{\{1\}} & 1,\ reit \\
4. & \qquad f_{\{1,2\}} & 2,3,\ \neg E \\
5. & \quad \neg\neg A_{\{1\}} & 2-4,\ \neg I \\
6. & A \to \neg\neg A & 1-5,\ \to I
\end{array}$$

And here is a proof of $(A \to B) \to (\neg B \to \neg A)$:

$$\begin{array}{lll}
1. & \quad A \to B_{\{1\}} & hyp \\
2. & \qquad \neg B_{\{2\}} & hyp \\
3. & \qquad\quad A_{\{3\}} & hyp \\
4. & \qquad\quad A \to B_{\{1\}} & 1,\ reit \\
5. & \qquad\quad B_{\{1,3\}} & 3,4,\ \to E \\
6. & \qquad\quad \neg B_{\{2\}} & 2,\ reit \\
7. & \qquad\quad f_{\{1,2,3\}} & 5,6,\ \neg E_1 \\
8. & \qquad \neg A_{\{1,2\}} & 3-7,\ \neg I \\
9. & \quad \neg B \to \neg A_{\{1\}} & 2-8,\ \to I \\
10. & (A \to B) \to (\neg B \to \neg A)_{\emptyset} & 1-9,\ \to I
\end{array}$$

7 Exercises

Try to prove the following:

1. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
2. $(A \to (A \to B)) \to (A \to B)$
3. $(A \to \neg A) \to \neg A$
4. $(A \to \neg B) \to (B \to \neg A)$
5. $(\neg A \to B) \to (\neg B \to A)$

8 Further Reading

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Lecture 4: Sequent Style Calculi

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1 Sequent-Style Natural Deduction Systems

Until now we have been studying Fitch-style natural deduction systems. In Fitch-style systems, the differences between some logical systems, like intuitionist and classical logic, are treated as differences between the rules governing the connectives. Thus, in the framework of Fitch-style natural deduction it would seem that these logics confer different meanings on the same connectives. In particular, it would seem that the difference between classical and intuitionist logic is that they give different meanings to negation.

In sequent-style systems – either Gentzen's sequent calculus or Greg Restall's sequent-style natural deduction systems (which we are studying here), the difference between logical systems is not in their treatment of the connectives. Rather, it is in the way that they treat the *notion of proof* itself. A sequent is a string

$$X \vdash Y$$

where X and Y are structures of formulas. We will see a rigorous definition of 'structure' soon. The structure to the left of the turnstile is a structure of premises and the structure to the right is a structure of conclusions. If this sequent is provable in a particular system, then it can be said that the conclusions Y follow from the premises X in that system. In Gentzen's sequent calculus for intuitionist logic (which he called **LJ**) admissible sequents have at most a single formula as the premise structure (it may also be empty). In the sequent calculus for classical logic (**LK**), structures with arbitrary numbers of formulas are allowed as conclusions. Apart from that, the primitive rules governing the connectives are the same for both systems. Thus, it can be said that the two systems different, not in the meanings of the connectives, but in the inferences that they take to be admissible.

In this lecture, we examine Restall's natural deduction systems. Today we will look in particular at the systems for the relevant logic **R** and intuitionist logic. And we will be able to compare and contrast them with regard to the different structural rules that they allow.

The disadvantage of Restall's sequent-style systems is that their treatment of negation is rather complicated. I will avoid this by considering negation (and f) -free systems.

2 Structural Rules and Meaning Rules

The set of structures is the smallest set such that:

- Every formula A is a structure;
- The symbol 0 is a structure;
- If X and Y are structures, so is $(X;Y)$.

A sequent is a string

$$X \vdash A$$

where X is a structure and A is a formula.

