Stability of the Contour

Chicago '09

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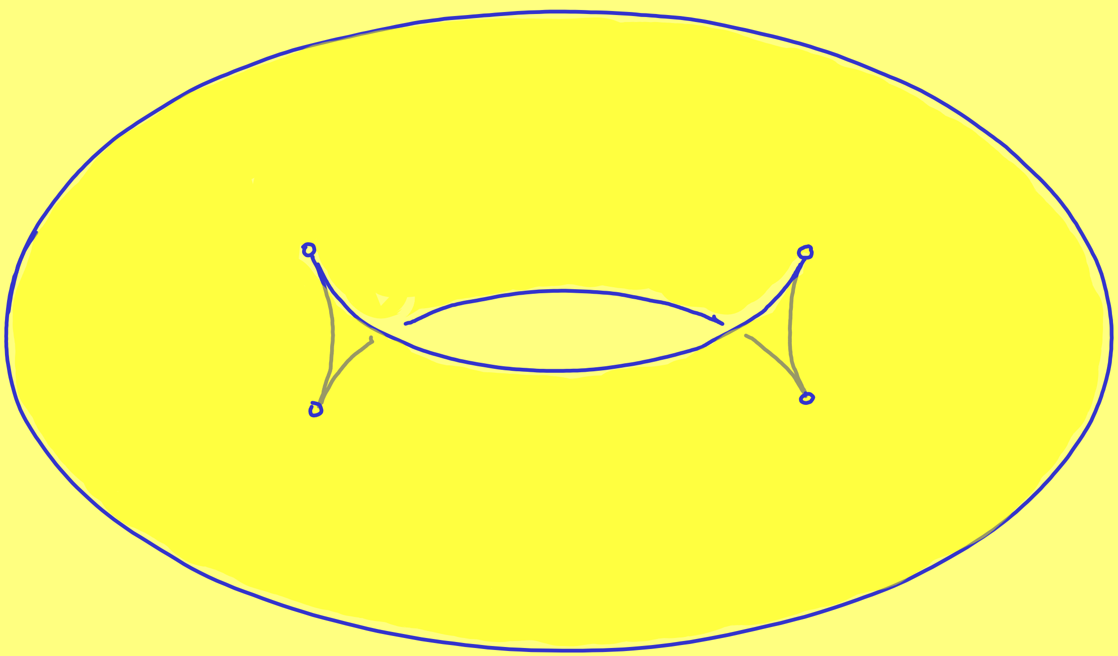
Edelsbrunner Duke
Morozov Stanford
Patel Duke

I Mappings

II Persistence

III Stress

I.1 The Contour



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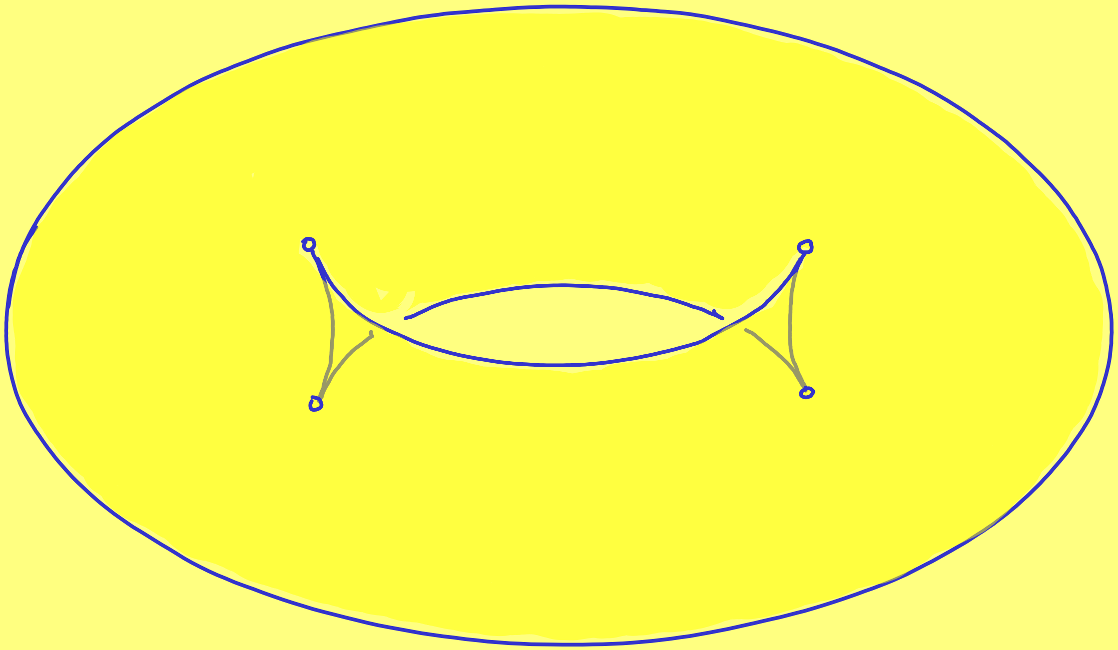
$$f : \mathbb{M} \to \mathbb{R}^2$$

generic smooth

compact, orientable 2-mfld w/o boundary



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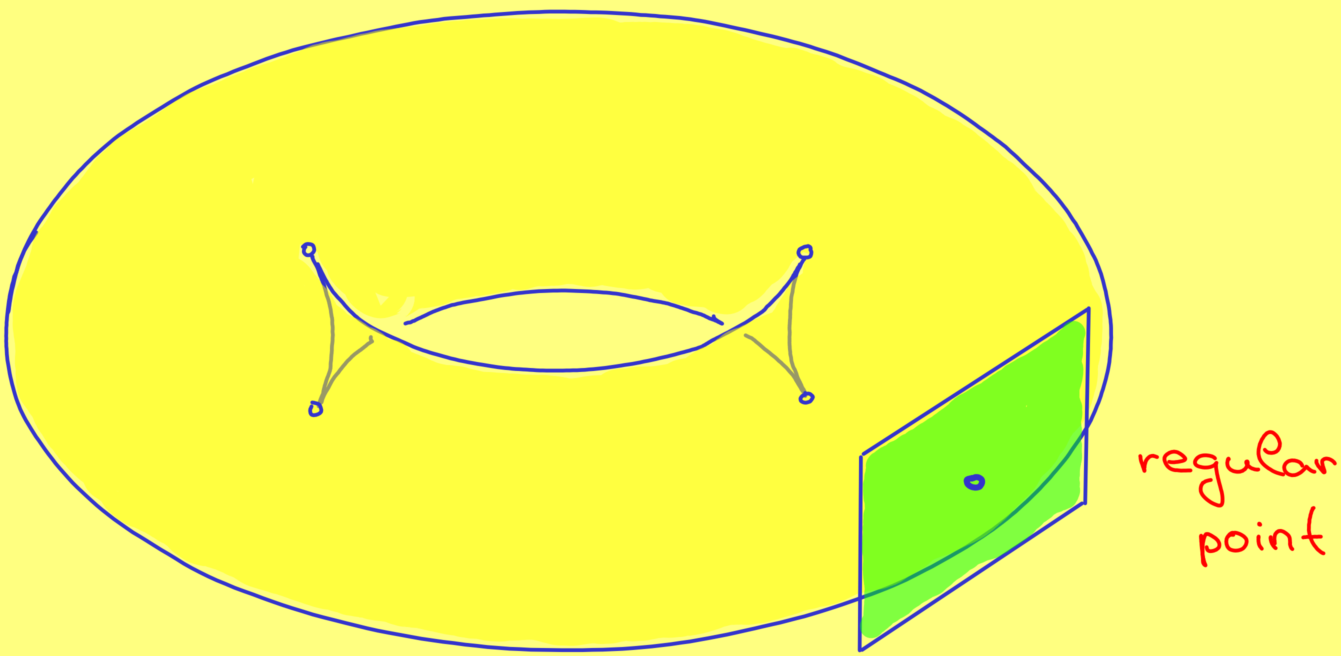

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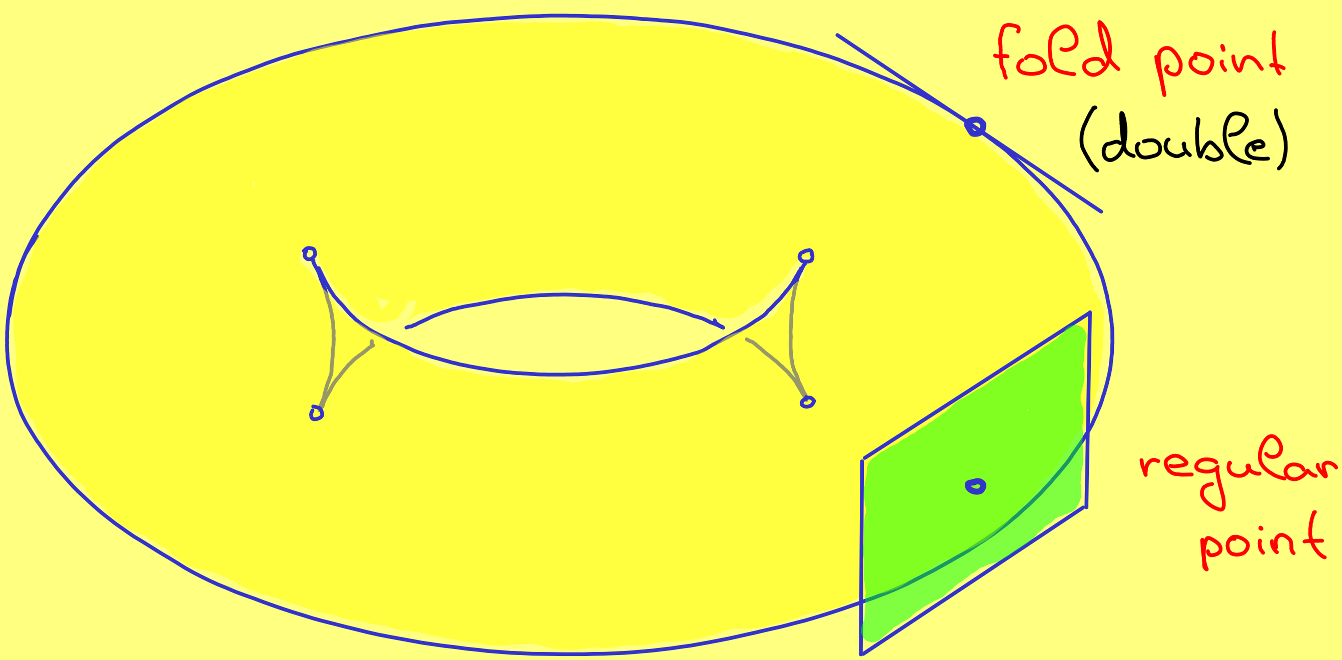

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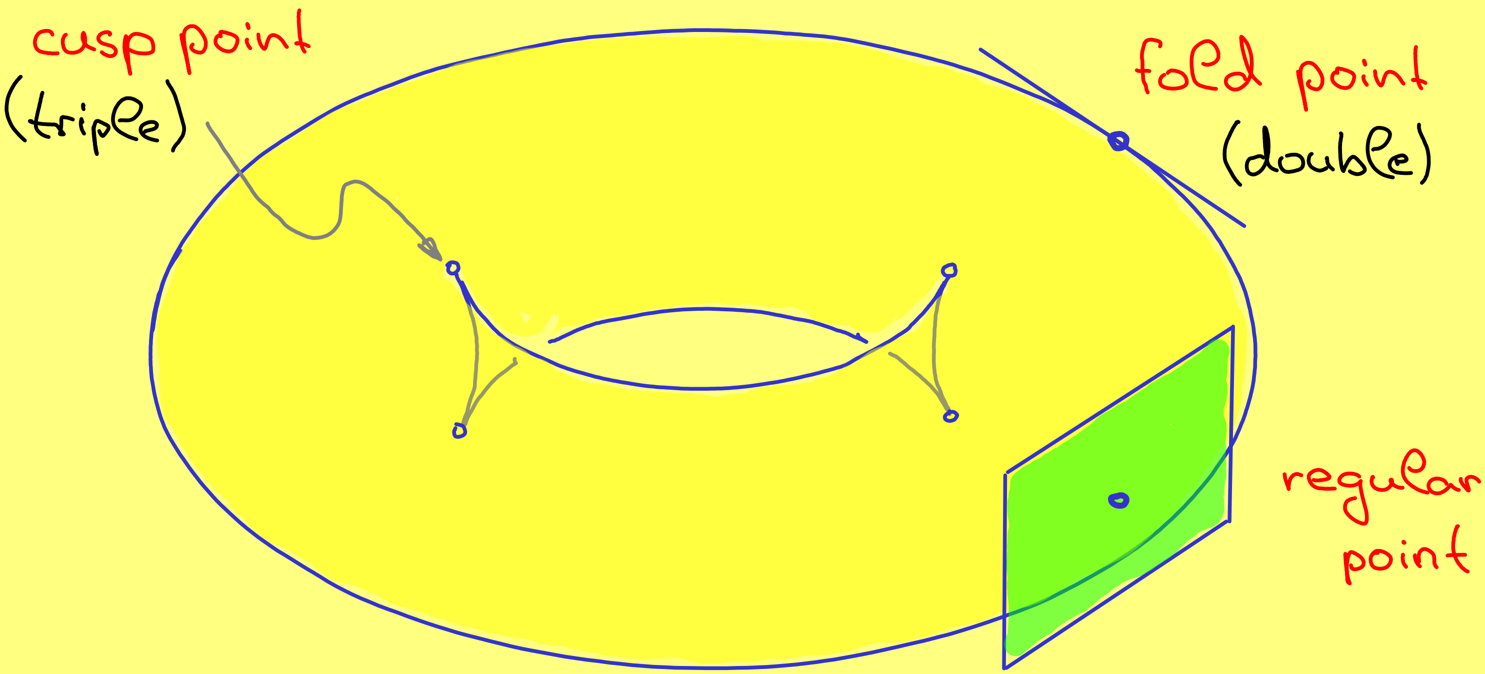

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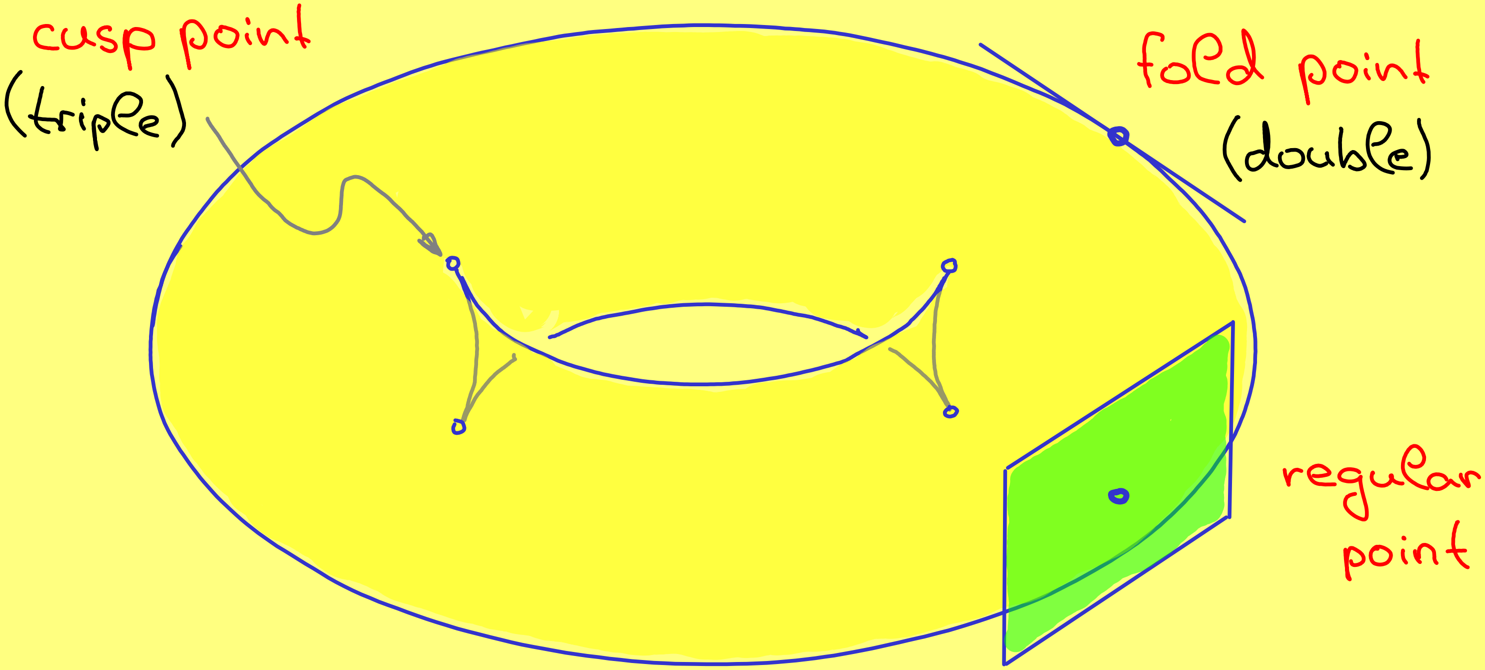

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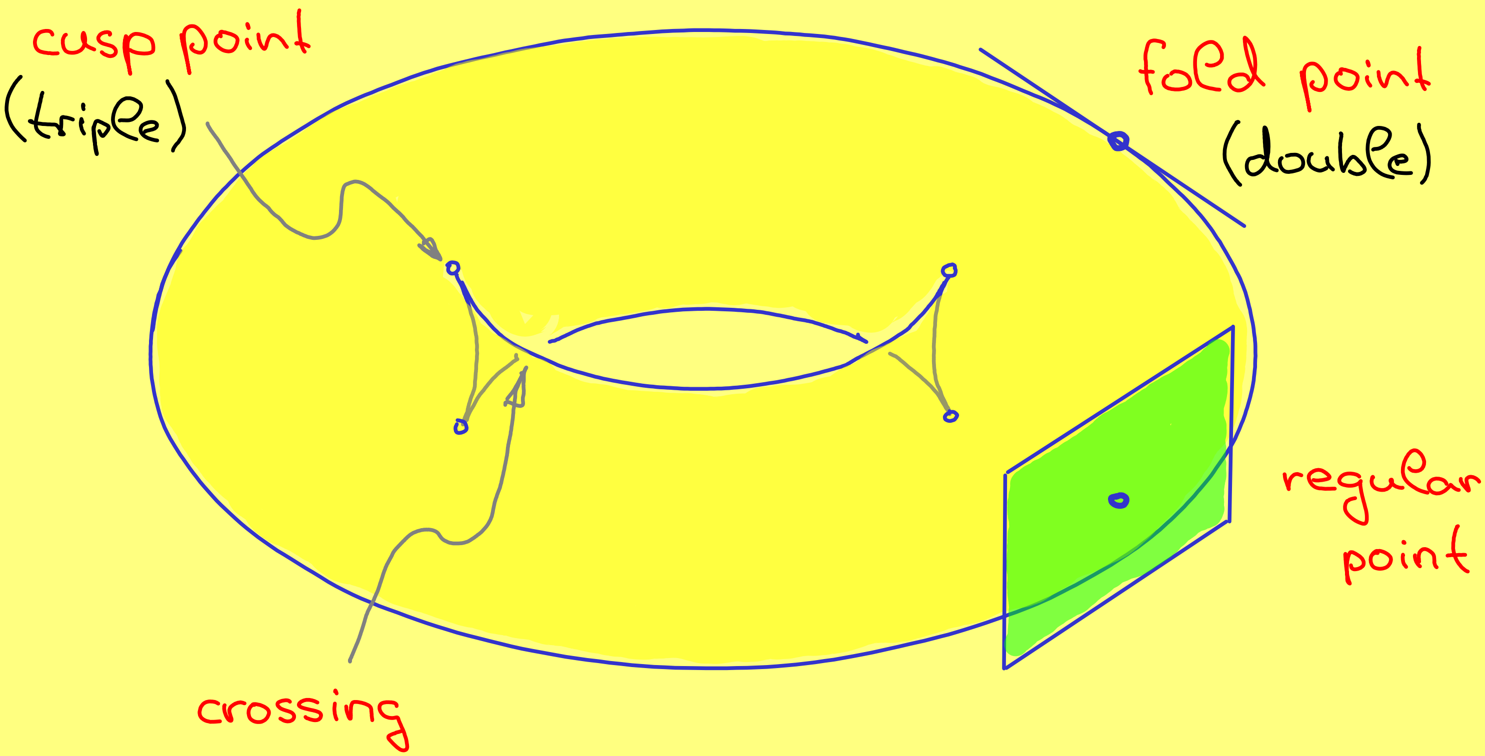

