Decidability of $\mathcal{SHI}$ with transitive closure of roles

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Example : Transitive Closure in Concept Axioms

- Devices have as their direct part a battery :
  $\texttt{Device} \sqsubseteq \exists \texttt{hasPart}.\texttt{Battery}$
- Devices have at some level of decomposition a battery :
  $\texttt{Device} \sqsubseteq \exists \texttt{hasPart}^+.\texttt{Battery}$

Example : Transitive Closure in Concept Axioms

- Devices have as their direct part a battery :
  $\texttt{Device} \sqcap \exists \texttt{hasPart.Battery}$
- Devices have at some level of decomposition a battery :
  $\texttt{Device} \sqcap \exists \texttt{hasPart}^{+}\texttt{.Battery}$

Remark :

If we define `hasPart` as a *transitive role*, we cannot distinguish the two concepts above

- OWL-DL is not expressive enough to describe these concepts

Problems

1. Decidability of OWL-DL ($\mathcal{SHOIN}$) with transitive closure of roles in concept axioms is known but there is not a practical algorithm ;
2. If we add transitive closure of roles to concept and role axioms even in $\mathcal{SHI}$, decidability of the resulting logic is still unknown

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1. Decidability of OWL-DL ($\mathcal{SHOIN}$) with transitive closure of roles in concept axioms is known but there is not a practical algorithm ;
2. If we add transitive closure of roles to concept and role axioms even in $\mathcal{SHI}$, decidability of the resulting logic is still unknown

Goal :

An algorithm for checking satisfiability in the logic $\mathcal{SHI}$ with transitive closure in concept and role axioms

Outline

1. The logic $\mathcal{SHI}_+$
2. Related Works : $\mathcal{SHIQ}, \mathcal{ALC}_{reg}$
3. Neighborhood and Normalization Tree
4. Normalization Tree with Cyclic Paths
5. Algorithm for concept satisfiability
6. Conclusion and Future Work

$\mathcal{SHI}_+$ = $\mathcal{SHI}$ with Transitive Closure of Roles

Syntax

- Concept names : N_C, role names : N_R ;
- Transitive closure of roles : $\{R^+ \mid R \in N_R\}$
- Inverse roles : $\{S^- \mid S \in N_R \cup \{R^+ \mid R \in N_R\}\}$,
- Role hierarchy $\mathcal{R} := \{R \sqsubseteq S\}$ where R, S are role names, transitive closures or inverses ($\mathcal{SHI}_+$-roles) ;
- Formulae inductively defined from N_C, $\mathcal{SHI}_+$-roles and logic constructors :
  $C := A \mid C \sqcap D \mid C \sqcup D \mid \neg C \mid \exists R.C \mid \forall R.C$
- Concept axioms $\mathcal{T} := \{C \sqsubseteq D\}$
- An ontology $\mathcal{O} := \mathcal{T} \cup \mathcal{R}$

$\mathcal{SHI}_+$ = $\mathcal{SHI}$ with Transitive Closure of Roles

Semantics

- An interpretation $\mathcal{I} = \langle \Delta, .^{\mathcal{I}} \rangle$ with $\Delta^{\mathcal{I}} \neq \emptyset$ and $.^{\mathcal{I}}$ a function s.t. $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$; $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$;
  $(C \sqcap D)^{\mathcal{I}}$:=$C^{\mathcal{I}} \cap D^{\mathcal{I}}$;$(C \sqcup D)^{\mathcal{I}}$:=$C^{\mathcal{I}} \cup D^{\mathcal{I}}$; $(\neg C)^{\mathcal{I}}$:=$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$;
  $(\exists R.C)^{\mathcal{I}}$:=$\{x \mid \exists y.y \in C^{\mathcal{I}} \wedge \langle x, y \rangle \in R^{\mathcal{I}}\}$;
  $(\forall R.C)^{\mathcal{I}}$:=$\{x \mid \langle x, y \rangle \in R^{\mathcal{I}} \Rightarrow y \in C^{\mathcal{I}}\}$;
  $R^{-\mathcal{I}}$:=$\{\langle x, y \rangle \mid \langle y, x \rangle \in R^{\mathcal{I}}\}$
  $P^{+\mathcal{I}}$:=$\bigcup_{n>0} (P^n)^{\mathcal{I}}$ with $(P^1)^{\mathcal{I}} = P^{\mathcal{I}}$, $(P^n)^{\mathcal{I}} = (P^{n-1})^{\mathcal{I}} \circ P^{\mathcal{I}}$;

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- An interpretation $\mathcal{I}$ which satisfies all axioms in $\mathcal{R}$ (resp. $\mathcal{T}$) is called a model of $\mathcal{R}$ (resp. $\mathcal{T}$), denoted $\mathcal{I} \models \mathcal{R}$ (resp. $\mathcal{I} \models \mathcal{T}$). A concept C is satisfiable w.r.t. $\mathcal{T}$ and $\mathcal{R}$ iff there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{R}$, $\mathcal{I} \models \mathcal{T}$ and $C^\mathcal{I} \neq \emptyset$;

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- Ontology consistency, subsumption $C \sqsubseteq D$ can be reduced to concept satisfiability.