Structural rules allow us to manipulate the premise structure. For example, one rule (not admissible in every system), tells us that if we have a substructure $X;Y$ in a premise structure Z, then we can have the same structure $Y;X$. We then say that in systems that admit this rule that the following is a valid inference:

$$\frac{Z(X;Y) \vdash A}{Z(Y;X) \vdash A}$$

We can use this rule, for example, in deriving the sequent $A \vdash (A \to B) \to B$:

$$\dfrac{\dfrac{\dfrac{A \to B \vdash A \to B \qquad A \vdash A}{A \to B; A \vdash B}{\scriptstyle \to E}}{A; A \to B \vdash B}{\scriptstyle CI}}{A \vdash (A \to B) \to B}{\scriptstyle \to I}$$

The rule that allows us to permute premises is called CI. Any sequent of the form $A \vdash A$ is an axiom of the system (and so can be a starting point for a proof). We will explain the rules of implication introduction and elimination soon.

The introduction and elimination rules for the connectives are called *meaning rules*, since they are supposed to govern the meanings of the connectives.

3 Meaning Rules

Restall's systems are natural deduction systems (and not Gentzen sequent systems) because each connective has an introduction and elimination rule, just as they do in Fitch-style systems. Here we will only treat implication, conjunction, and disjunction. Negation is rather more difficult.

The implication introduction rule is the following:

$$\frac{X; A \vdash B}{X \vdash A \to B} \to I$$

This tells us that if A is the final premise (last undischarged hypothesis), we can push it through the turnstile (discharge it) to make an implication. Clearly, this mirrors the implication introduction rule of the Fitch-style system rather closely. Similarly, the implication elimination rule is rather like that of the Fitch-style system in that they both are closely modelled on modus ponens.

$$\frac{X \vdash A \quad Y \vdash A \to B}{X; Y \vdash B} \to E$$

Like the $\to E$ rule of the Fitch-style system for **R**, this rule tells us that the conclusion B was derived from the combination of the premises that gave us A and $A \to B$.

The conjunction rules are rather simple and are also very similar to the rules for the Fitch-style system:

$$\frac{X \vdash A \quad X \vdash B}{X \vdash A \wedge B} \wedge I$$

$$\frac{X \vdash A \wedge B}{X \vdash A} \qquad \frac{X \vdash A \wedge B}{X \vdash B} \wedge E$$

Like the Fitch-style system, we cannot use conjunction introduction unless both conjuncts are derived from the same hypotheses.

The disjunction introduction rules are very familiar looking as well:

$$\frac{X \vdash A}{X \vdash A \vee B} \qquad \frac{X \vdash B}{X \vdash A \vee B} \vee I$$

The disjunction elimination rule may not look familiar, but it is very reasonable:

$$\frac{Y(A) \vdash C \quad Y(B) \vdash C \quad X \vdash A \vee B}{Y(X) \vdash C} \vee E$$

This tells us that if we can get C from using either A or B in a given context and we can obtain $A \vee B$ from X, then we can replace A or B in that context with X and still obtain C.

4 The Structural Rules

There are three structural rules that will include in all our systems. The first is the cut rule, viz.,

$$\frac{X \vdash A \quad Y(A) \vdash B}{Y(X) \vdash B} \text{cut}$$

The cut rule is not like the other structural rules, since it is about combining premise structures, whereas the others are about modifying single premise structures. The cut rule is useful, and we will use it, but we could do without taking it as primitive, since it can be derived using the meaning rules:

$$\cfrac{\cfrac{X \vdash A}{X \vdash A \vee A}\vee I \quad Y(A) \vdash B \quad Y(A) \vdash B}{Y(X) \vdash B}\vee E$$

This proof requires the use of the disjunction rules. Sometimes, however, it is useful to talk about fragments of our logics that do not contain disjunction. Then we appeal to the cut rule as a primitive. For example, in a logic that contains only conjunction the following rule can be derived:

$$\frac{X(A) \vdash C}{X(A \wedge B) \vdash C}$$

Here's an easy proof:

$$\cfrac{\cfrac{A \wedge B \ \vdash \ A \wedge B}{A \wedge B \vdash A}\wedge E \qquad X(A) \vdash C}{X(A \wedge B) \vdash C}\text{cut}$$

The other structural rules that we will include are called 'left push' and 'left pop'. These have to do with the zero-place structural connective 0. As we shall see, 0 is important for the understanding of a theorem in these systems. The left push and pop rules tell us that 0 is a "left identity" in the algebraic sense:

$$\text{(left push)} \quad X \Longleftarrow 0; X$$

$$\text{(left pop)} \quad 0; X \Longleftarrow X.$$

Here is a list of the other structural rules taken verbatim from Restall (2000) p 26:

Name	Label	Rule		
Associativity	B	X;(Y;Z)	$\Longleftarrow$	(X;Y);Z
Twisted Associativity	B′	X;(Y;Z)	$\Longleftarrow$	(Y;X);Z
Converse Associativity	B^C	(X;Y);Z	$\Longleftarrow$	X;(Y;Z)
Strong Commutativity	C	(X;Y);Z	$\Longleftarrow$	(X;Z);Y
Weak Commutativity	CI	X;Y	$\Longleftarrow$	Y;X
Strong Contraction	W	(X;Y);Y	$\Longleftarrow$	X;Y
Weak Contraction	WI	X;X	$\Longleftarrow$	X
Mingle	M	X	$\Longleftarrow$	X;X
Weakening	K	X	$\Longleftarrow$	X;Y
Commuted Weakening	K′	X	$\Longleftarrow$	Y;X

The reason why the arrow used to state the rule is backwards is that in constructing sequent style proofs we always start from the bottom, from what we want to prove. Proofs are much easier to construct in that way.

The basic system, without any structural rules (apart from cut, which can be eliminated) is called **B** (for the "basic system"). It was originally discovered by Richard Routley and Robert Meyer when they were constructing model theoretic semantics for relevant logics. We will return to the topic of **B** and other weak relevant logics in our final lecture tomorrow.

5 Relevant Logic (with and without Distribution)

We obtain a logic very close to **R** with the rules B, B′, C, and W (together with cut, and left push and pop). But in this logic we cannot derive the distribution rule, i.e.,

$$A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$$

As we shall prove in section 6 below, the Fitch-style and Restall-style systems for R without distribution make exactly the same inferences valid.

But how can we make valid distribution? In the Fitch-style system, we just added it as a primitive rule. Here another method will be used, due (independently) to Gregor Mints and J.M. Dunn. The trick is to add an extra structural connective , and to have it governed by different rules than ; obeys. So, let us add , together with the formation rule, if X and Y are structures, then so is X, Y, the following structural rules (this table is taken verbatim from Restall (2000) p 36):

Name	**Label**		**Rule**	
Associativity	eB	$X,(Y,Z)$	$\Longleftarrow$	$(X,Y),Z$
Commutativity	eCI	X,Y	$\Longleftarrow$	Y,X
Contraction	eWI	X,X	$\Longleftarrow$	X
Weakening	eK	X	$\Longleftarrow$	X,Y

We can now derive distribution as follows. First we do the following derivations:

$$\dfrac{\dfrac{\dfrac{B \vdash B}{B,\ A \wedge (B \vee C) \vdash B}{\scriptstyle eK} \qquad \dfrac{\dfrac{A \wedge (B \vee C)\ \vdash\ A \wedge (B \vee C)}{A \wedge (B \vee C) \vdash A}{\scriptstyle \wedge E}}{B, A \wedge (B \vee C) \vdash A}{\scriptstyle eK}}{B, A \wedge (B \vee C) \vdash A \wedge B}{\scriptstyle \wedge I}}{B, A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)}{\scriptstyle \vee I}$$

and

$$\dfrac{\dfrac{\dfrac{C \vdash C}{C,\ A \wedge (B \vee C) \vdash C}{\scriptstyle eK} \qquad \dfrac{\dfrac{A \wedge (B \vee C)\ \vdash\ A \wedge (B \vee C)}{A \wedge (B \vee C) \vdash A}{\scriptstyle \wedge E}}{B, A \wedge (B \vee C) \vdash A}{\scriptstyle eK}}{B, A \wedge (B \vee C) \vdash A \wedge C}{\scriptstyle \wedge I}}{B, A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)}{\scriptstyle \vee I}$$

But we also can easily derive

$$A \wedge (B \vee C) \vdash B \vee C$$

So, by disjunction elimination we obtain

$$A \wedge (B \vee C), A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$$

and, by contraction, we get

$$A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$$

which is what we want.