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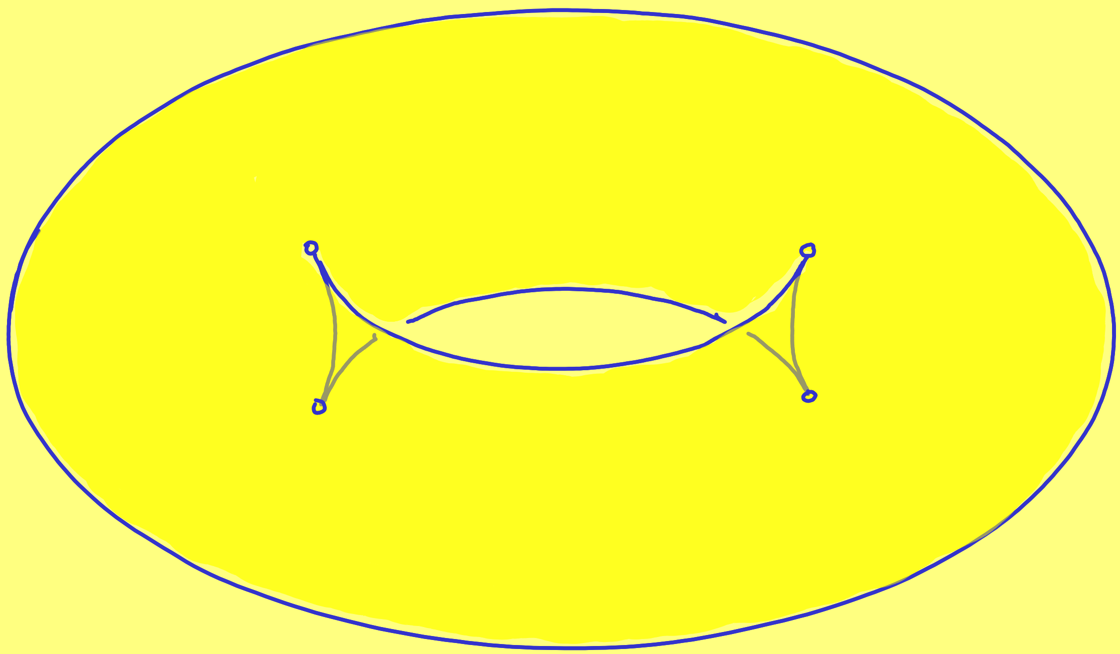

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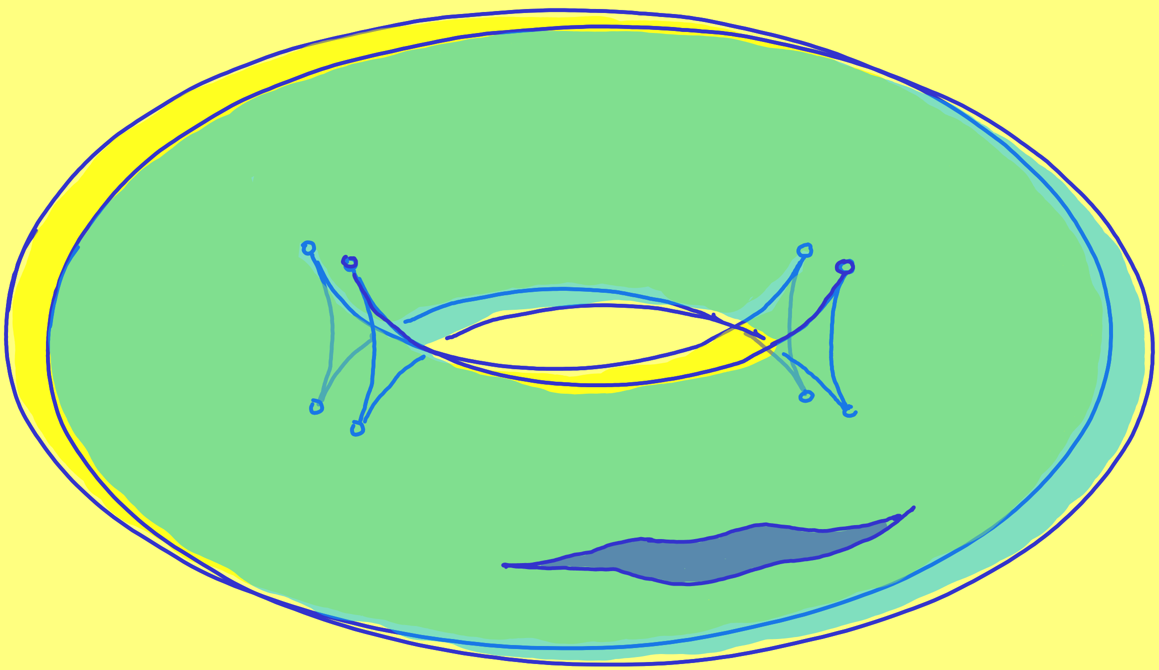

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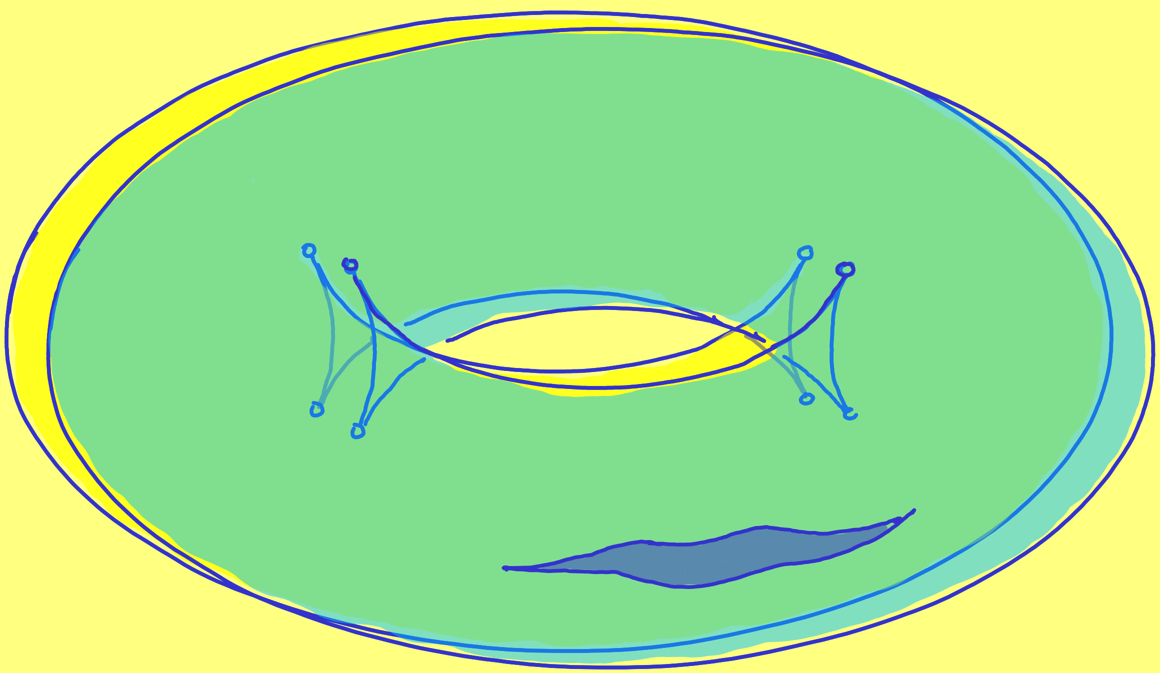

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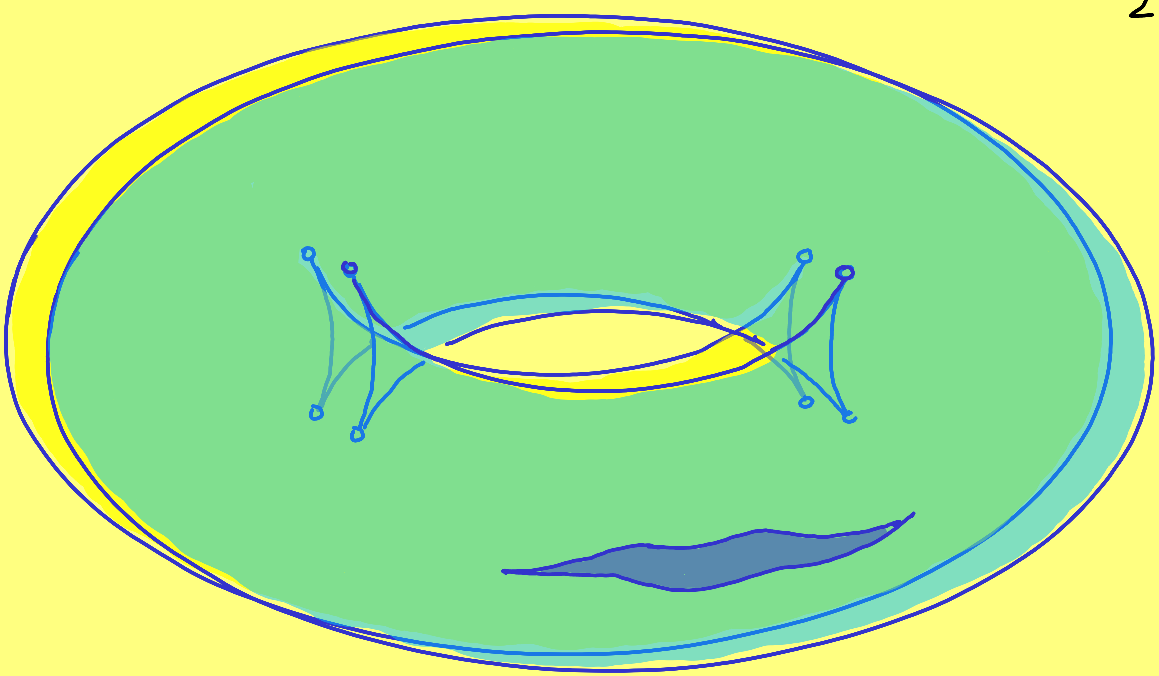

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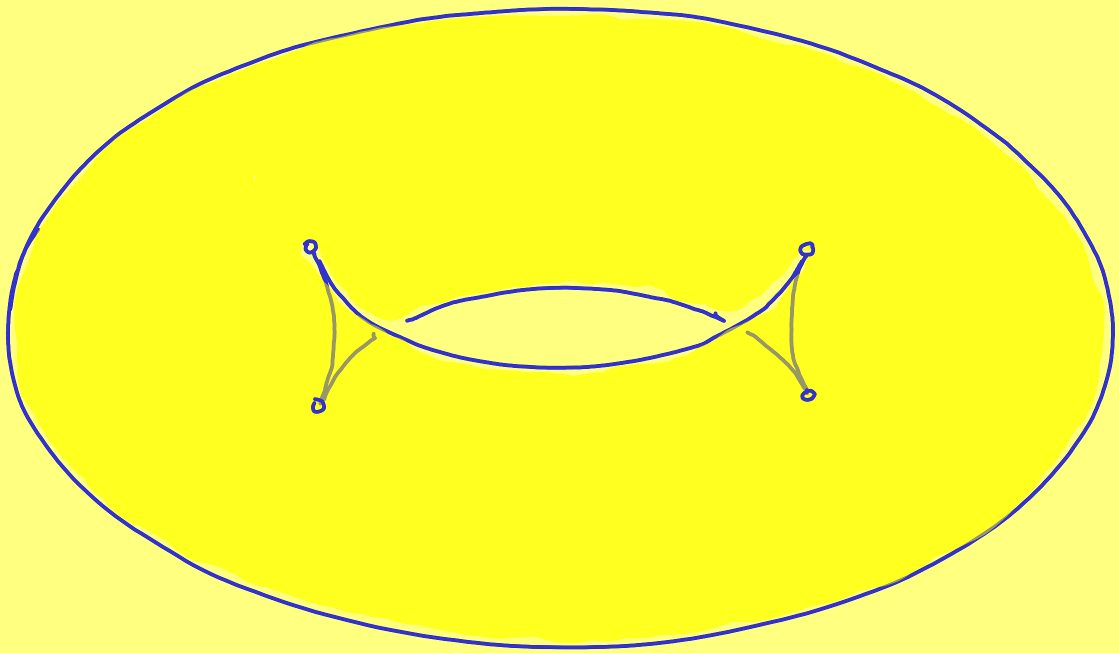

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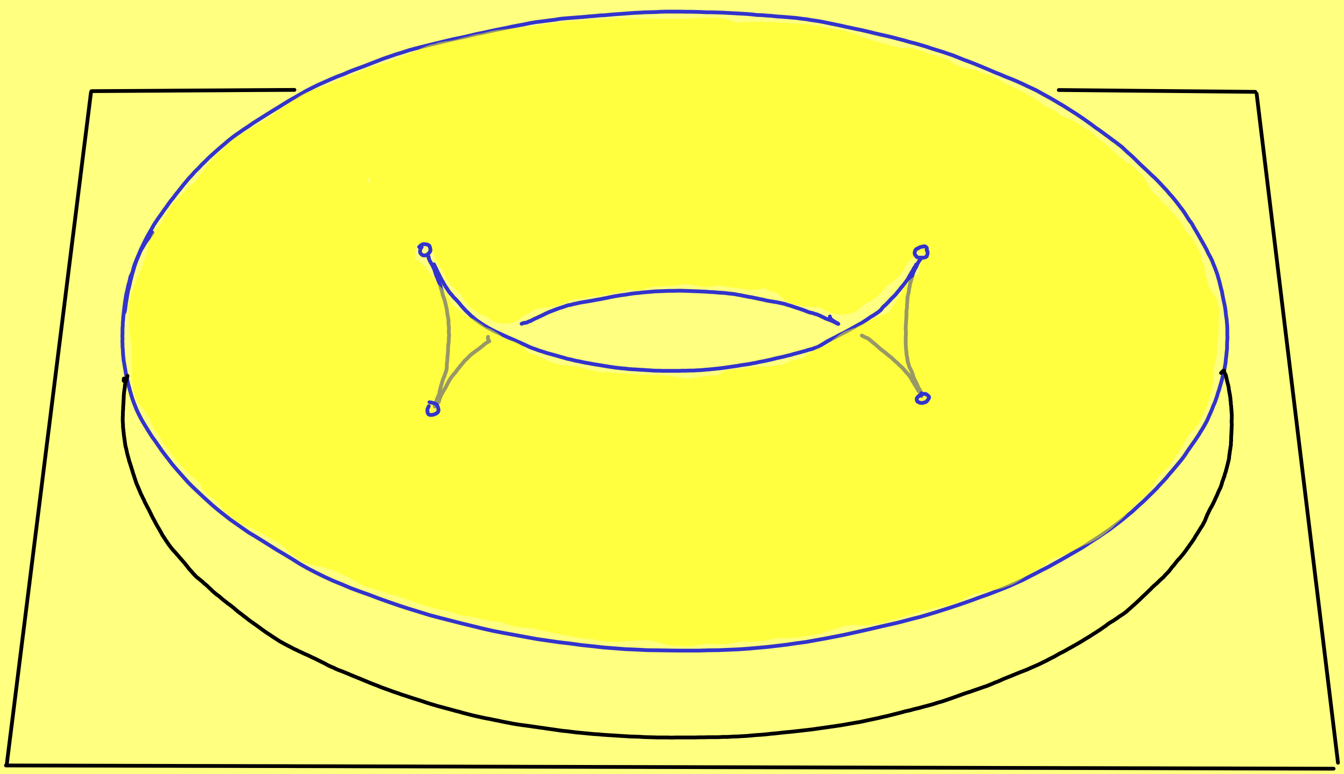

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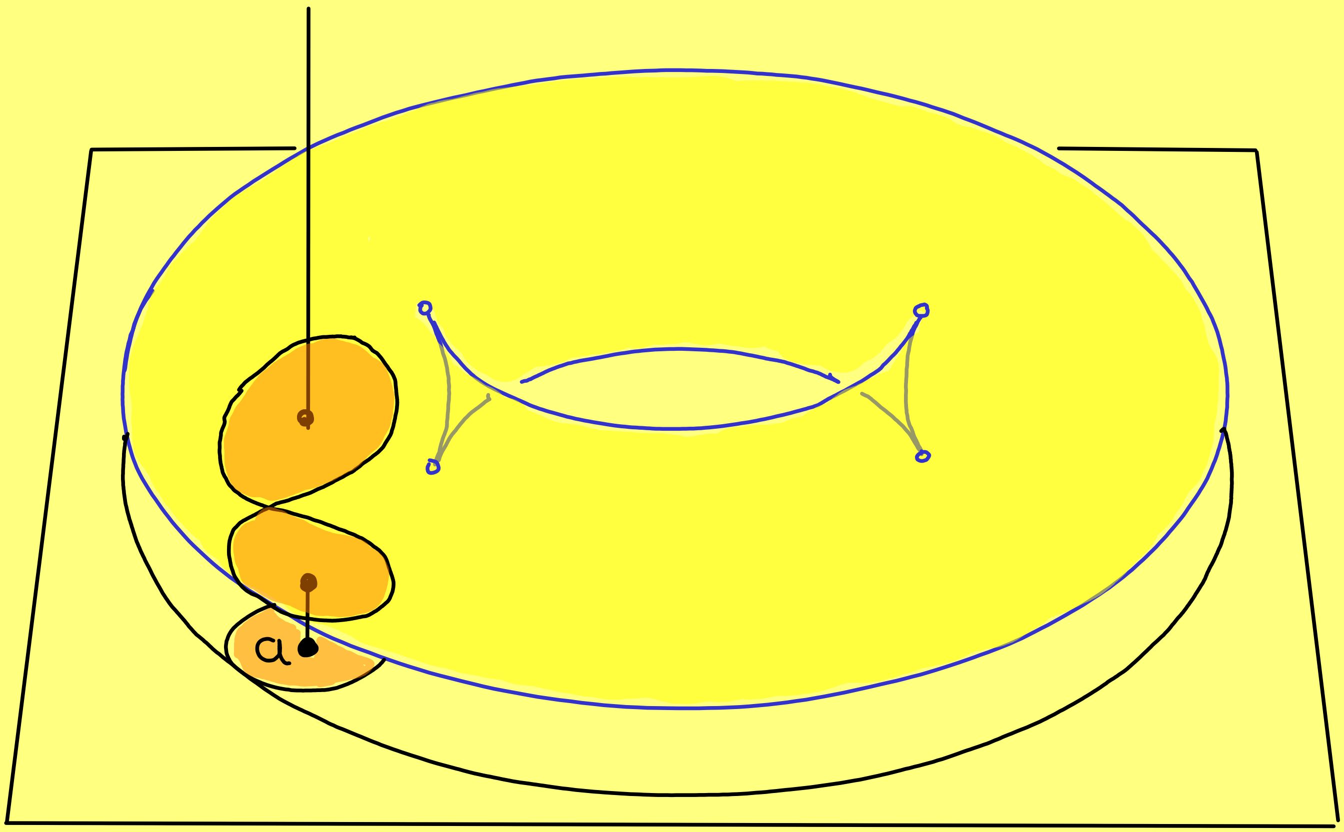

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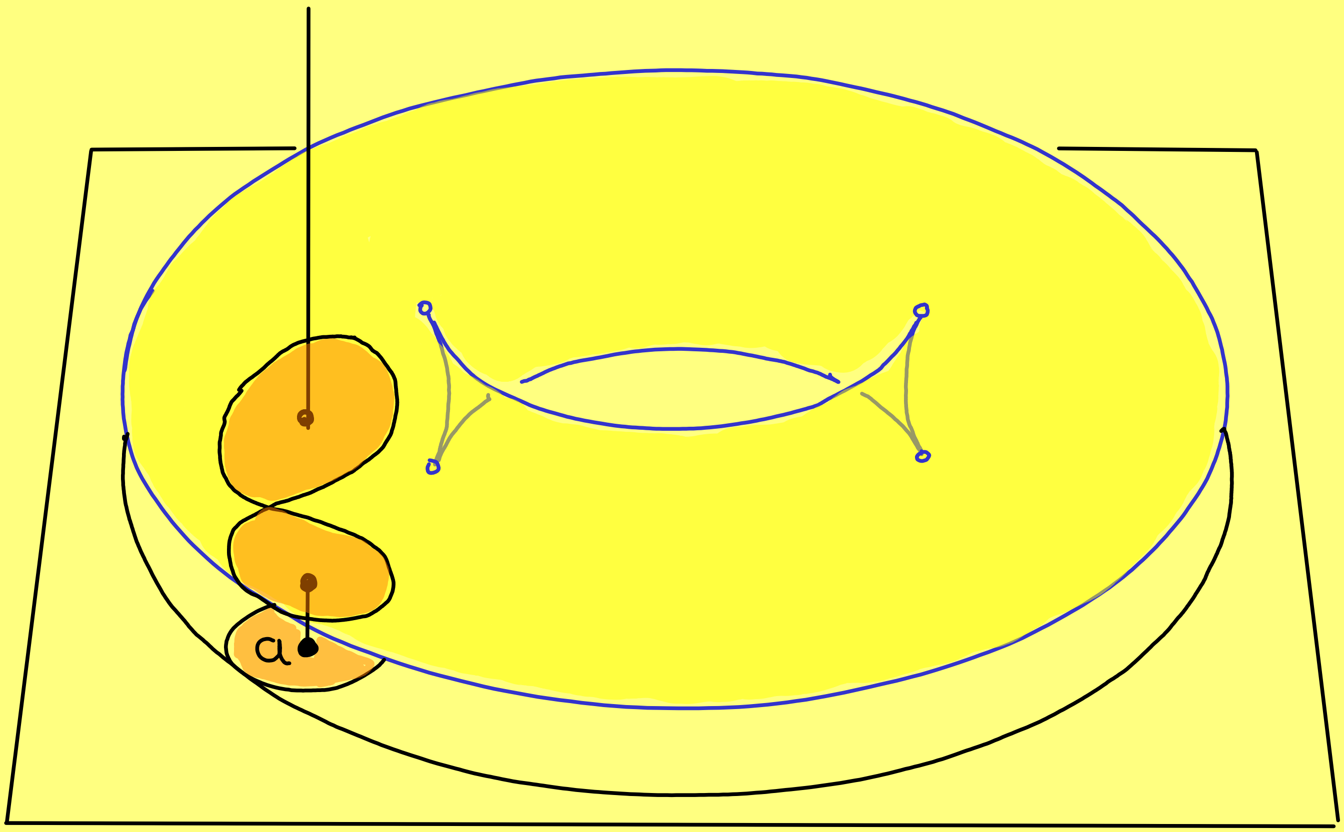