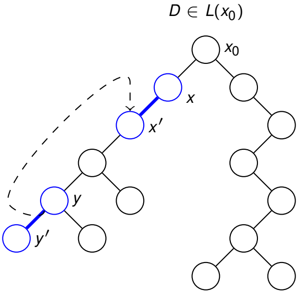
Related works : tableaux-based algorithms ($\mathcal{SHIQ}$ [Horrocks et al.], $\mathcal{ALC}_{reg}$ [Baader])

1. Tableaux :
   - being a possibly infinite graph whose nodes and edges are labelled ;
   - expressing as local properties the semantic constraints imposed by labels
2. Completion Trees :
   - Using expansion rules to express tableaux properties ;
   - Being built by applying expansion rules ;
   - Using blocking condition to ensure termination ;
   - Providing a finite representation of possibly infinite models ;

Why the usual blocking condition fails ?

Blocking condition :

$$L(x) = L(y)$$
$$L(x') = L(y')$$
$$L(\langle x, x' \rangle) = L(\langle y, y' \rangle)$$



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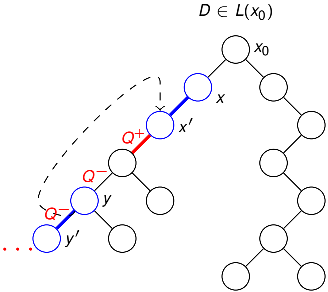

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Remark

If transitive closure is added to $\mathcal{SHI}$ then :

- Global properties are needed in tableaux
- The blocking condition is no longer sufficient

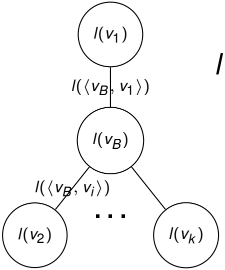
Key ideas of our approach to satisfiability in $\mathcal{SHI}_+$

- New tableaux :
  - Introducing a global property for satisfying transitive closures
- New construction of completion trees
  - Introducing neighborhood notion to capture all expansion rules for $\mathcal{SHI}$;
  - Tiling neighborhoods together to build a normalization tree by using the usual blocking condition ;
  - Satisfying transitive closure is translated into selecting a "good" normalization tree

Neighborhoods

D	:	$\mathcal{SHI}_+$ concept
$\mathtt{sub}(D)$	:	set of all sub-concepts of D
$\mathcal{T}, \mathcal{R}$	:	concept axioms and role hierarchy in $\mathcal{SHI}_+$
$\mathbf{R}$	:=	set of roles R occurring in $\mathcal{T}, \mathcal{R}, D$ with R^- and R^+
$\mathtt{sub}(\mathcal{T}, \mathcal{R})$	:=	set of all sub-concepts formed from $\mathtt{nnf}(\neg C \sqcup D)$ w.r.t. $\mathcal{R}$ where $C \sqsubseteq D \in \mathcal{T}$

$(v_B, N_B, l) =$

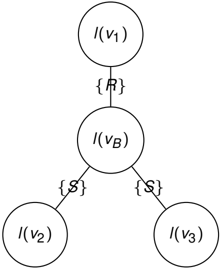


$$l(v) \in 2^{\mathtt{sub}(\mathcal{T},\mathcal{R})}, v \in N_B \cup \{v_B\}$$

$$l(\langle v_B, v_i \rangle) \in 2^{\mathbf{R}}, v_i \in N_B$$

all semantic constraints satisfied at v_B : valid neighborhood

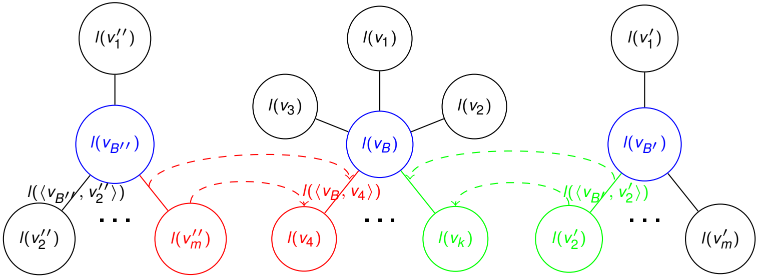
Valid neighborhood : example



$$l(v_B) = \{\exists R.C, \exists S.C, \exists S.D, E\}$$
$$l(v_1) = \{C, \forall R^{-}.E\}$$
$$l(v_2) = \{C, \exists R.C\}$$
$$l(v_3) = \{C\}$$