6 An Equivalence Proof

Lemma 1 *If there is a valid Fitch-style proof in the system for* $\boldsymbol{R}$ *without distribution of the form*

$$\begin{array}{|l} A^1_{\{i\}} \\ \quad \begin{array}{l} \\ \end{array} \\ \cdots \begin{array}{|l} A^n_{\{k\}} \\ \quad\vdots \\ C_{\{i,\dots,k\}} \end{array} \end{array}$$

then a sequent $A^1, ..., A^n \vdash C$ *is provable in the sequent system for* $\boldsymbol{R}$ *without distribution and if there is a Fitch-style proof of* $C_\emptyset$ *then* $0 \vdash C$ *is provable.*

Proof. We take a valid Fitch-style proof and construct a deduction of the corresponding sequent. for each line of the Fitch-style proof, B_α, we construct the sequent $\bar{\alpha} \vdash B$, where $\bar{\alpha}$ is the multiset $[A_j, ..., A_l]$, and each A_p (for $p \in \alpha$) is the corresponding hypothesis in the Fitch-style proof and $\bar{\emptyset} = 0$. Then we show, by induction on the length of the Fitch-style proof, that $\bar{\alpha} \vdash B$ is provable.

Case 1. $\alpha = \{j\}$ and $B_{\{j\}}$ is a hypothesis. Then we have for $\bar{\alpha} \vdash B$, $B \vdash B$, which is an axiom and so is certainly provable.

Inductive hypothesis: for all previous steps, the sequents $\bar{\beta} \vdash D$ are provable.

Case 2a. B_α results from previous steps by conjunction introduction. Thus, $B = E \wedge F$ for some formulas E and F and E_α and F_α are proven on previous lines of the Fitch-style proof. By the inductive hypothesis, $\bar{\alpha} \vdash E$ and $\bar{\alpha} \vdash F$ are provable. So, by the rule of conjunction introduction, $\bar{\alpha} \vdash E \wedge F$ is provable, hence $\bar{\alpha} \vdash B$ is provable.

Case 2b. B_α results from a previous step by conjunction elimination. Easy.

Case 3a and 3b. B_α results from a previous step by disjunction introduction. B_α results from previous steps by disjunction elimination. Try on your own.

Case 4a. B_α results from previous steps by implication introduction. Then, $B = E \to F$. Since B_α results from implication introduction, we have, in the

Fitch-style proof,

$$\begin{array}{l|l} & E_{\{j\}} \\ & \quad \vdots \\ & F_{\alpha \cup \{j\}} \\ \multicolumn{2}{l}{E \to F_\alpha} \end{array}$$

By the inductive hypothesis, $\overline{\alpha \cup \{j\}} \vdash F$ is provable in the sequent system. This is the same as $\bar{\alpha} \sqcup \overline{[j]} \vdash F$. But $\overline{[j]} = [E]$, so we have $\bar{\alpha} \sqcup [E] \vdash F$. By implication introduction for the sequent system, then, we also have

$$\bar{\alpha} \vdash E \to F$$

which is what we want.

Case 4b. B_α results from previous steps by implication elimination. Thus, there are previous steps in the Fitch-style proof, $E \to B_\beta$ and E_γ such that $\beta \cup \gamma = \alpha$. Then, by the inductive hypothesis, $\bar{\beta} \vdash E \to B$ and $\bar{\gamma} \vdash E$ are provable. By the rule of implication elimination, we can thus prove

$$\bar{\beta}; \bar{\gamma} \vdash B$$

But, because of the associativity and commutativity of ; for **R**, this is the same as

$$\overline{\beta \cup \gamma} \vdash B,$$

which is what we want, since $\beta \cup \gamma = \alpha$. ■

We now prove the converse.

Lemma 2 *If the sequent $A^1, ..., A^n \vdash C$ is provable, there is a valid Fitch-style proof of the form*

$$\begin{array}{|l} A^1_{\{i\}} \\ \cdots \begin{array}{|l} A^n_{\{k\}} \\ \quad \vdots \\ C_{\{i,\ldots,k\}} \end{array} \end{array}$$

Proof. Here is a sketch of how this proof works (the fully worked out proof is extremely long). We prove this lemma by an induction on the length of the derivation of $A^1, ..., A^n \vdash C$.