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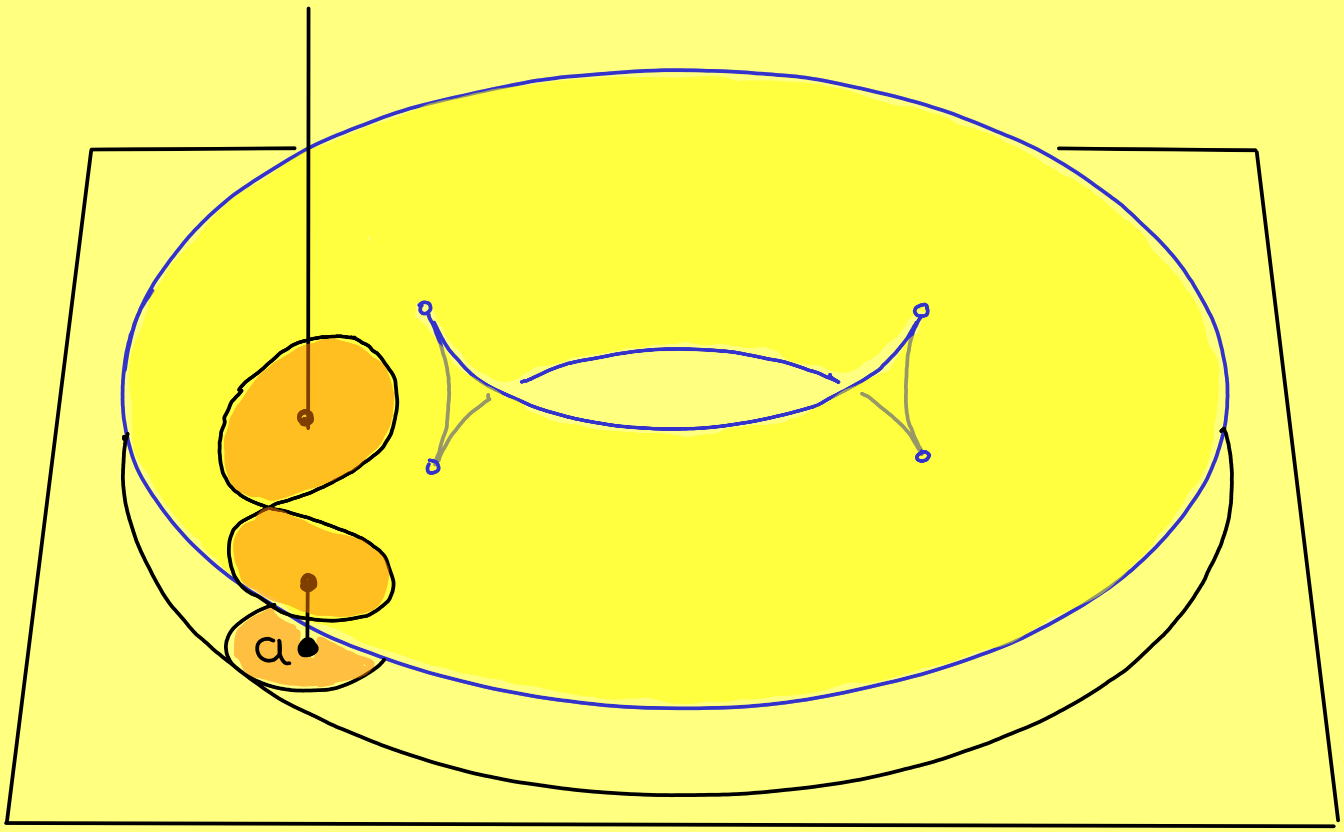

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x crit. for f_a

$\Downarrow$

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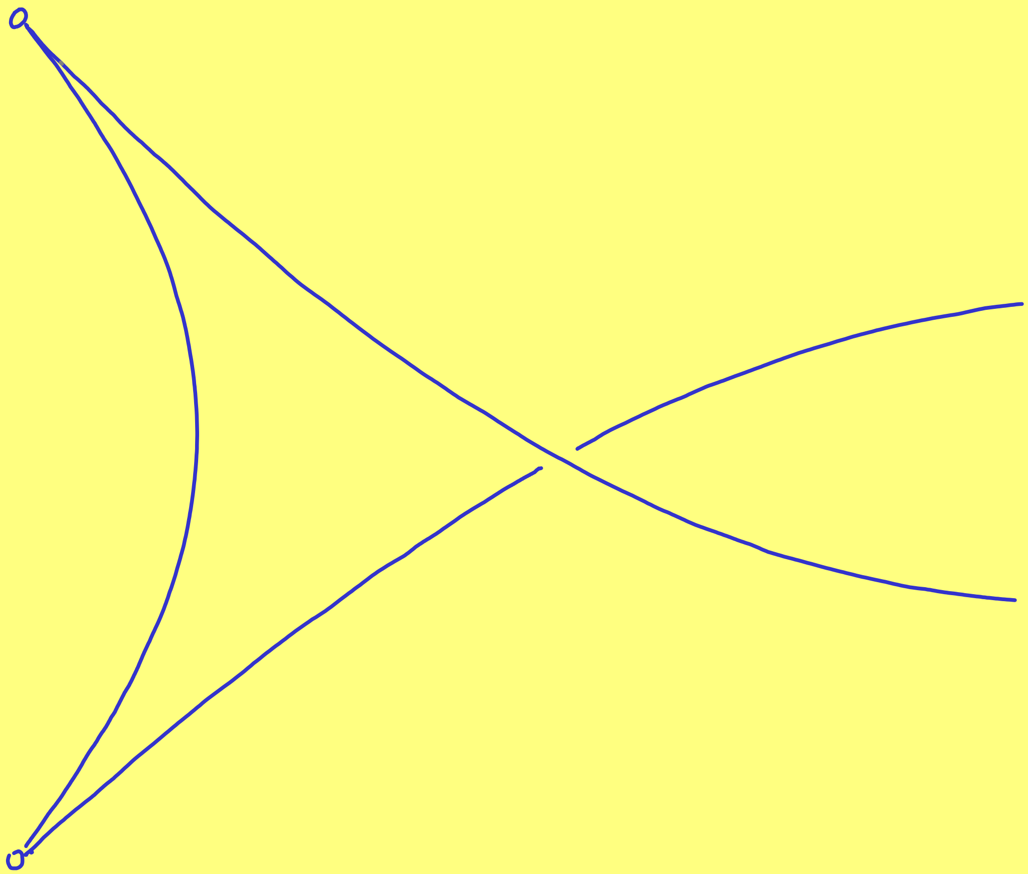
(III) $\mathrm{Contour}(f)$ has finite # of cusps + crossings.

I Mappings

II Persistence

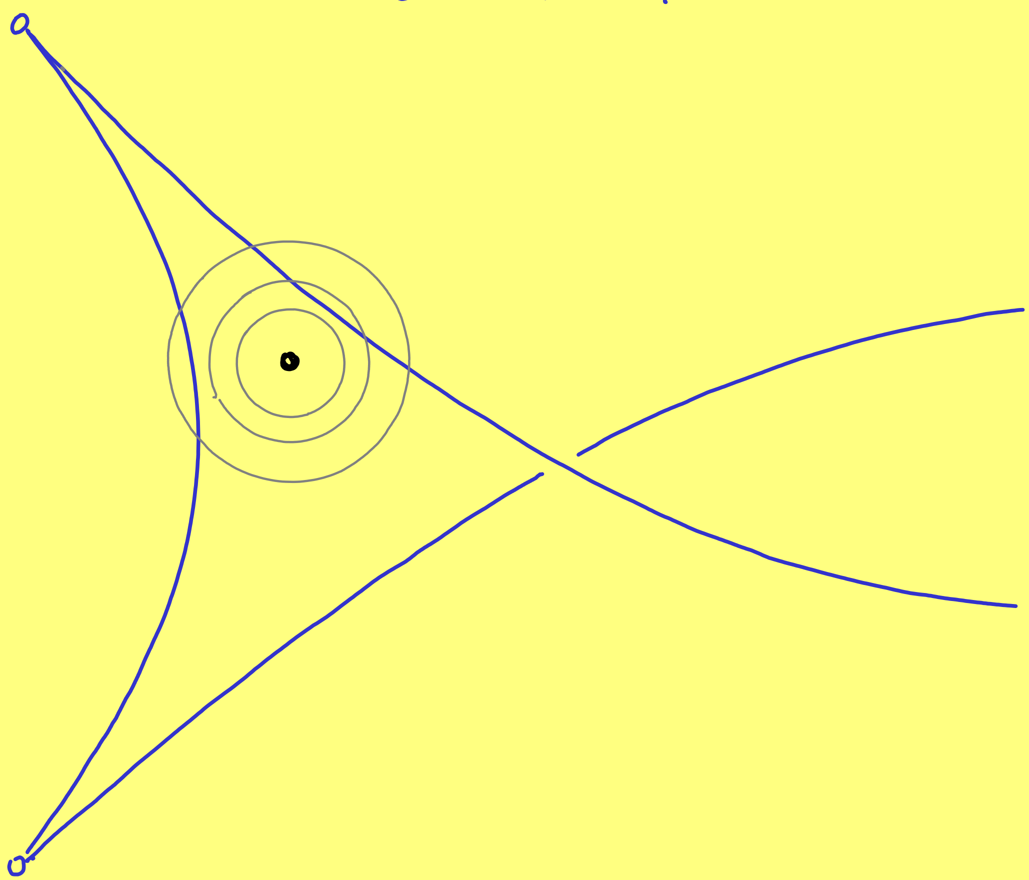
III Stress

II.1 FILTRATION



II.1 Filtration

$F_r = H_0(\mathbb{M}_r(a))$ indexed by radius



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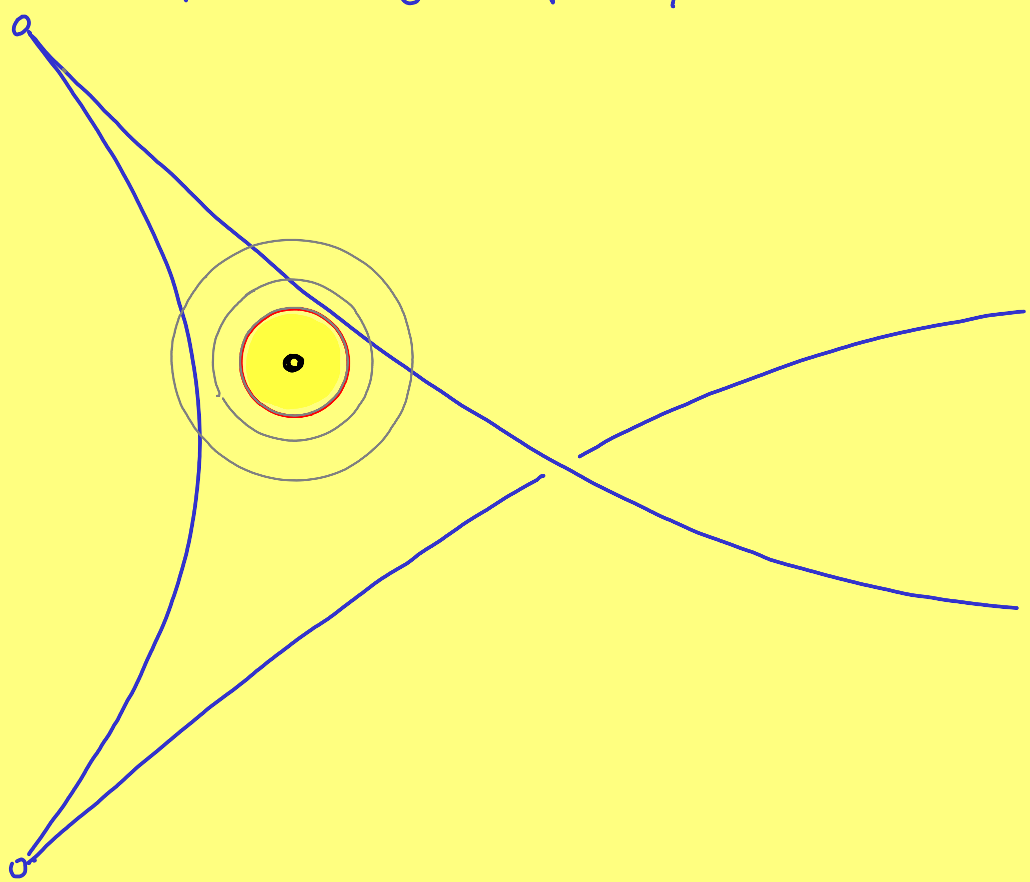

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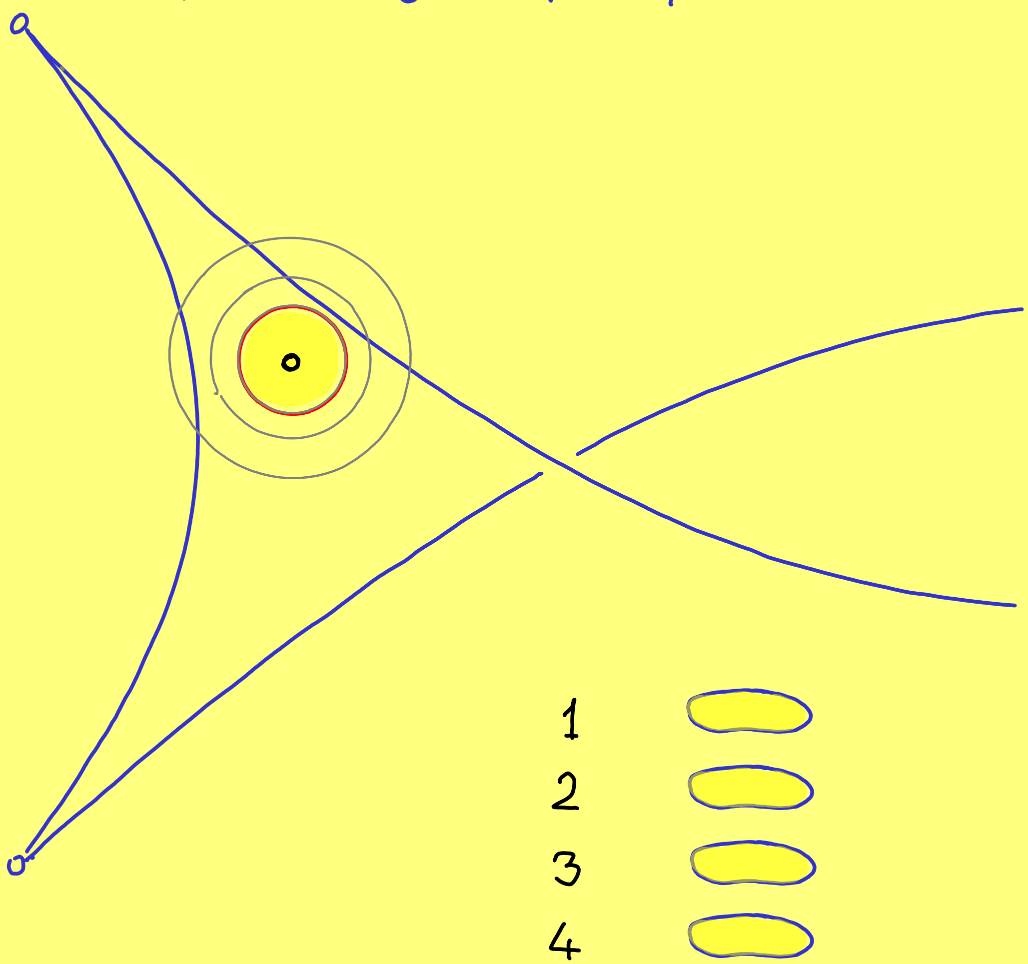

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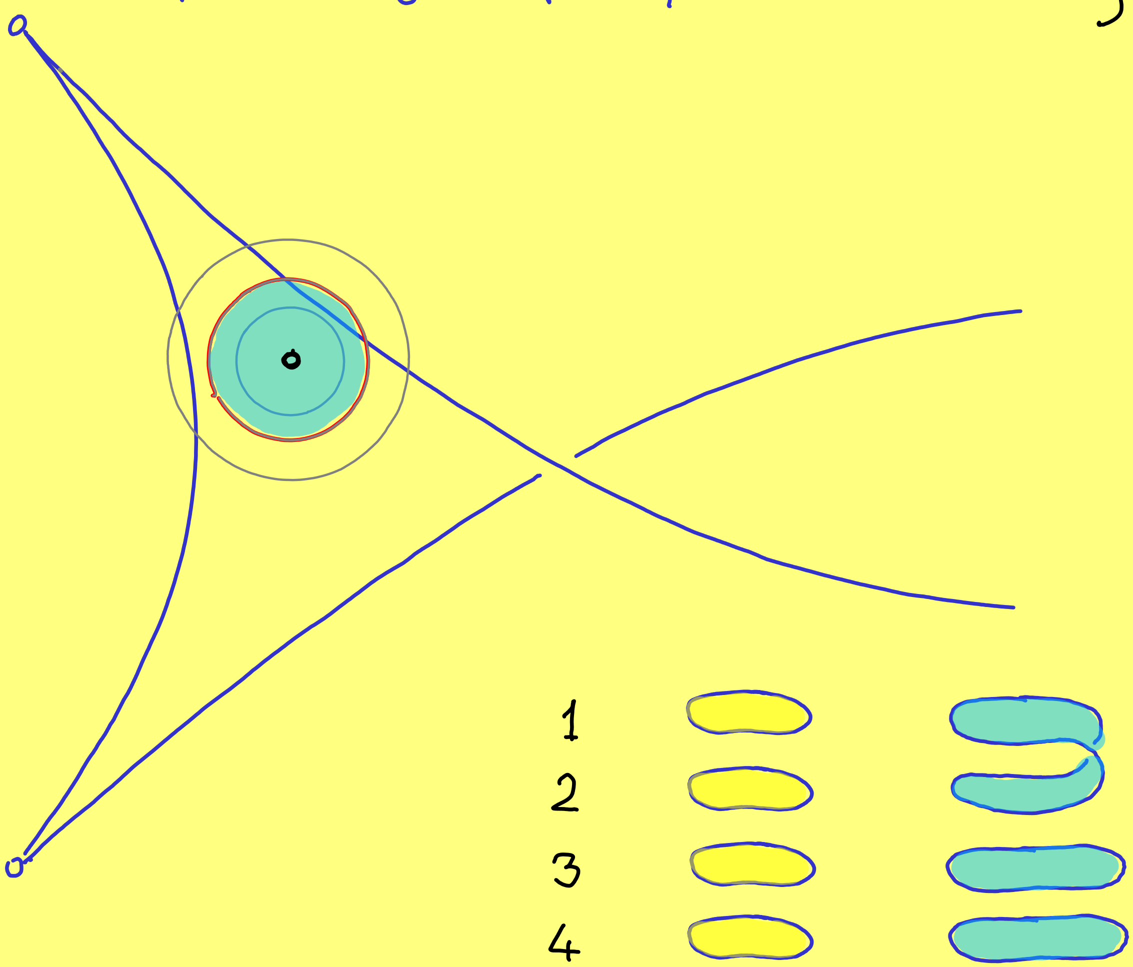

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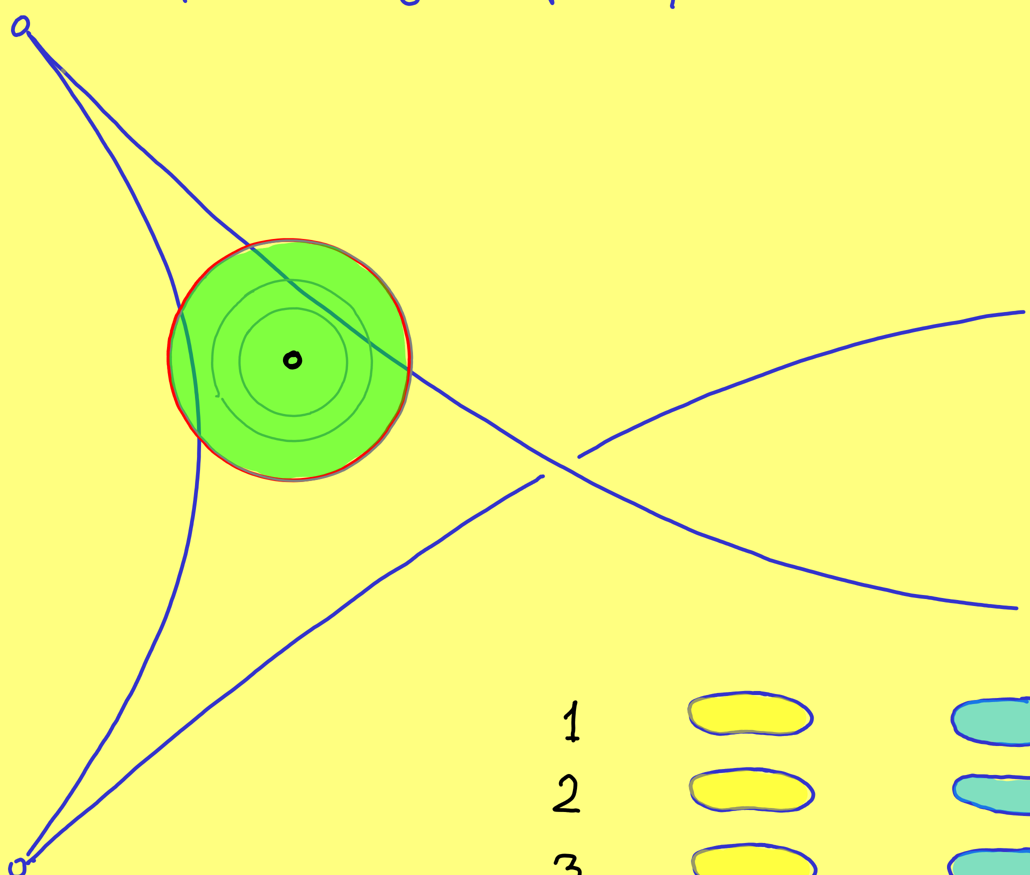