Saturated neighborhoods

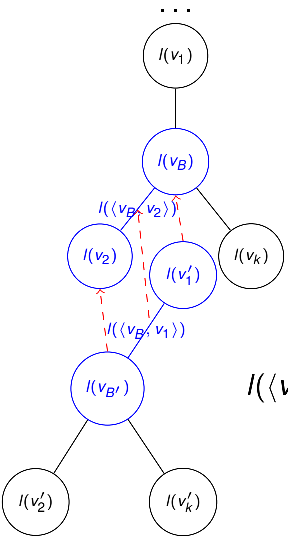
- A valid neighborhood (v_B, N_B, l) is *saturated* if for each valid neighborhood $(v_{B'}, N_{B'}, l)$ with $l(v_{B'}) = l(v_B)$ and for each $v' \in N_{B'}$, there exists $v \in N_B$ such that $l(v) = l(v')$ and $l(\langle v_B, v\rangle) = l(\langle v_{B'}, v'\rangle)$



We denote $\mathbb{B}$ for the set of saturated neighborhoods

Tiling saturated neighborhoods

$\mathbb{B} \ni (v_B, N_B, l) =$



$\mathbb{B} \ni (v_{B'}, N_{B'}, l) =$

Conditions for tiling :

$$l(v_B) = l(v_1')$$
$$l(v_{B'}) = l(v_2)$$
$$l(\langle v_B, v_2 \rangle) = \texttt{Inv}(\ l(\langle v_{B'}, v_1' \rangle)\ )$$

Normalization Tree

Let D be an $\mathcal{SHI}_+$ concept w.r.t. $\mathcal{T}$ and $\mathcal{R}$.

- A normalization tree **T** is built by tiling saturated neighborhoods,
- Tiling terminates at a node (it becomes a leaf) if the blocking condition is satisfied

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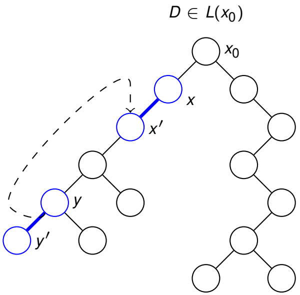
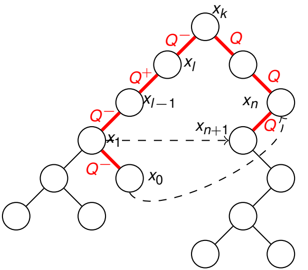


Normalization Tree with Cyclic Paths

$\langle x_0, x_1, \cdots, x_k, \cdots, x_n, x_{n+1} \rangle$ is a cyclic path for $\langle x_l, x_{l-1} \rangle \in E$ with $Q^+ \in L(\langle x_l, x_{l-1} \rangle)$ and $2 \le l \le k$ if

$$
\begin{gathered}
Q \in L(\langle x_h, x_{h+1} \rangle), n \ge h \ge l \\
Q^- \in L(\langle x_h, x_{h-1} \rangle), 1 \le h \le l-1 \\
L(x_1) = L(x_{n+1}) \\
L(x_0) = L(x_n) \\
L(\langle x_1, x_0 \rangle) = \mathtt{Inv}(\ L(\langle x_n, x_{n+1} \rangle)\ )
\end{gathered}
$$

$\mathbf{T} = (V, E, L)$



Decidability of $\mathcal{SHI}_+$

- **Theorem :** Let D be an $\mathcal{SHI}_+$ concept w.r.t. $\mathcal{T}$ and $\mathcal{R}$. D is satisfiable iff there is a normalization tree $\mathbf{T} = (V, E, L)$ such that for each $\langle x, y\rangle \in E$ with $Q^+ \in L(\langle x, y\rangle), Q \notin L(\langle x, y\rangle)$ there is a cyclic path for $\langle x, y\rangle$.
- **Algorithm (sketch) :**
  1. From $D, \mathcal{T}, \mathcal{R}$, finding $\mathbb{B}$ which is a set of saturated neighborhoods ;
  2. From $\mathbb{B}$, tiling neighborhoods to obtain a normalization tree $\mathbf{T} = (V, E, L)$;
  3. Building cyclic paths on $\mathbf{T}$.

Conclusion and Future Work

- Conclusion
  - An algorithm for deciding concept satisfiability in $\mathcal{SHI}_+$
    1. Separation of satisfying expansion rules for $\mathcal{SHI}$ from satisfying transitive closures by introducing neighborhood notion
    2. Translation of non-determinism caused by transitive closures into selection from normalization trees
  - Complexity : double exponential

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    1. Separation of satisfying expansion rules for $\mathcal{SHI}$ from satisfying transitive closures by introducing neighborhood notion
    2. Translation of non-determinism caused by transitive closures into selection from normalization trees
  - Complexity : double exponential
- Future Work
  - A goal-oriented algorithm
  - Adding qualifying number restriction ($\mathcal{Q}$) and nominals ($\mathcal{O}$) to $\mathcal{SHI}_+$ (i.e. adding transitive closure of roles to OWL-DL)

Questions ?

Thank you

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