Case 1. Suppose that $A^1, ..., A^n \vdash C$ is an axiom, i.e. is $C \vdash C$. Then we have a proof in the Fitch-style system, i.e.

$$\begin{array}{|ll} C_{\{i\}} & hyp \\ C_{\{i\}} & reit \end{array}$$

Inductive Hypothesis: All of the previous sequents in the derivation of $A^1, ..., A^n \vdash C$ are provable.

Case 2. Suppose that $A^1, ..., A^n \vdash C$ follows by a meaning rule. This is straightforward and like what we did in the proof of lemma 1.

Case 3. Suppose that $A^1, ..., A^n \vdash C$ follows from other provable sequents by a structural rule. We have the rules B, C, and W to consider. For B and C, you need to convince yourself that if C_α is a step in a valid Fitch-style proof, it does not matter what order the hypotheses are given. We can prove this by showing that we can just rewrite all the undischarged hypotheses at the beginning in whatever order and then use the reiteration rule to reconstruct the proof. Similarly, the fact that we can reiterate freely into subproofs means that we never have to assume the same formula twice to prove a given formula. We could have always proven it by assuming it just once. This takes care of W. (Note that you should be careful. Recall our proof of $\neg\neg(\neg\neg A \supset A)$ in intuitionist logic in lecture 2. In this proof we assumed $\neg\neg A$ more than once. But we needed to do so to discharge it more than once. Here we are only talking about undischarged hypotheses.) ■

Now we can state our equivalence theorem.

Theorem 3 (Equivalence) *There is a valid Fitch-style proof in the system for **R** without distribution of the form*

$$\begin{array}{|l} A^1_{\{i\}} \\ \quad \begin{array}{l|l} \ldots & A^n_{\{k\}} \\ & \quad \vdots \\ & C_{\{i,\ldots,k\}} \end{array} \end{array}$$

*if and only if a sequent $A^1, ..., A^n \vdash C$ is provable in the sequent system for **R** without distribution and if there is a Fitch-style proof of $C_\emptyset$ then $0 \vdash C$ is provable.*

Proof. Follows directly from lemmas 1 and 2. ■

7 Intuitionist Logic

The Restall system for intuitionist logic includes all of the rules of his table together with left pop and left push. Alternatively, we can define intuitionist logic as the logic that admits K and the following rule:

$$\text{Self-Distribution S } (X;Z);(Y;Z) \Longleftarrow (X;Y);Z$$

From S, we can derive the key axiom of intuitionist logic (formulated as a Hilbert-style axiom system), $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$. The following is a derivation of this axiom:

$$
\dfrac{
\dfrac{A \to (B \to C) \vdash A \to (B \to C) \qquad A \vdash A}{A \to (B \to C); A \vdash B \to C}{\scriptstyle \to E}
\qquad
\dfrac{A \to B \;\vdash\; A \to B \qquad A \vdash A}{A \to B; A \;\vdash\; B}{\scriptstyle \to E}
}{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{(A \to (B \to C); A);\; (A \to B; A) \;\vdash C}{(A \to (B \to C);\; A \to B); A \vdash C}{\scriptstyle S}
}{A \to (B \to C);\; A \to B \vdash A \to C}{\scriptstyle \to I}
}{(A \to (B \to C)) \vdash ((A \to B) \to (A \to C))}{\scriptstyle \to I}
}{0; (A \to (B \to C)) \vdash ((A \to B) \to (A \to C))}{\scriptstyle \text{left push}}
}{0 \vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))}{\scriptstyle \to I}
}{\scriptstyle \to E}
$$

From S, K, left push, and left pop, we can derive all the other rules. For example, here are derivations of four of the rules. B:

$$
\begin{array}{lll}
X; (Y; Z) & & \\
\Longleftarrow & (X; Z); (Y; Z) & \mathsf{K} \\
\Longleftarrow & (X; Y); Z & \mathsf{S}
\end{array}
$$

C:

$$
\begin{array}{lll}
(X; Y); Z & & \\
\Longleftarrow & (X; Y); (Z; Y) & \mathsf{K} \\
\Longleftarrow & (X; Z); Y & \mathsf{S}
\end{array}
$$