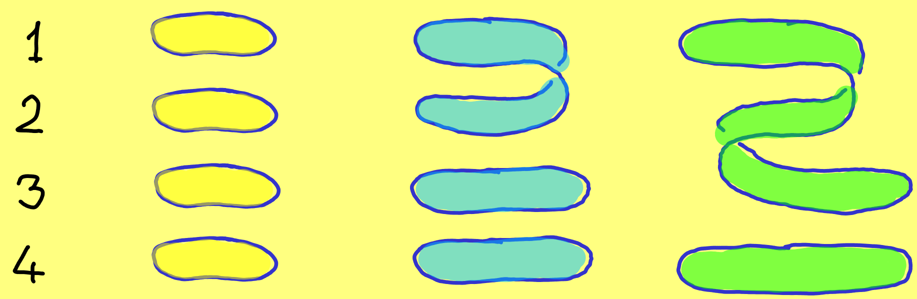

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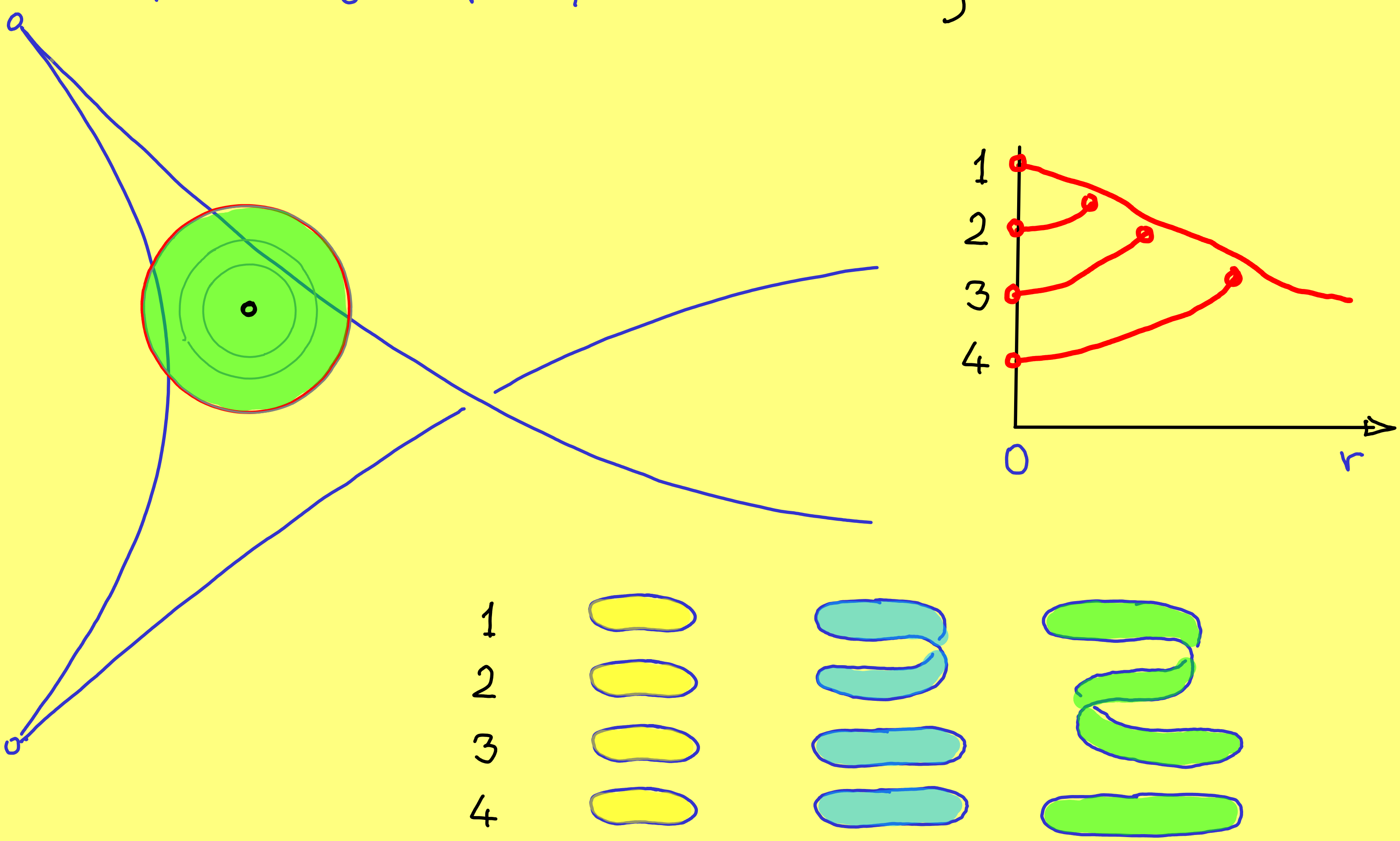

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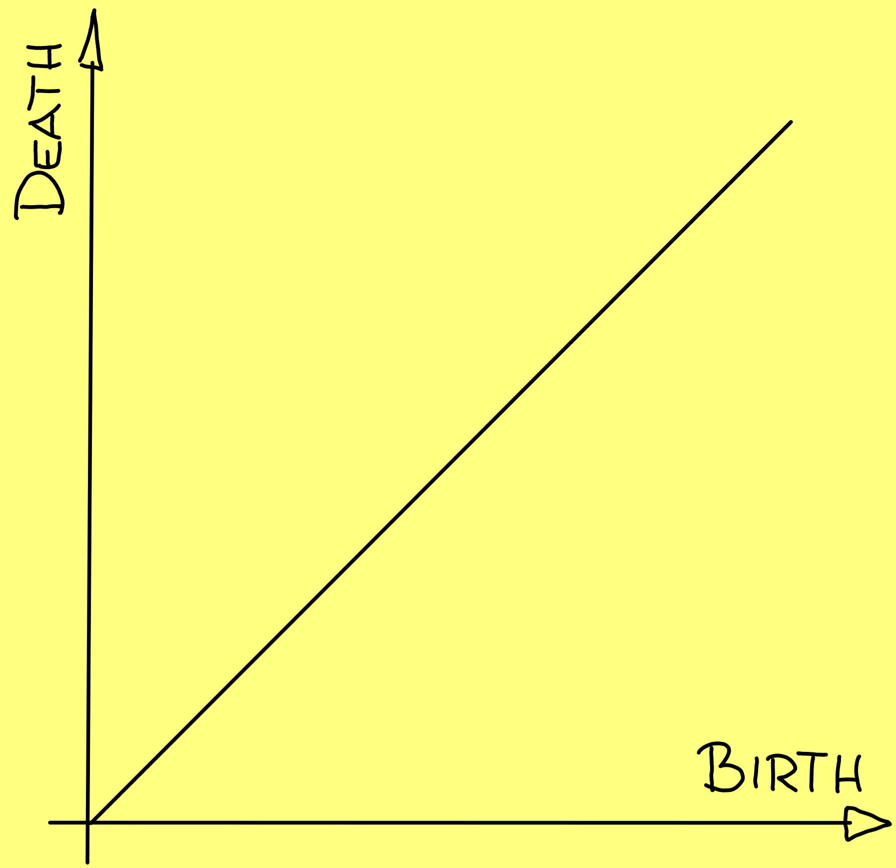

$\mathrm{Dgm}(f_a)$

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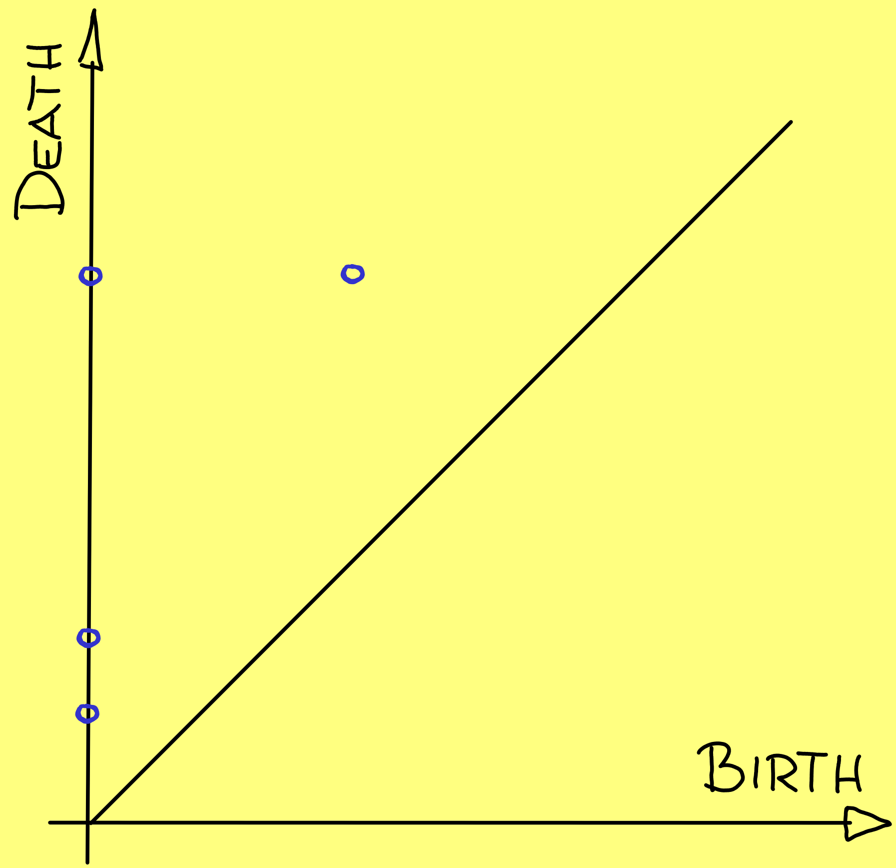

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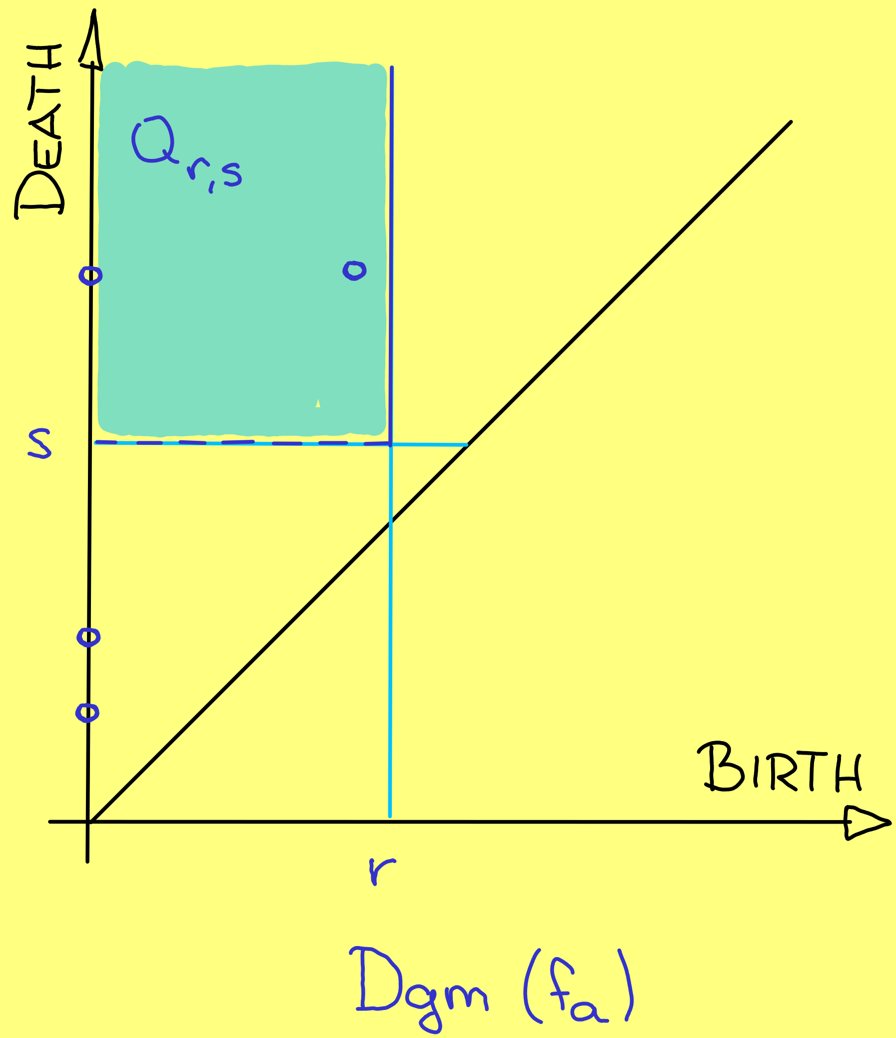

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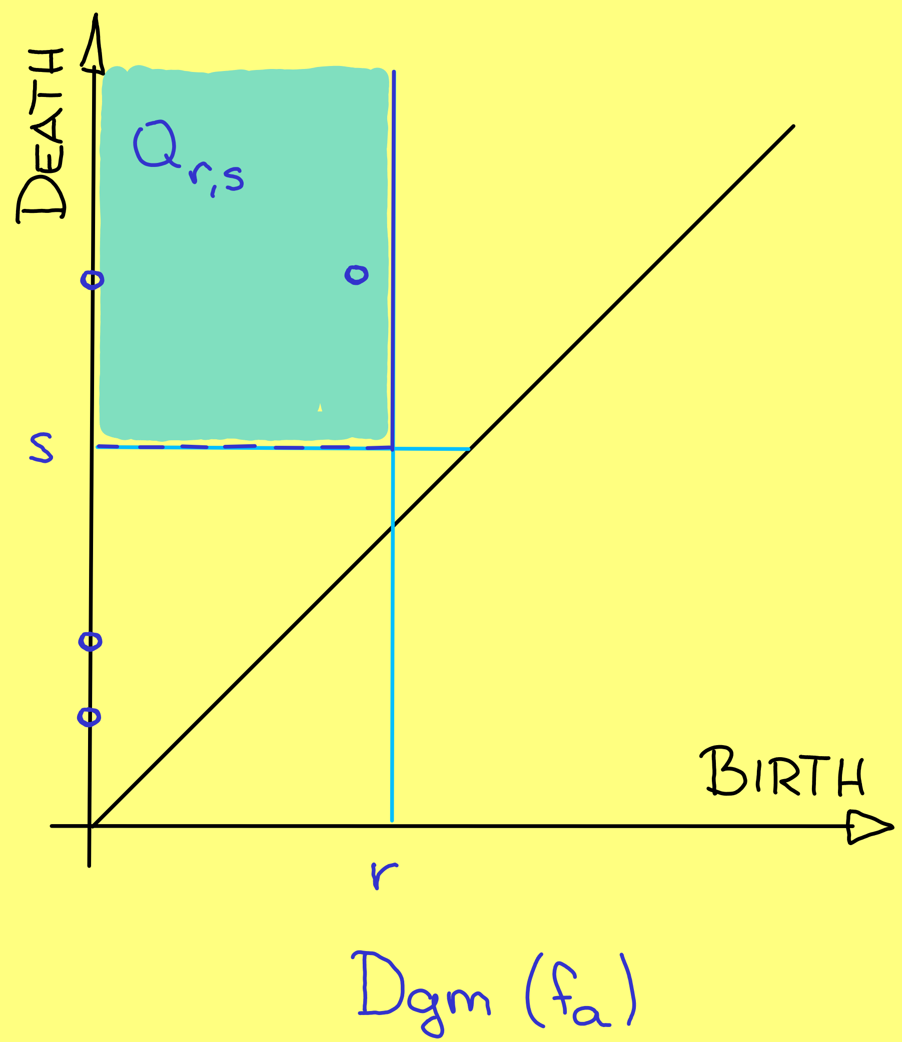
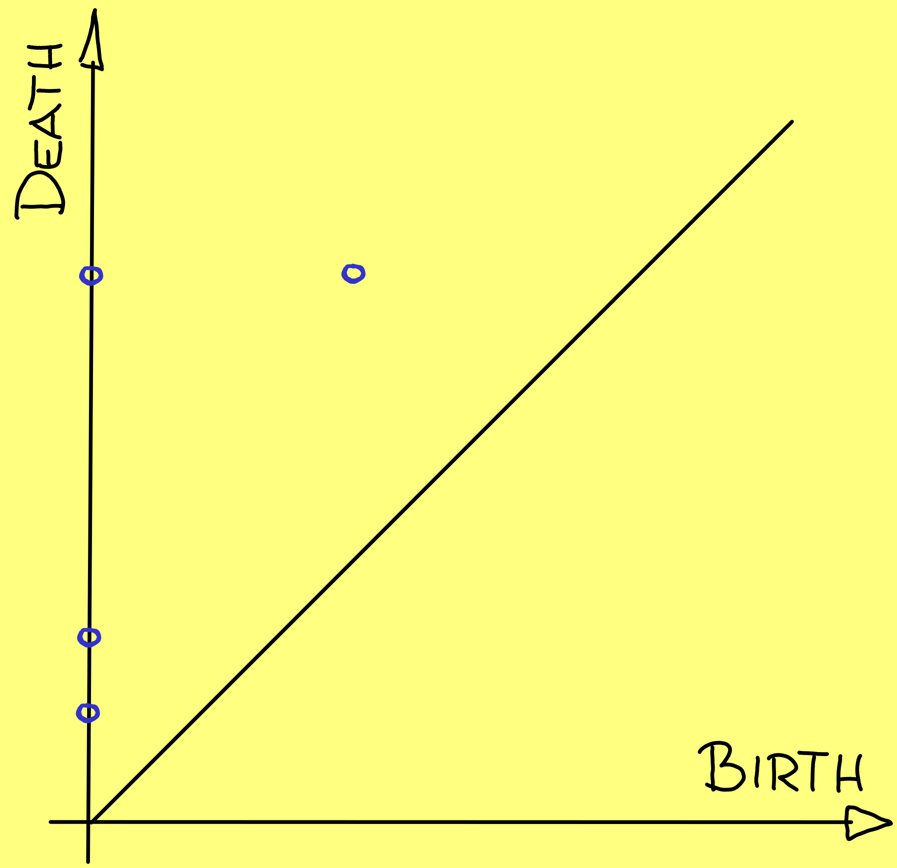


Fund. Lemma Pers. Hom.

#points in $Q_{r,s}$ is rank of image $F_r \to F_s$.