CI:

$$
\begin{array}{lll}
X; Y & & \\
\Longleftarrow & (0; X); Y & \text{left push} \\
\Longleftarrow & (0; Y); X & \mathsf{C} \\
\Longleftarrow & Y; X & \text{left pop}
\end{array}
$$

W:

$$
\begin{array}{lll}
(X; Y); Y & & \\
\Longleftarrow & (X; Y); (0; Y) & \text{left push} \\
\Longleftarrow & (X; 0); Y & \mathsf{S} \\
\Longleftarrow & (0; X); Y & \mathsf{CI} \\
\Longleftarrow & X; Y & \text{left pop}
\end{array}
$$

As an exercise, prove all the other rules from the table, using S

8 Further Reading

- J.R. Hindley and J.P. Seldin, *Introduction to Combinators and λ-Calculus*, Cambridge: Cambridge University Press, 1986
- Greg Restall, *An Introduction to Substructural Logics*, London: Routlege, 2000

ANU Logic Summer School
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Non-Classical Logics
Lecture 5: Substructural Logics

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1 Substructural Logics

In lecture 4 we looked at only two systems – intuitionist logic and **R**. But clearly there are many other systems we can construct using these rules. Any system that rejects one or more of the structural rules is called a *substructural logic*. **R** is a substructural logic, but intuitionist logic is not. In this lecture we will briefly survey some substructural logics other than **R**.

2 Weaker Relevant Logics

A relevant logic, roughly, is a logic that rejects the paradoxes of material and strict implication. So, every subsystem of a relevant logic is a relevant logic. Thus, there are many relevant logics other than **R**. Here we will look at a motivation for a class of weak relevant logics.

When Georg Cantor and Gottlob Frege first presented their different versions of modern set theory, they adopted variants of the *naïve axiom of comprehension*. This axiom says that for any formula (with at most x free),

$$\exists x \forall y (y \in x \leftrightarrow A).$$

As Ernst Zermelo and Bertrand Russell first showed, this axiom (together with classical logic) gets us into a lot of trouble. They derived what has become known as Russell's paradox. That is, they showed that from this axiom one could prove that there is a set that belongs to itself if and only if it does not belong to itself. Given the law of excluded middle (and other classical principles) this entails a contradiction.

Here we are not as interested in Russell's paradox as we are in Curry's paradox. For some relevant logicians, the derivation of a contradiction is not by itself a worry. These logicians – called *dialetheists* (for example, Graham Priest and Richard Sylvan) – believe that there are some true contradictions. In relevant logic, it is not the case that we can derive every proposition from a contradiction. It is triviality, not inconsistency, that worries dialetheists. So, Russell's paradox is not by itself a worry. Curry's paradox, on the other hand, does threaten to trivialize logical systems.

Here is a derivation of Curry's paradox. Suppose that we accept naïve comprehension. In terms of our sequent systems, this licences the following axioms:

$$\text{SC } t \in \{x : A(x)\} \vdash A(t)$$

$$\text{SA } A(t) \vdash t \in \{x : A(x)\}$$

'SC' stands for 'set conversion' and 'SA' stands for 'set abstraction'. Let p be an arbitrary proposition ('the moon is made of green cheese', 'pigs fly', 'New Zealand will win the next soccer world cup', ...). Using SC and SA in the context of the Restall system for **R**, we can prove that p. Let c be the set $\{x : x \in x \to p\}$. The naive comprehension principle allows us to postulate the existence of a set corresponding to any formula. Here the formula is $x \in x \to p$. Now we have

$$\cfrac{\cfrac{\cfrac{}{c \in c \ \vdash \ c \in \{x : x \in x \to p\}}{=} \qquad \cfrac{}{c \in \{x : x \in x \to p\} \ \vdash \ c \in c \to p}{SC}}{c \in c \vdash \ c \in c \to p}{\text{cut}} \qquad c \in c \vdash c \in c}{c \in c; c \in c \vdash p}{\to E}$$

$$\cfrac{c \in c; c \in c \vdash p}{c \in c \vdash p}{WI}$$

We now can use left pop and $\to I$ to prove

$$0 \vdash c \in c \to p$$

And then we use set abstraction to prove that $0 \vdash c \in \{x : x \in x \to p\}$, which is just

$$0 \vdash c \in c$$

But we already have a derivation of $c \in c \vdash p$. So, by cut we have

$$0 \vdash p.$$

Therefore, we have proven that any arbitrary formula is a theorem. *This is a bad thing.*