II.3 Bottleneck Stability



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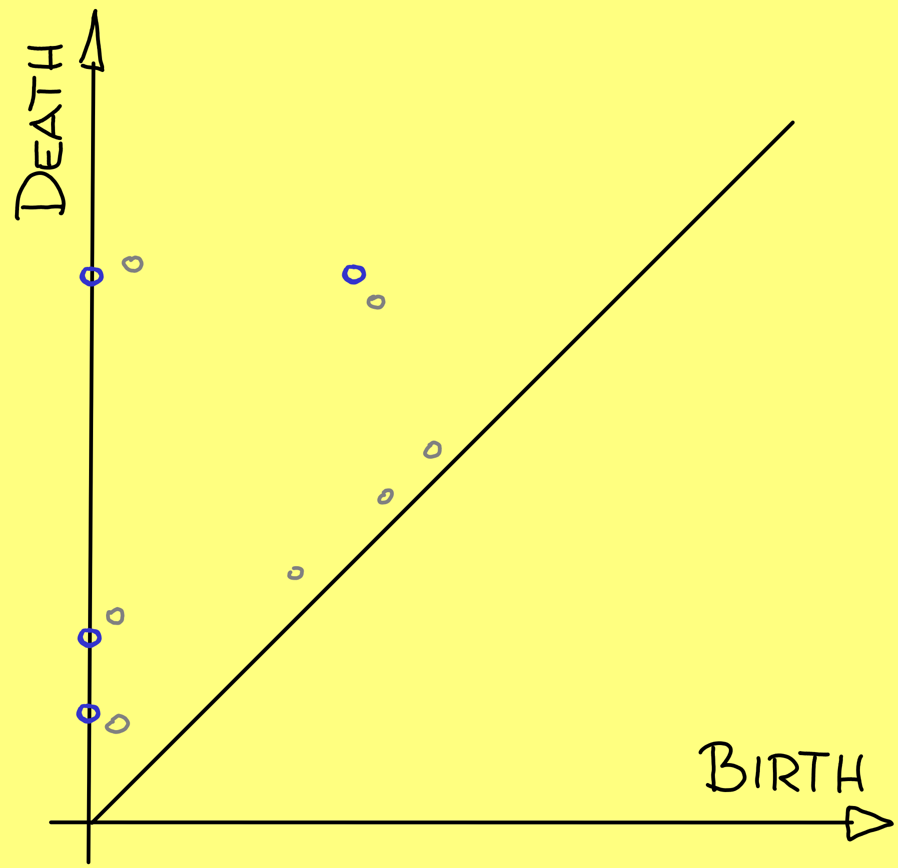

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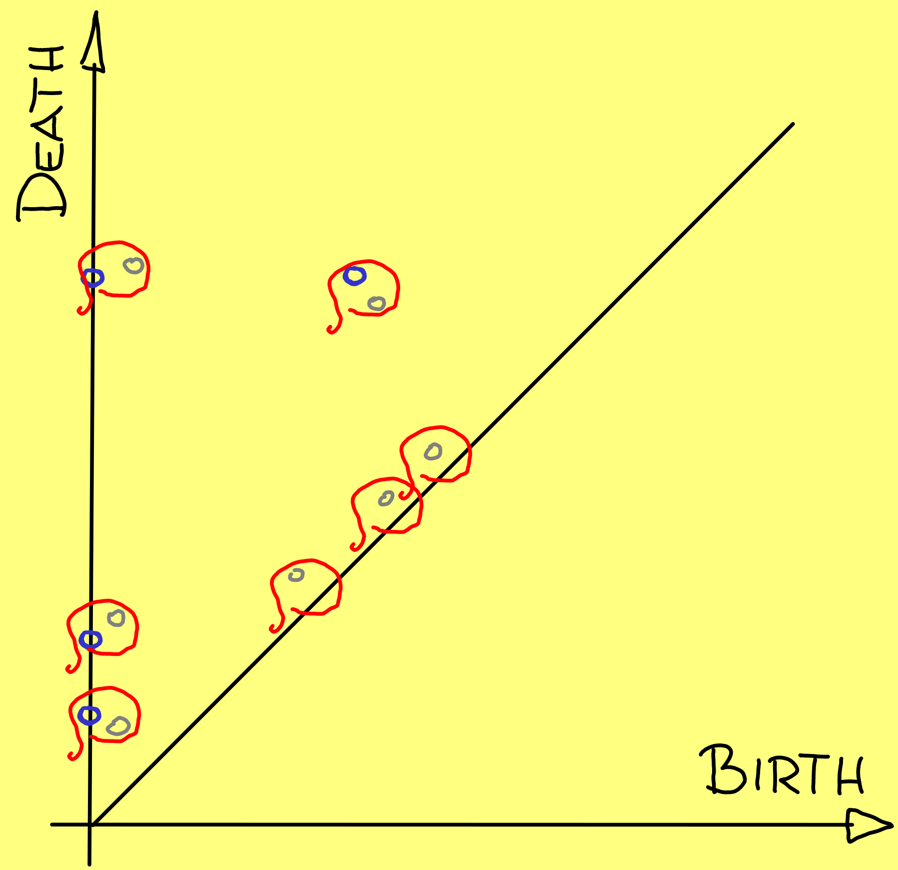

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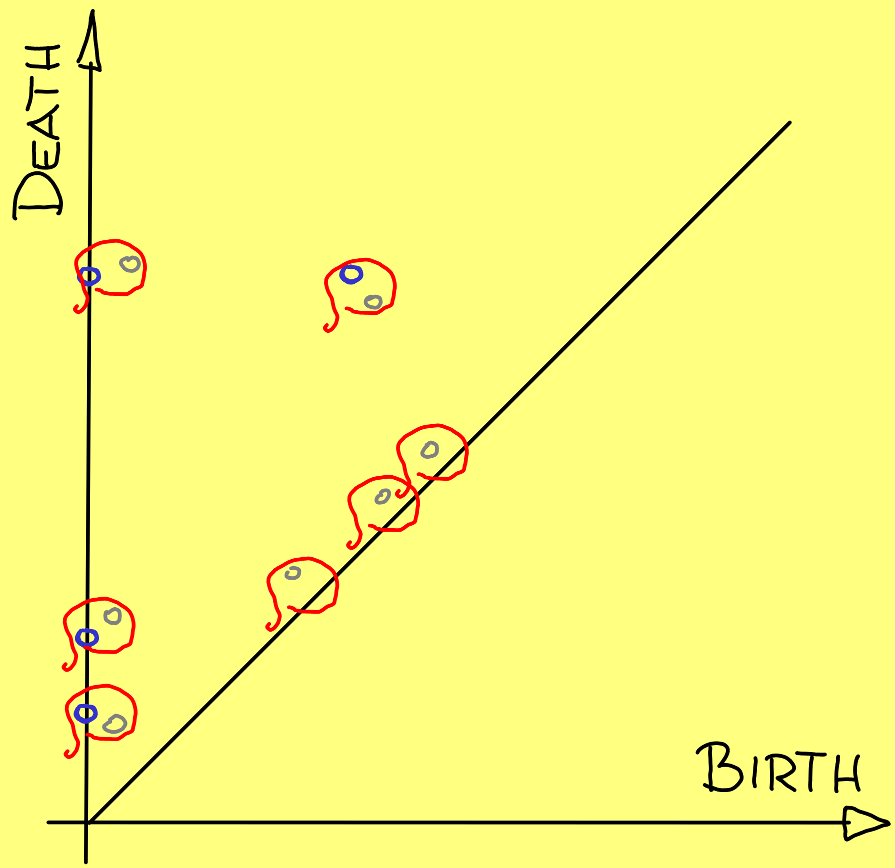
$$W_\infty(D, D_0) = \inf_{\gamma: D \to D_0} \sup_{x \in D} \| x - \gamma(x) \|_\infty$$



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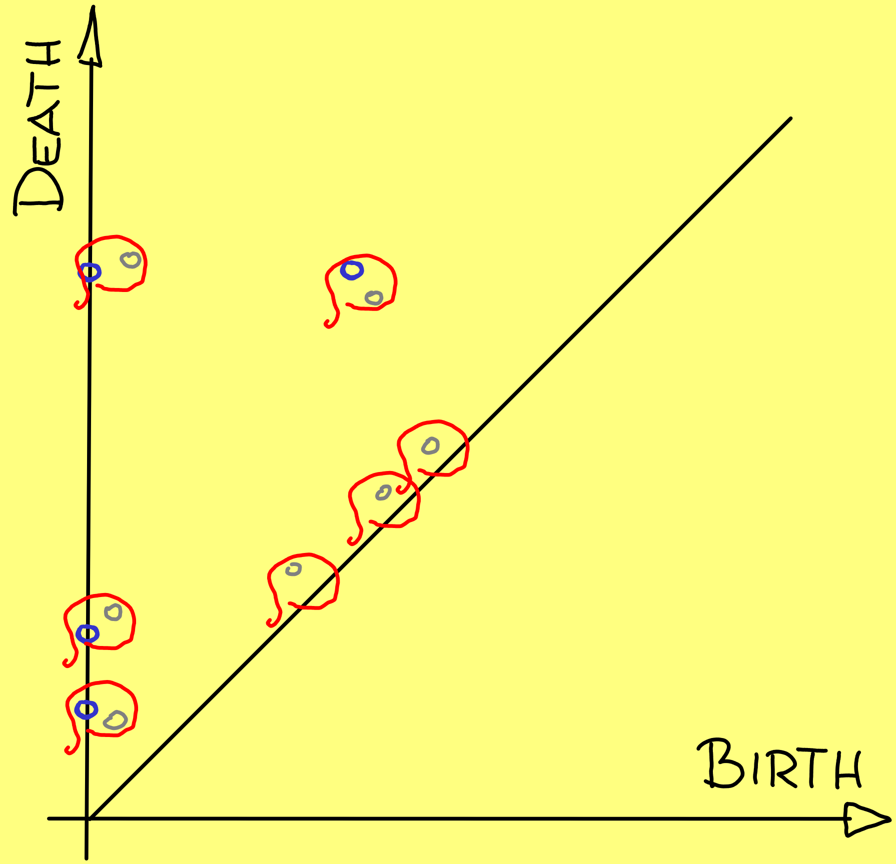

L_∞-STAB. THM.

Tame functions $f_a, g_a : \mathbb{M} \to \mathbb{R}$.

$$W_\infty(\mathrm{Dgm}(f_a), \mathrm{Dgm}(g_a)) \leq \|f_a - g_a\|_\infty .$$

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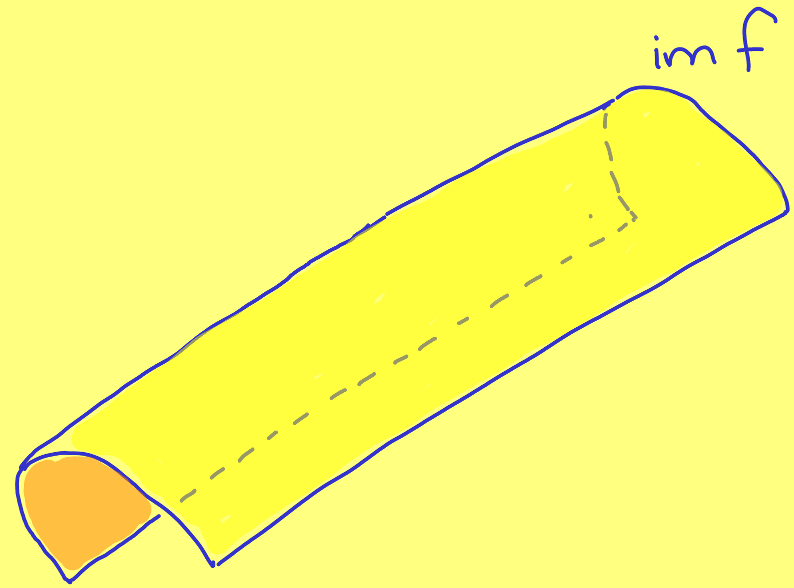

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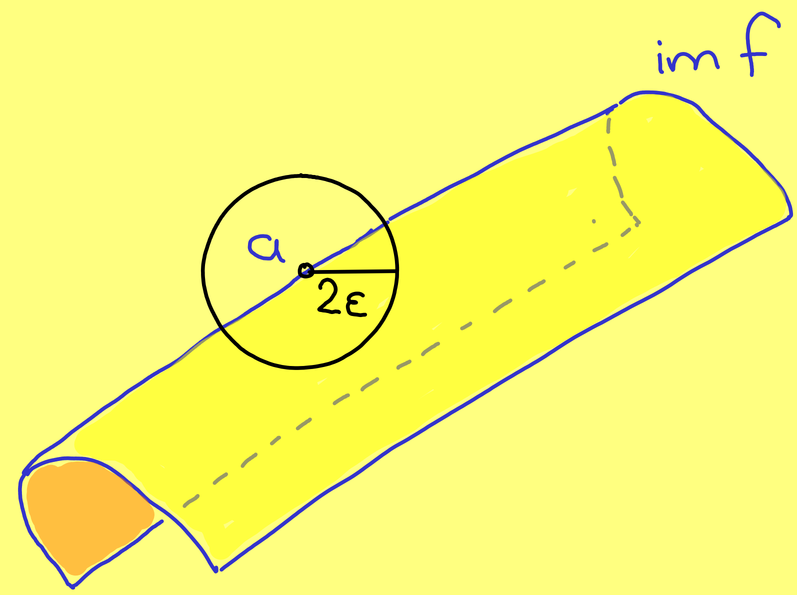

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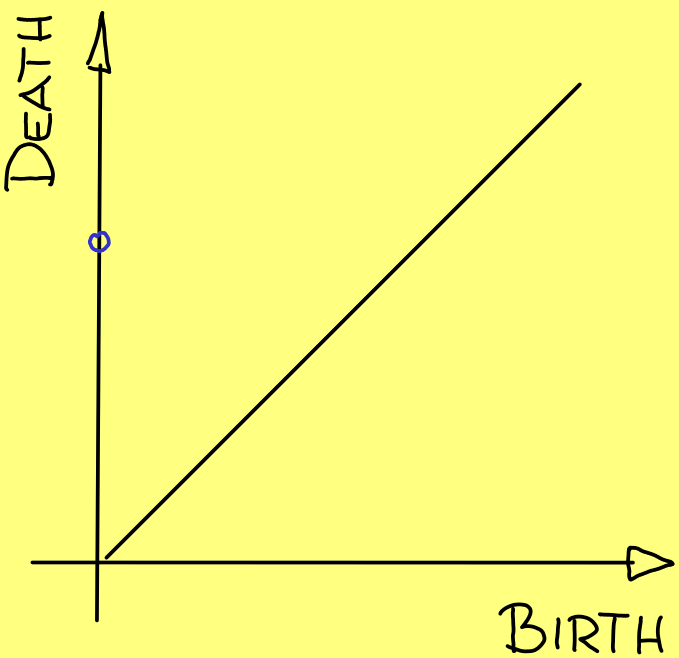

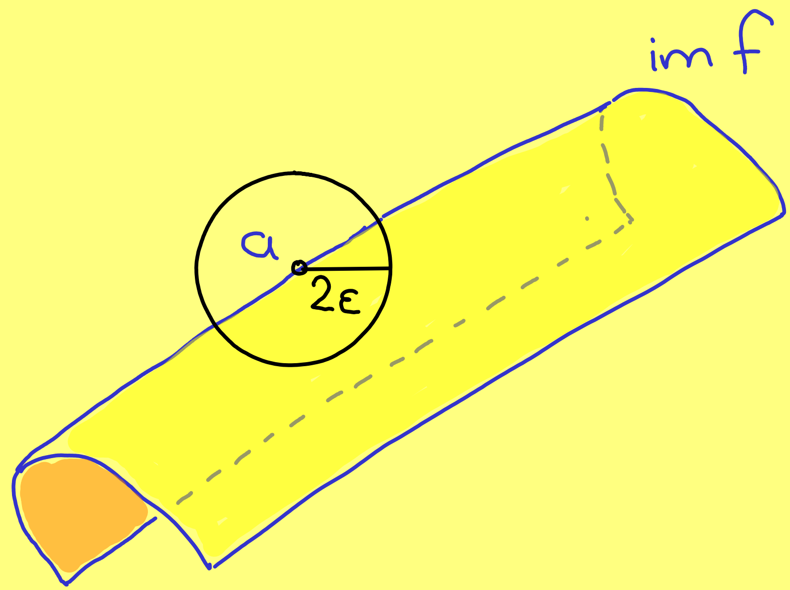

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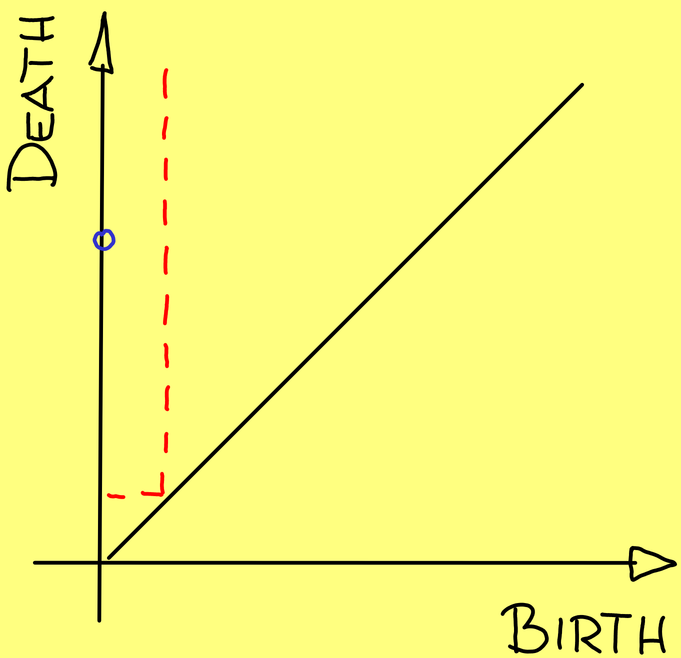

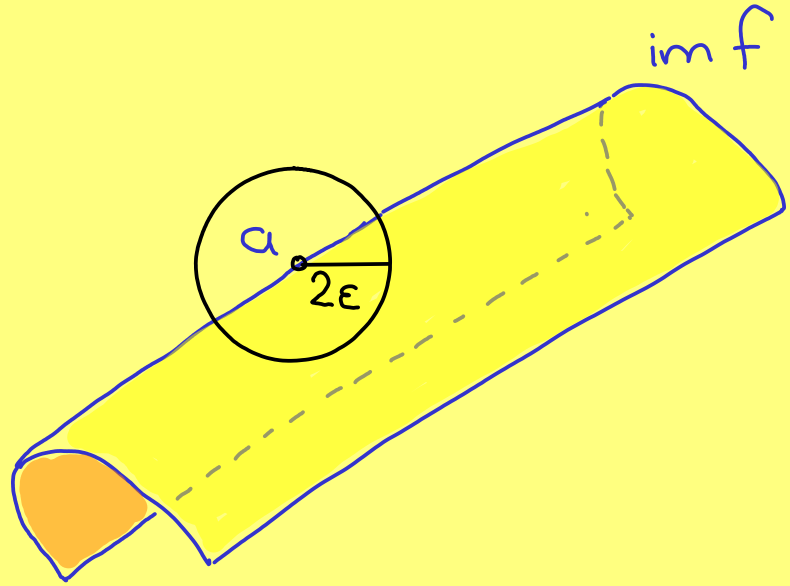

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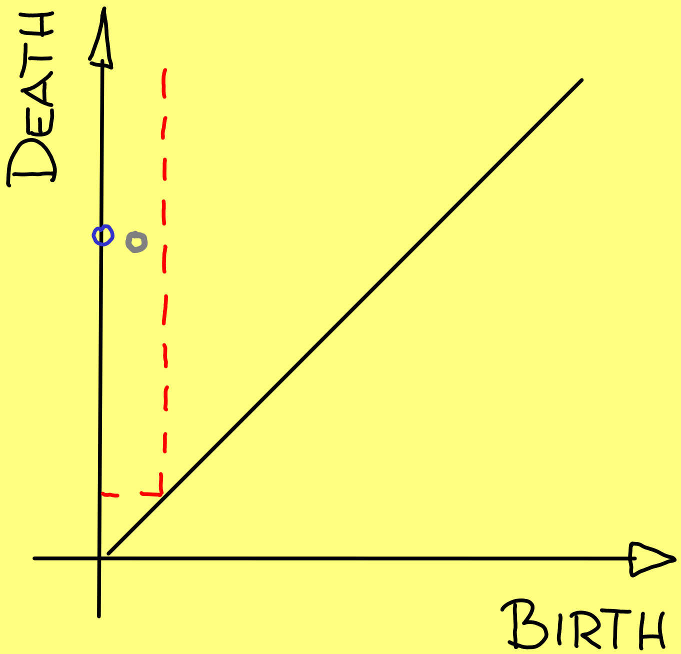

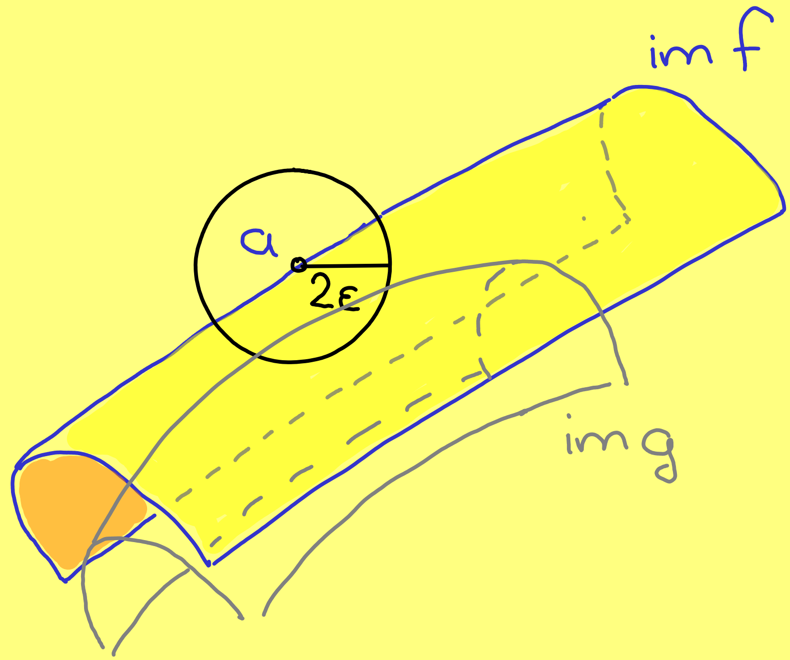

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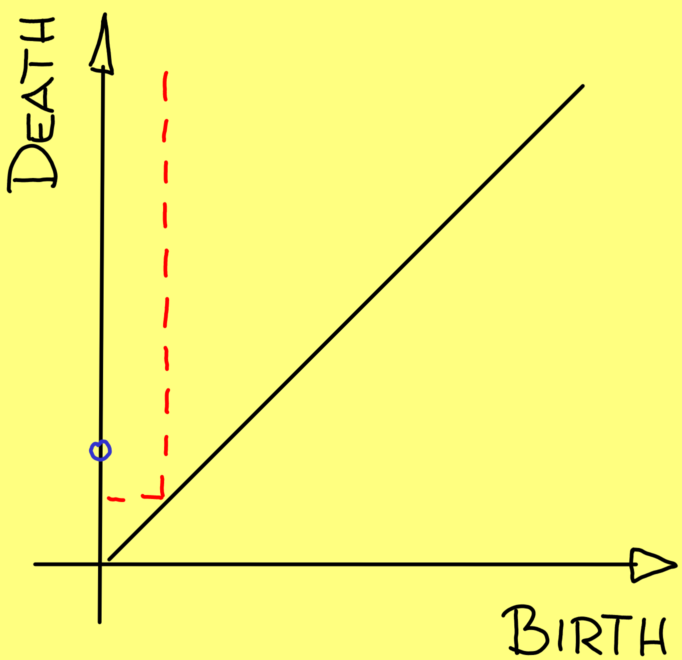

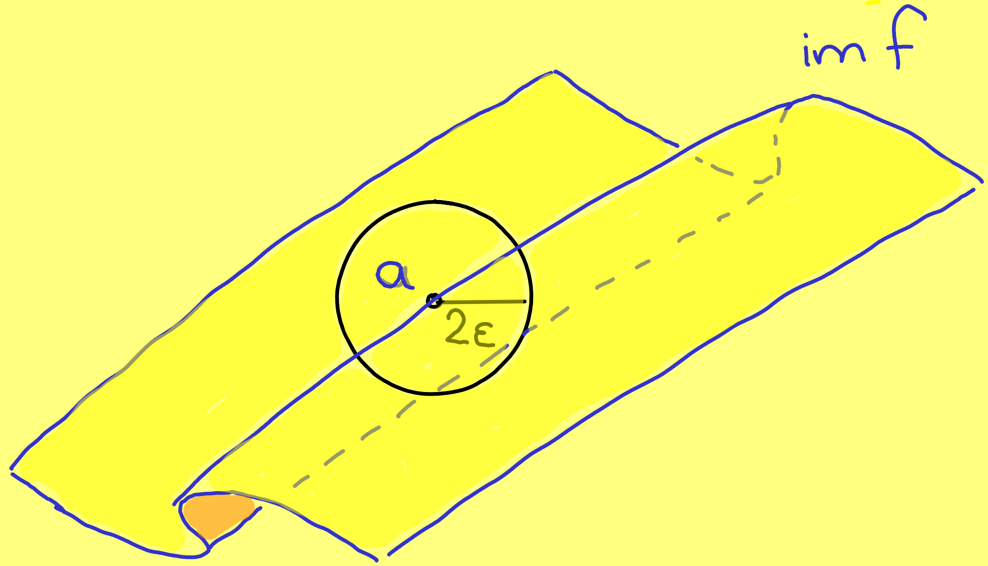

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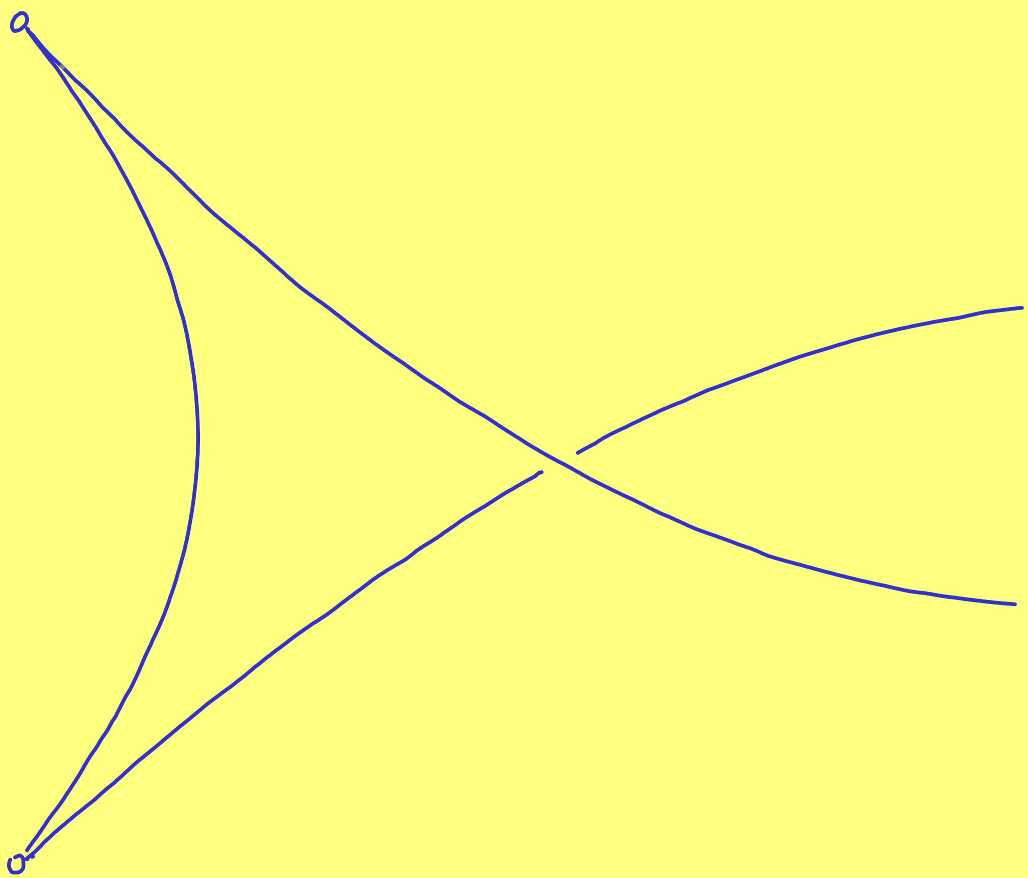

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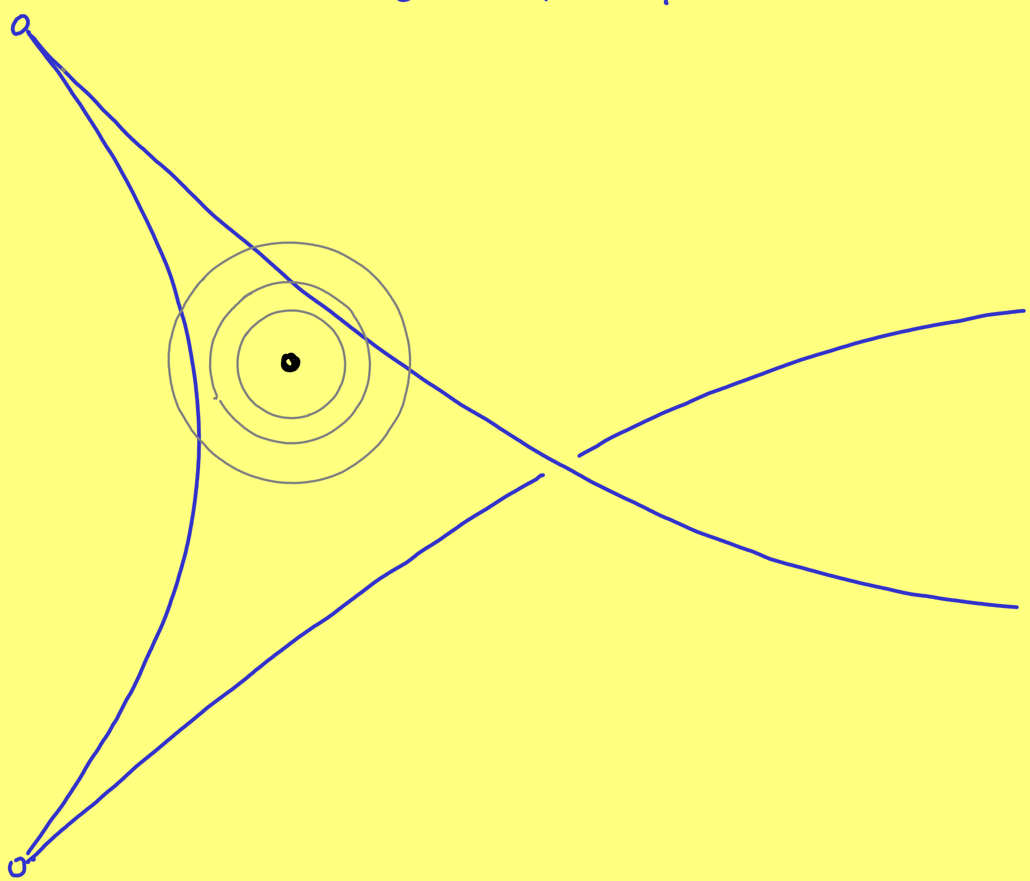

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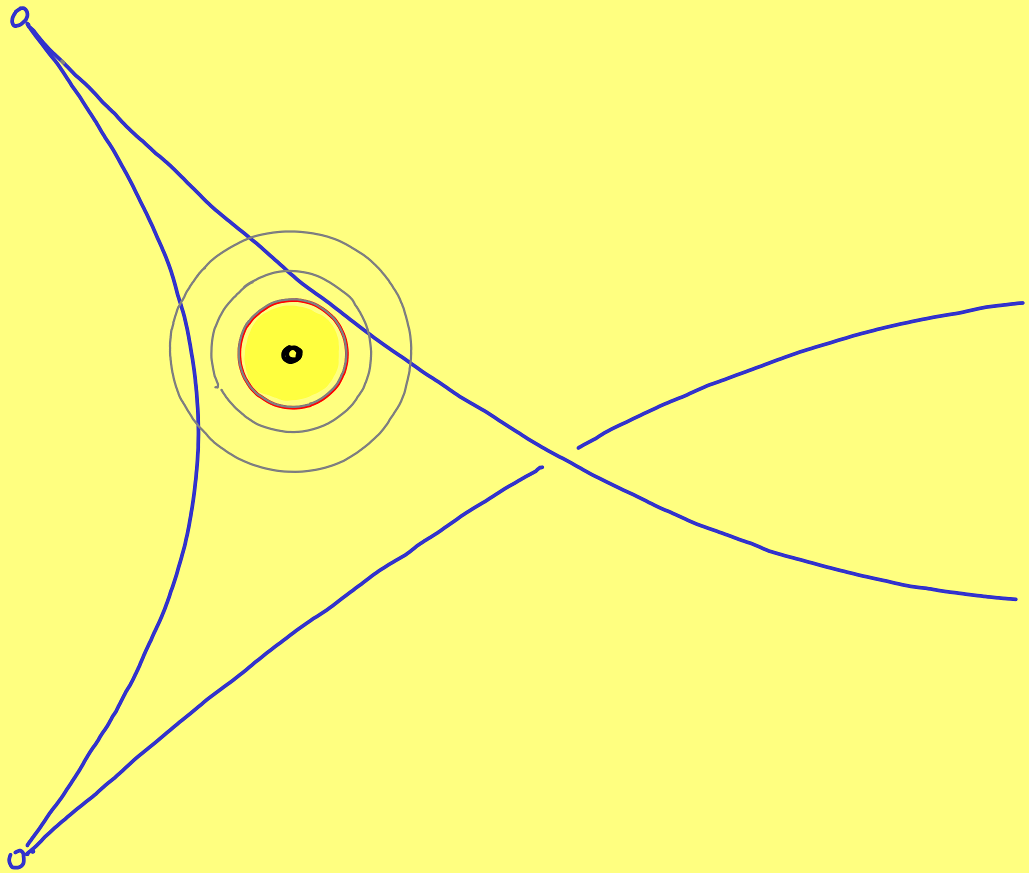

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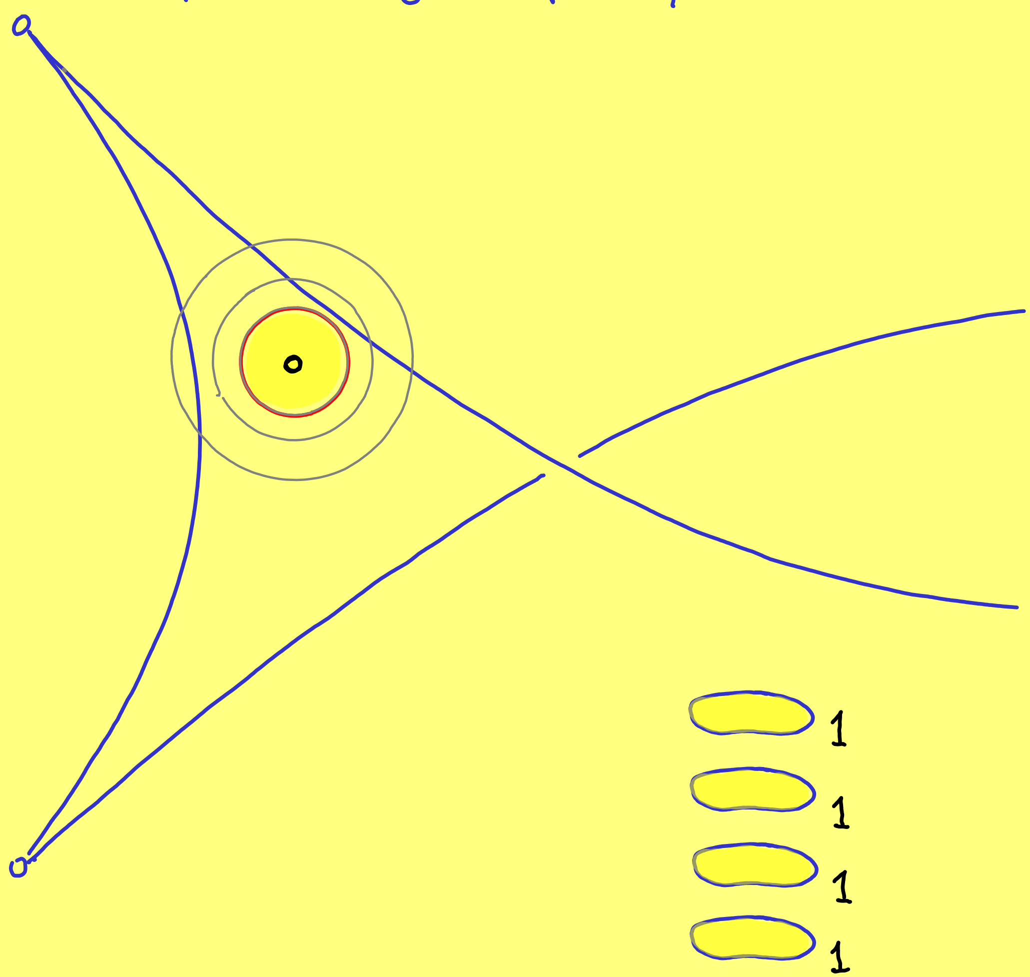

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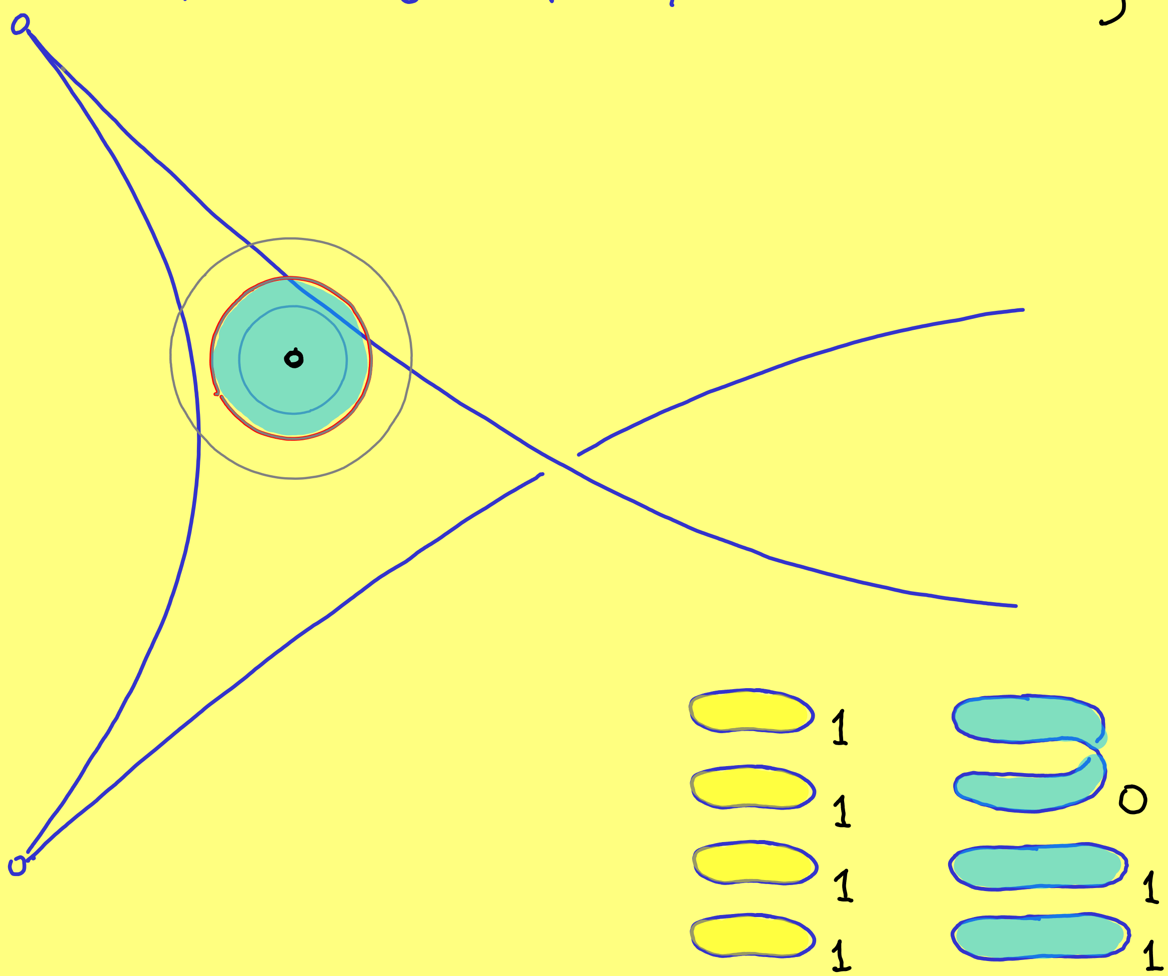

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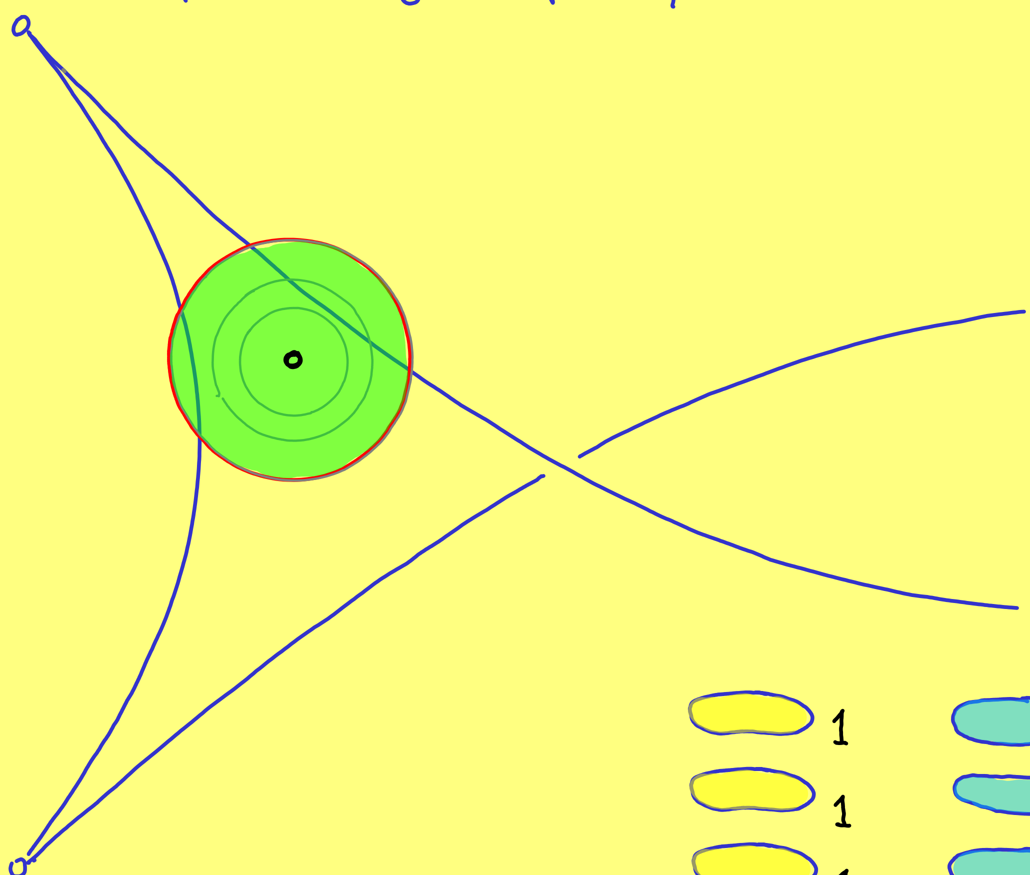