Note that cut (which is derivable in all our systems) and WI are the only structural rules used in the proof. Thus, in order to formulate a non-trivial set theory with naïve comprehension, we need to reject WI. In fact, the rejection of WI by itself is not sufficient (there are other structural rules that are close enough to WI that they allow alternative proofs of the paradox). But, as Ross Brady has shown, there are a large number of weak relevant logics that can support a non-trivial naïve set theory.

3 Linear Logic

Not all logical systems are meant to deal with inferences about propositions. Linear logic was created as a logic of computational resources. Suppose that you know that if you have a certain amount of free memory you can run a particular program, call it 'program 1'. Suppose that you know that this amount of free memory will allow you to run another program, program 2. So, in slightly mixed notation we have

$$M \vdash Run(1)$$

and

$$M \vdash Run(2)$$

But there is a sense in which we definitely do *not* want to infer

$$M \vdash Run(1) \textit{ and } Run(2).$$

Linear logic adds an intensional form of conjunction, $\circ$ (fusion), to capture this notion (both ... and). And it rejects contraction. So we have

$$\frac{M \vdash Run(1) \quad M \vdash Run(2)}{M; M \vdash Run(1) \ \circ \ Run(2)}$$

This tells us that if we have two lots of free memory of that particular size, then we can run both programs. And this seems right. If we had WI, we would then be able to infer that one lot of memory would do the trick, but we know that's false. So contraction (and WI, which we can infer from contraction) must go.

4 Categorial Grammars

In the 1950s, Joe Lambek constructed what we now think of as substructural logics for the rules of natural language syntax. These have become known as *Lambek calculi*, and are also known under the more general heading of *categorial grammars*.

In order to understand how these systems work, let us look at a simple example. It is standard in most natural languages that if we put a noun phrase together in the right way with a verb phrase, we get a sentence. So we have the following rule:

$$NP + VP \longrightarrow S$$

Now, let's consider a particular noun phrase, *Lola*, and a particular verb phrase, *is asleep*. We use fusion (see the section on linear logic above) to mean concatenation. So, we can use the above rule to give us:

$$\textit{Lola} \circ \textit{is asleep} \vdash S$$

Fusion acts like a sort of intensional conjunction. But it is very weak. For example, it is not commutative (we don't want to say that *is asleep* $\circ$ *Lola* is a sentence).

We also add two types of implication, \ and /. For each of these, there are two sets of introduction and elimination rules:

$$\frac{\mathit{Lola} \circ \mathit{is\ asleep} \vdash S}{\mathit{Lola} \vdash VP/S} /I$$

$$\frac{\mathit{Lola} \circ \mathit{is\ asleep} \vdash S}{\mathit{is\ asleep} \vdash NP\backslash S} \backslash I$$

And

$$\frac{\mathit{Lola} \vdash VP/S \quad \mathit{is\ asleep} \vdash VP}{\mathit{Lola} \circ \mathit{is\ asleep} \vdash S} /E$$

$$\frac{\mathit{Lola} \vdash NP \quad \mathit{is\ asleep} \vdash NP\backslash S}{\mathit{Lola} \circ \mathit{is\ asleep} \vdash S} \backslash E$$

Notice that *Lola* and *is asleep* each have more than one syntactic type. For any expression, we can show that it has infinitely many types, through a process called *type raising*, that you can see at work in the use of the introduction rules for \ and /.

5 Further Reading

- Ross Brady, *Universal Logic*, Stanford: CSLI, 2006 (for weak relevant logic)
- Jean-Yves Girard, *The Blind Spot*, 2006, http://iml.univ-mrs.fr/~girard/coursang/coursang0.pdf.gz (for linear logic)
- Greg Restall, *An Introduction to Substructural Logics*, London: Routledge, 2000