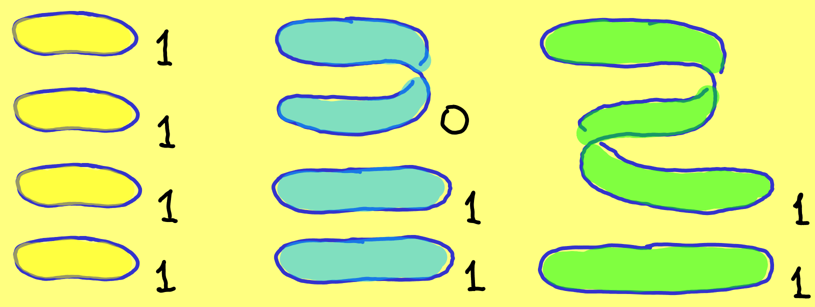

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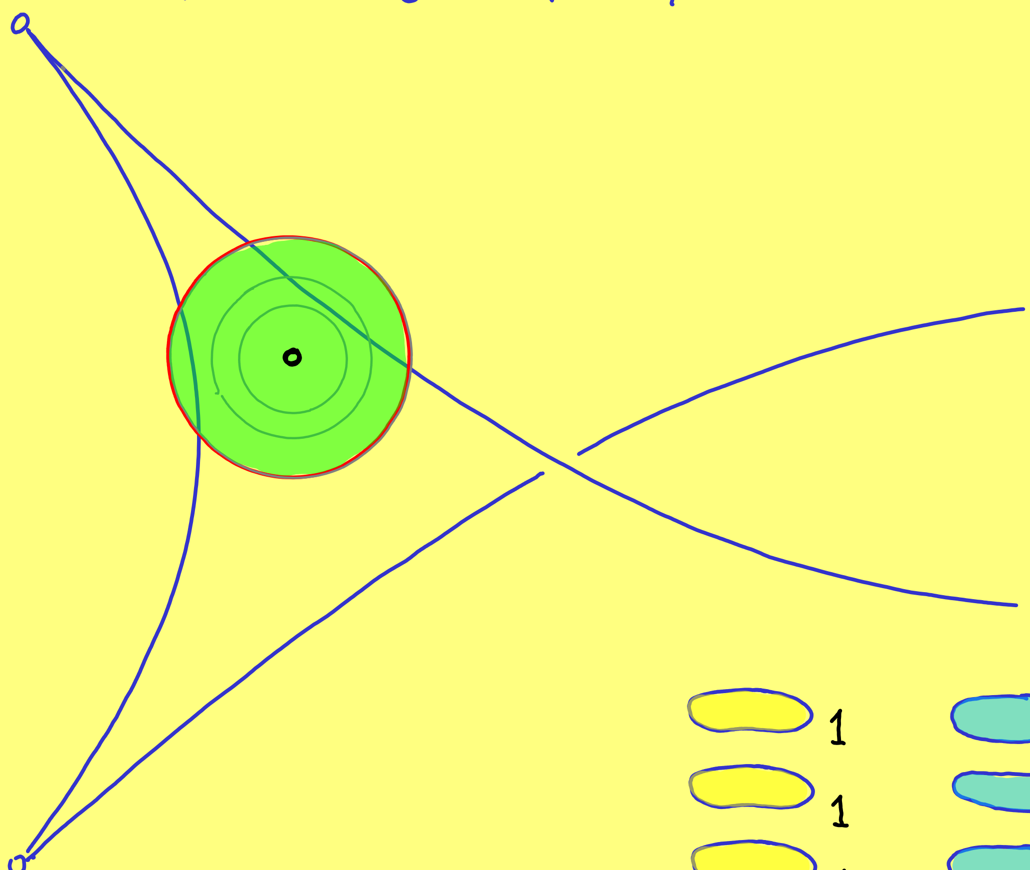
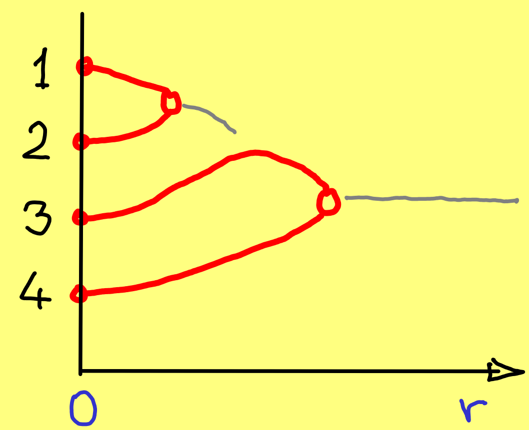
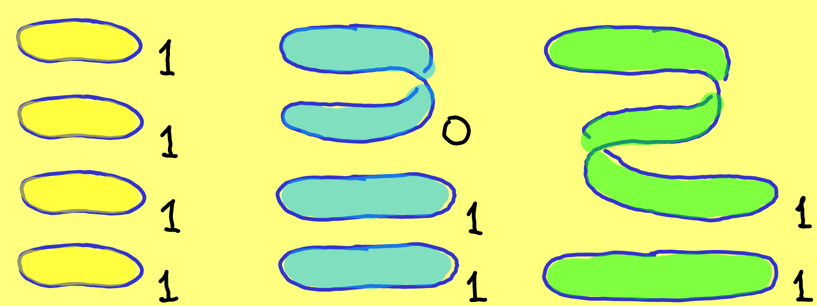
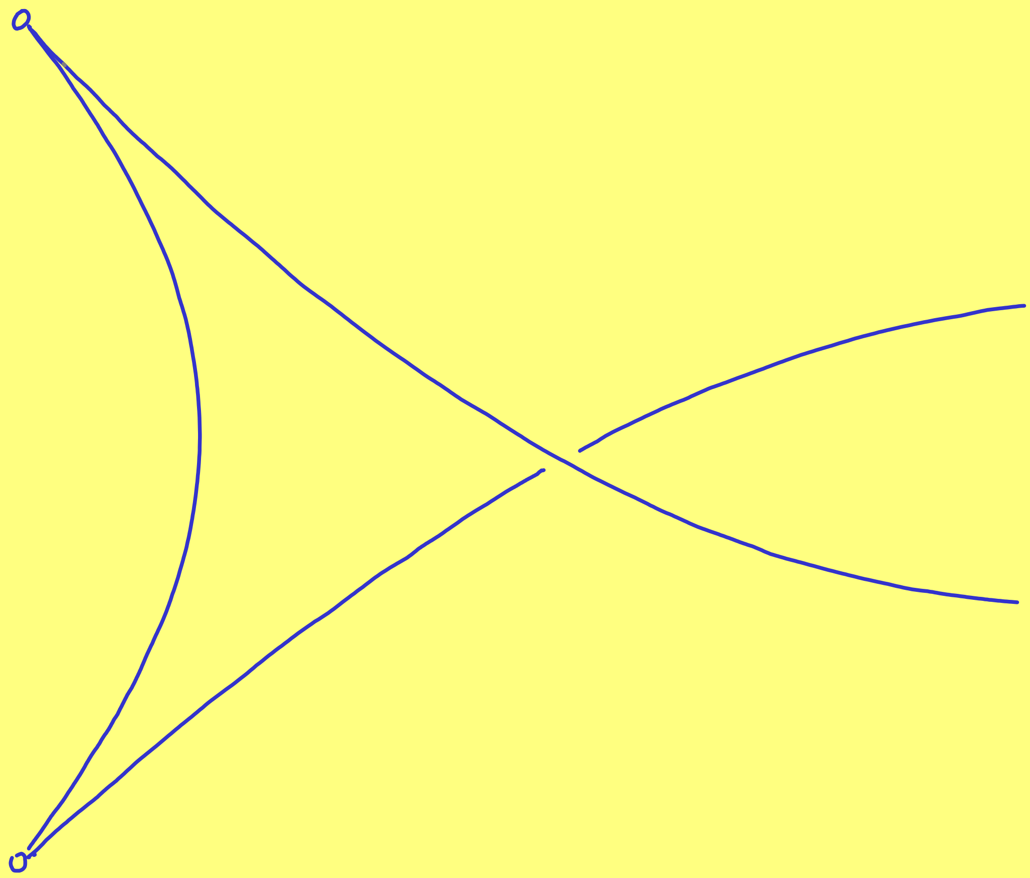




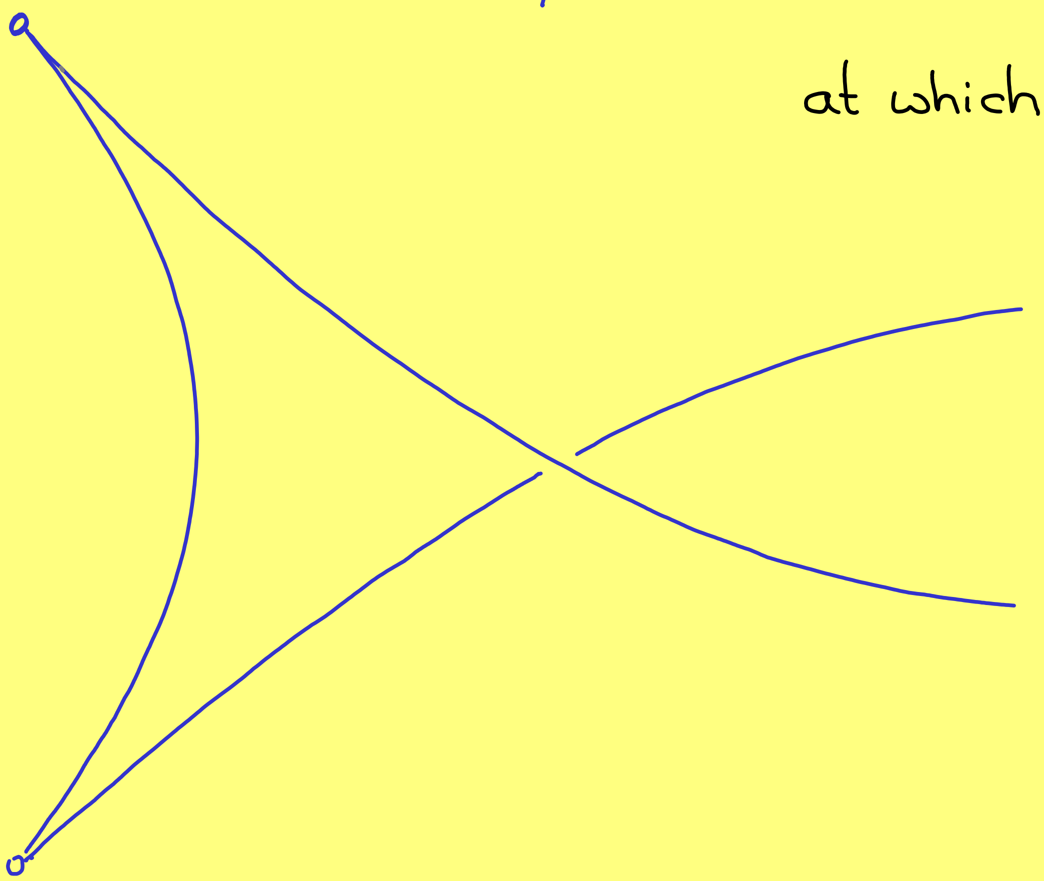


III.2 Stress Function



III.2 Stress Function

$\varphi : \mathbb{M} \to \mathbb{R}$ with $\varphi(x)$ the threshold at which component falls ill.



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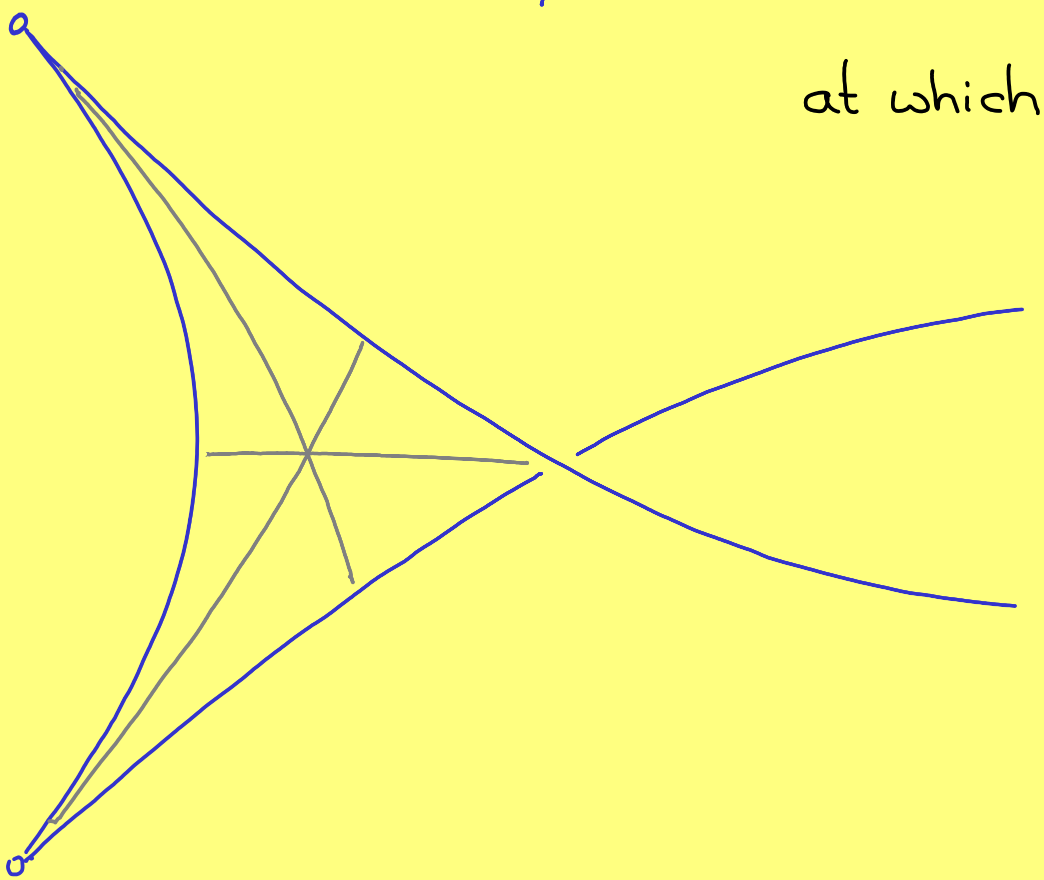
$\varphi : \mathbb{M} \to \mathbb{R}$ with $\varphi(x)$ the threshold at which component falls ill.



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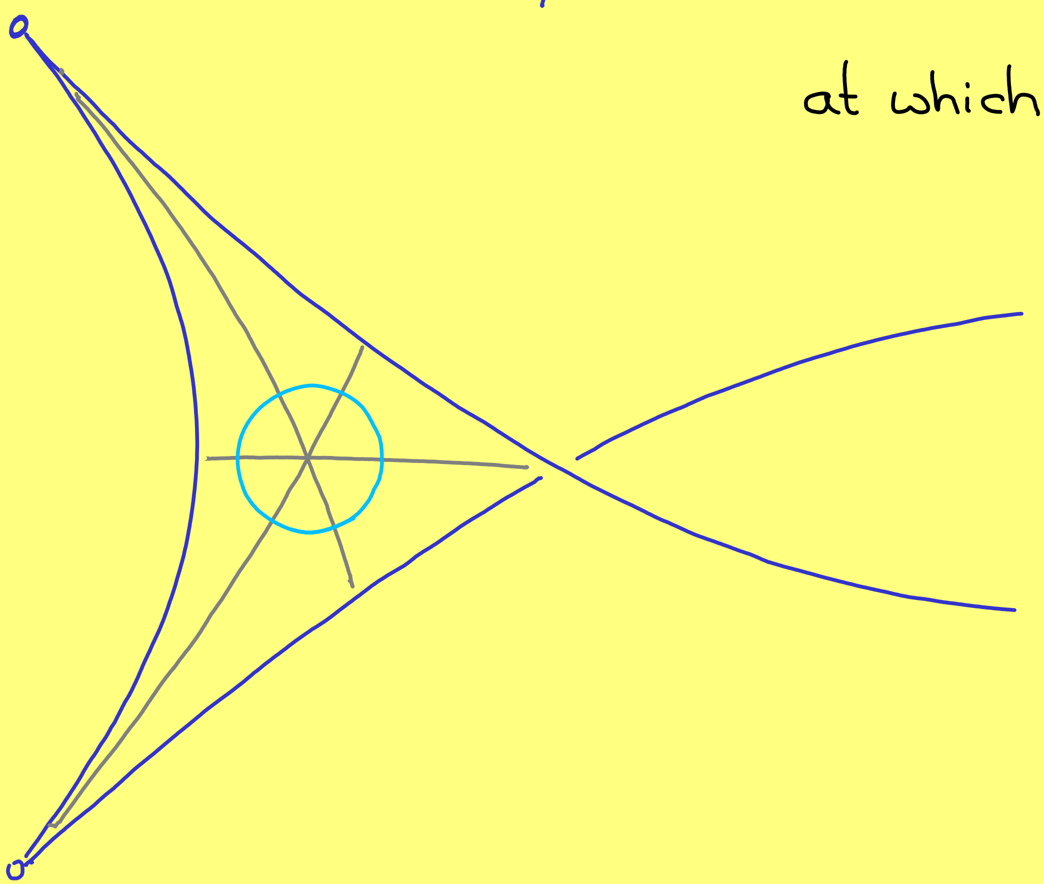

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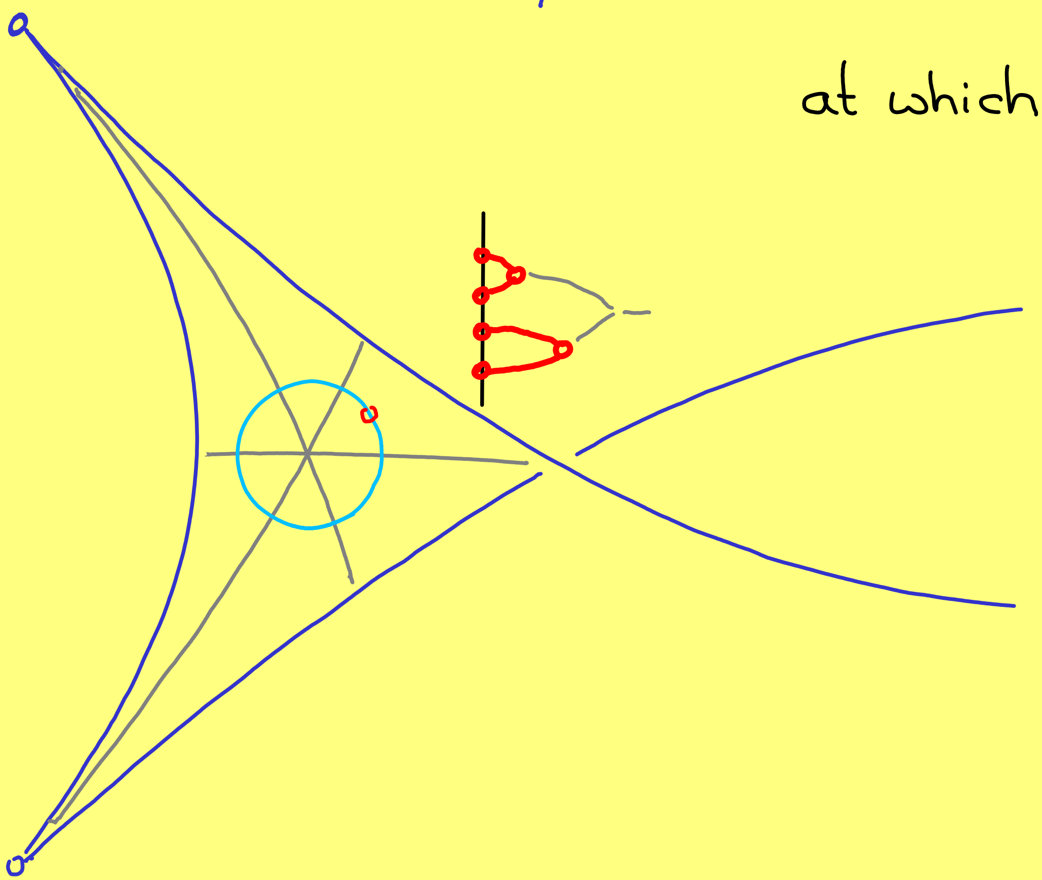

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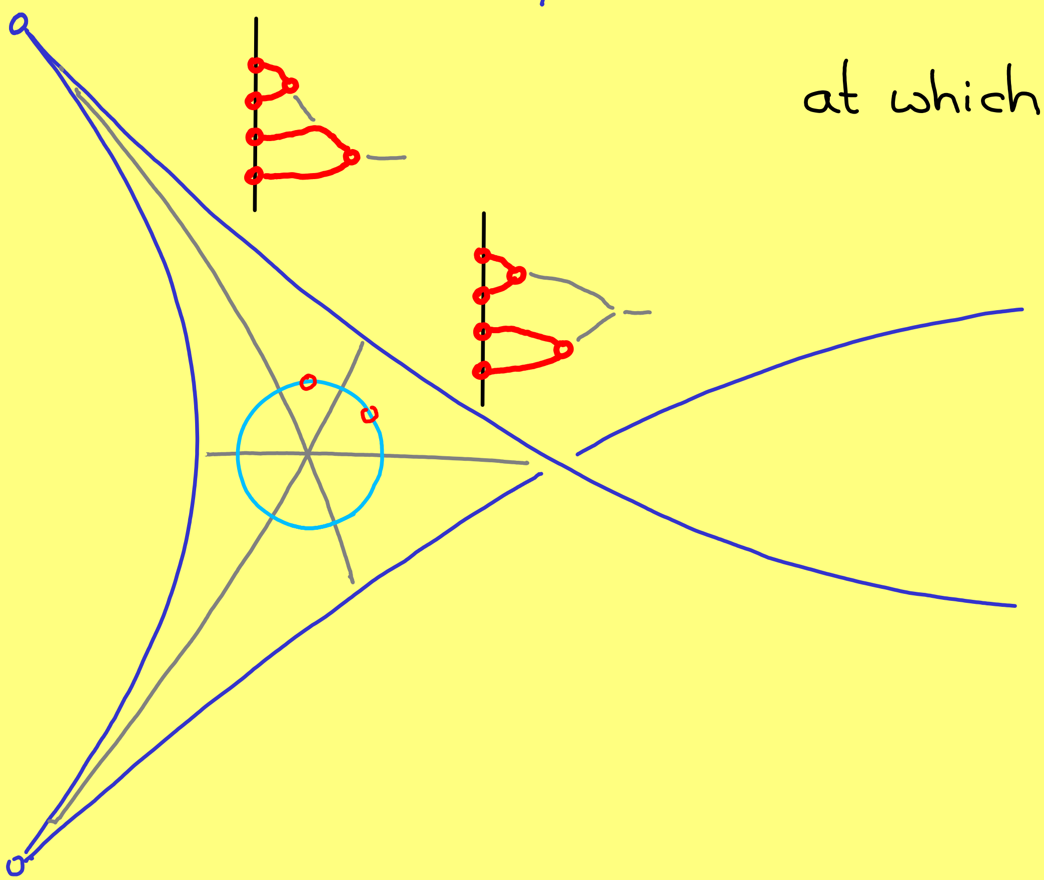

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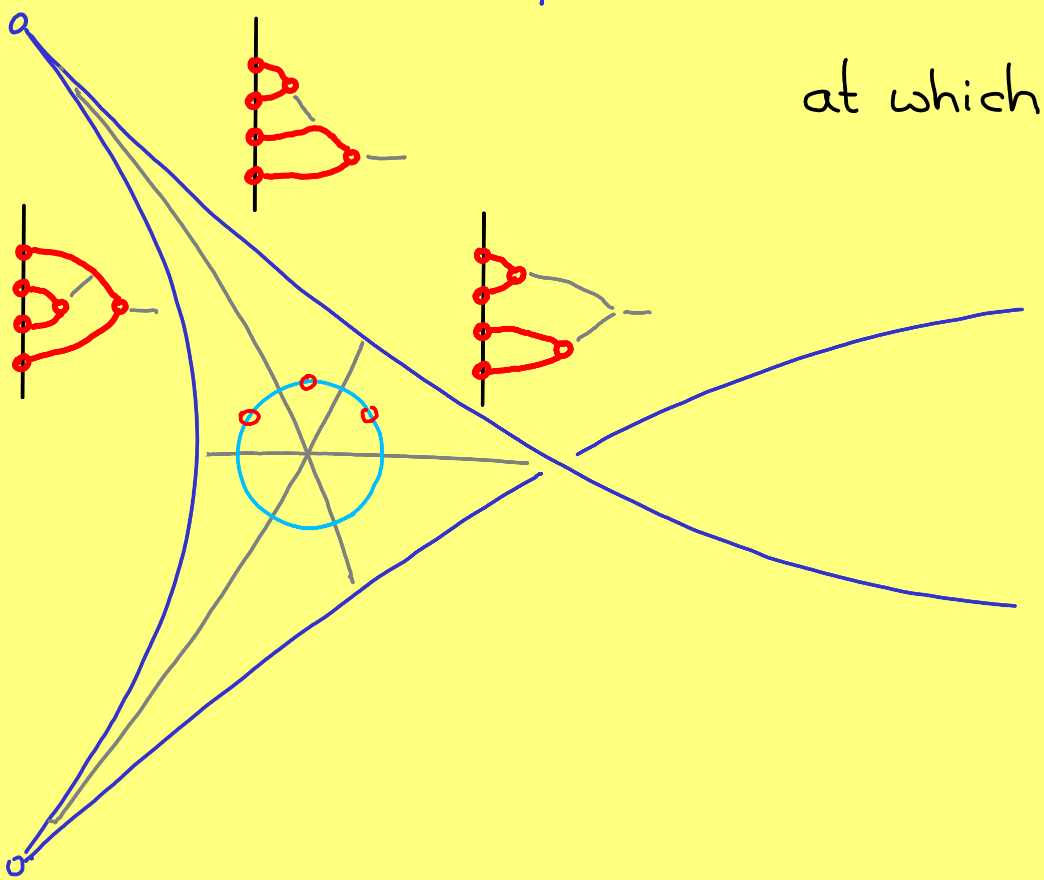

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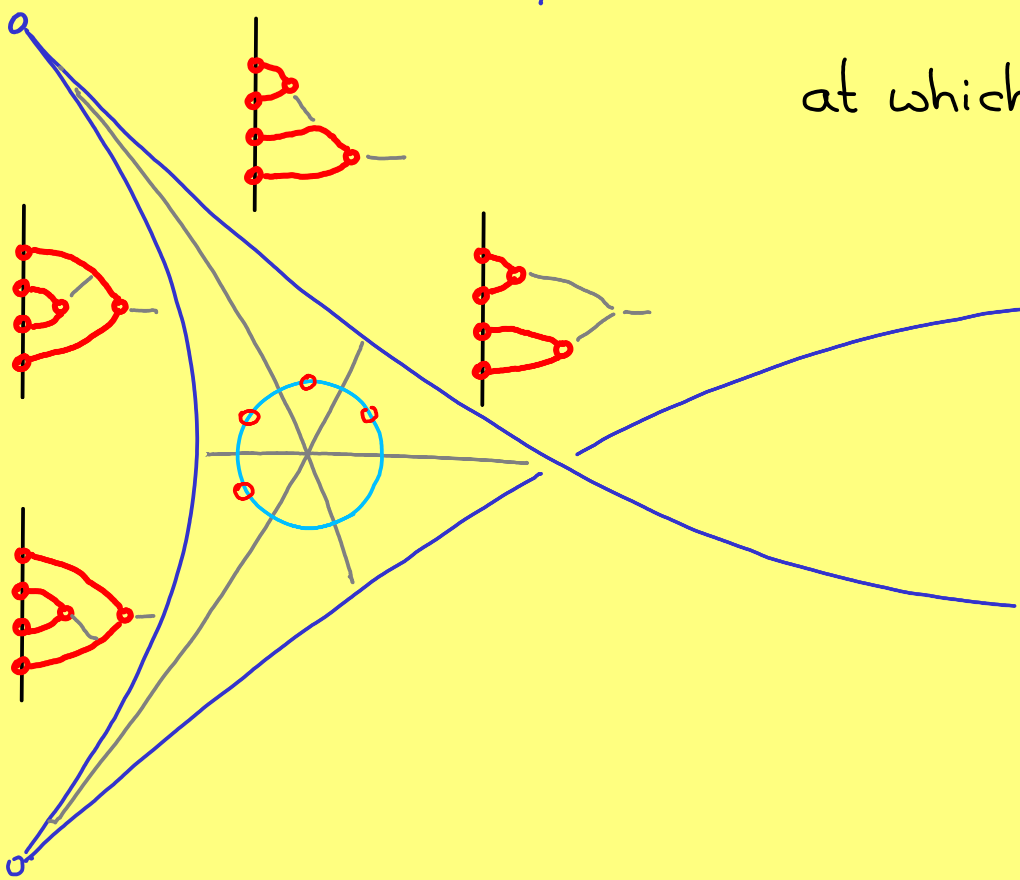

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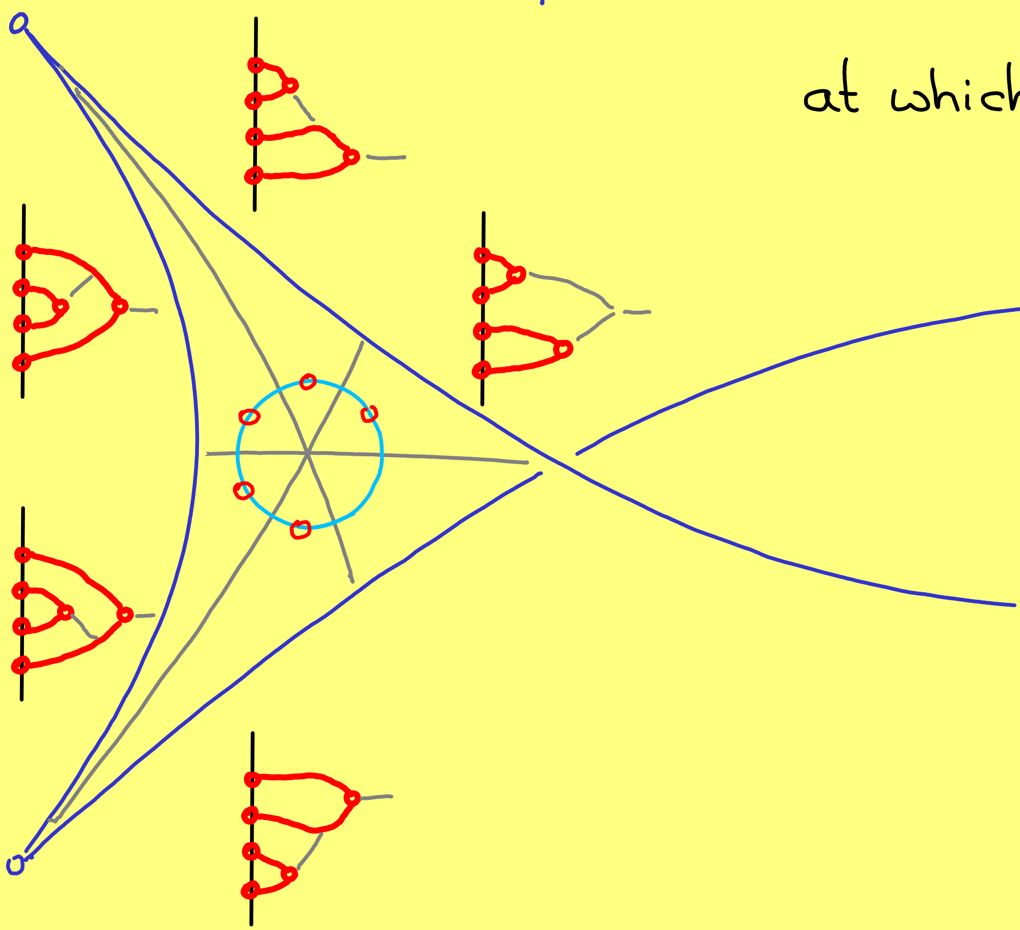

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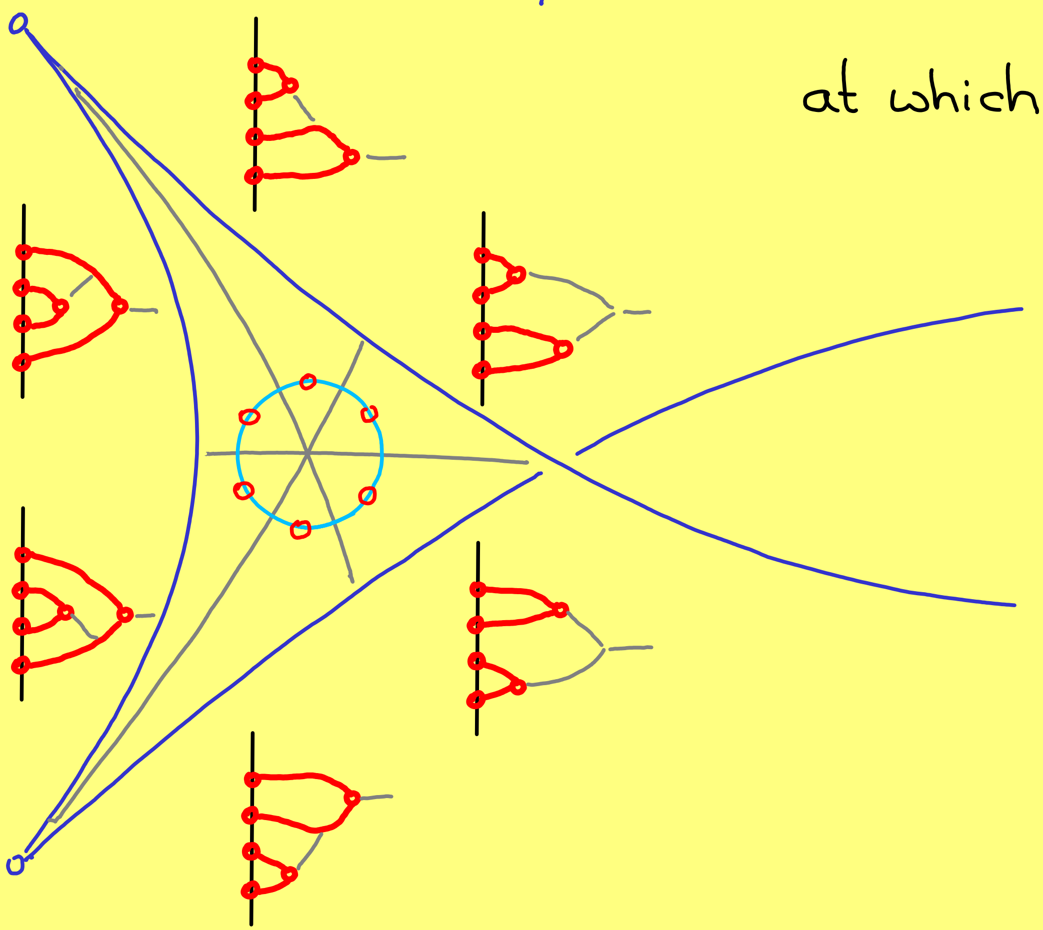

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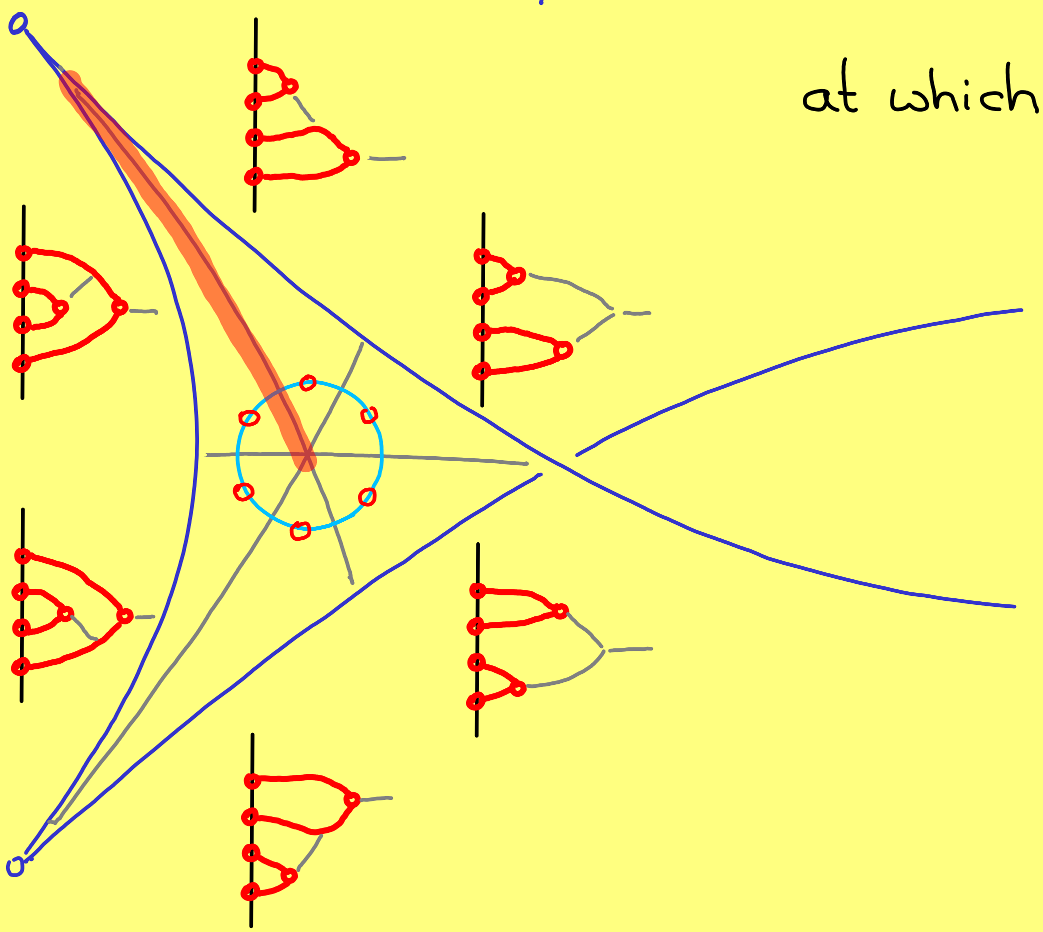
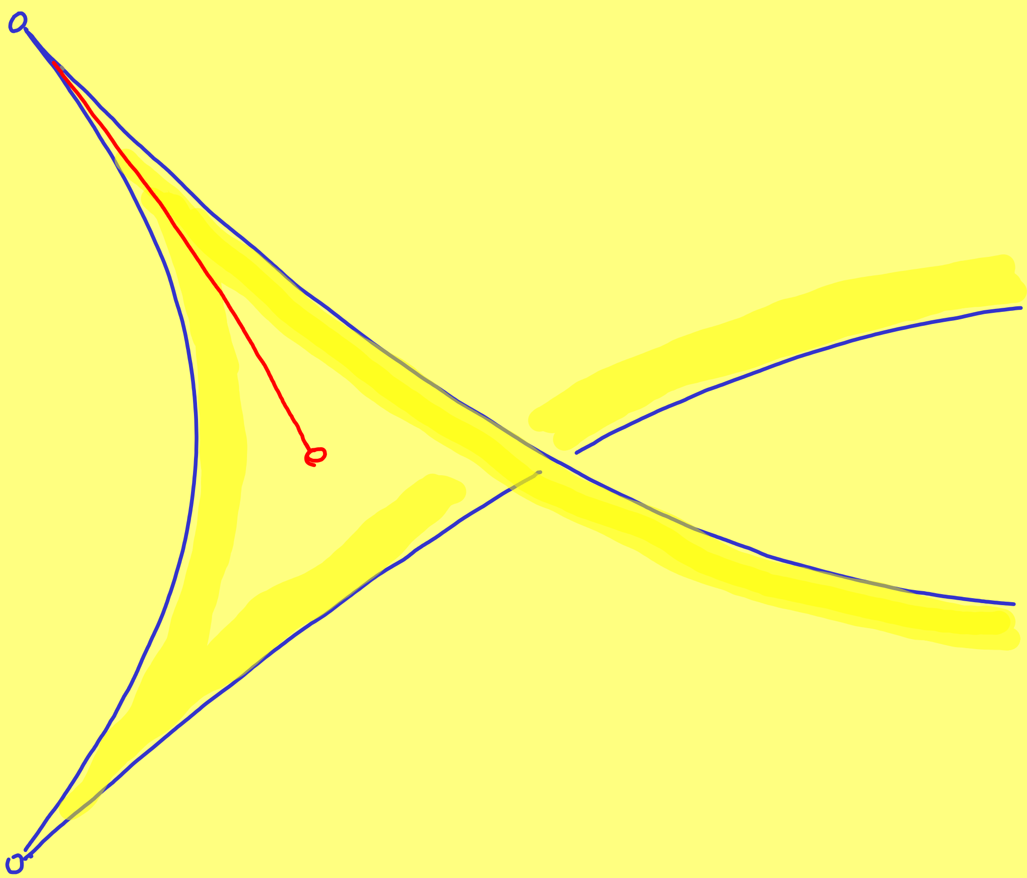


III.3 Erosion



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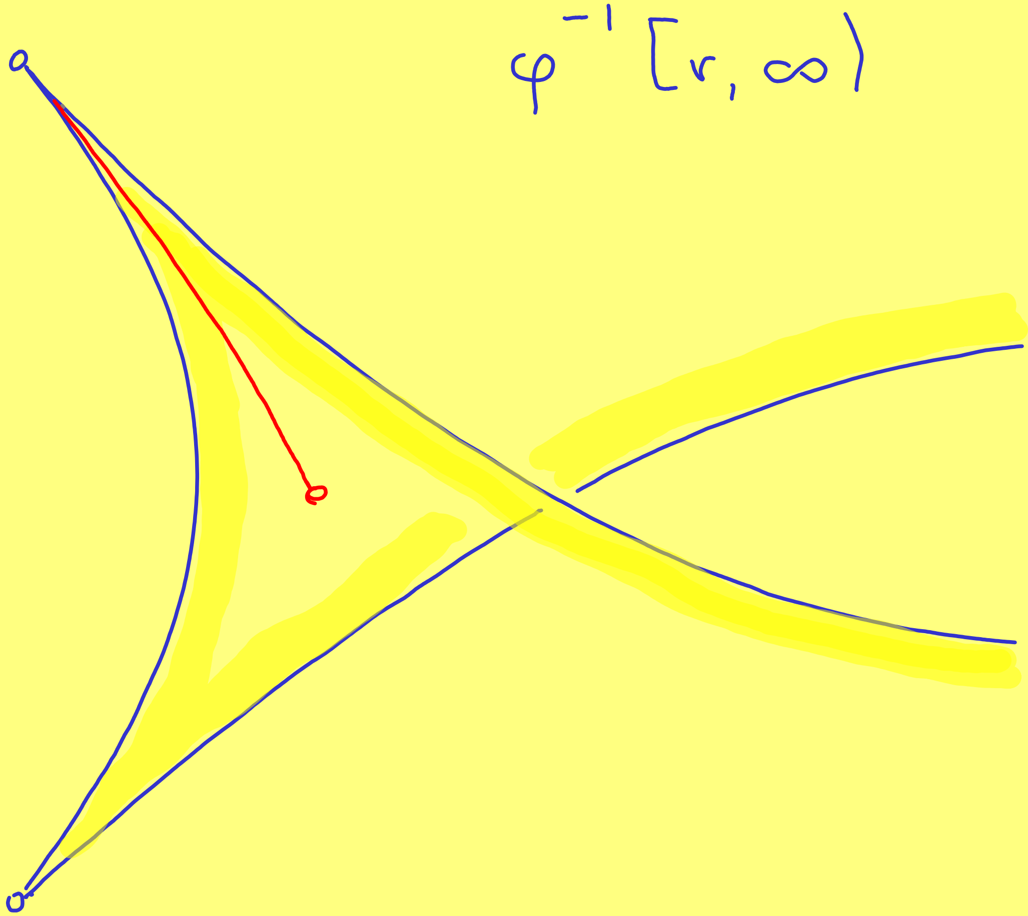
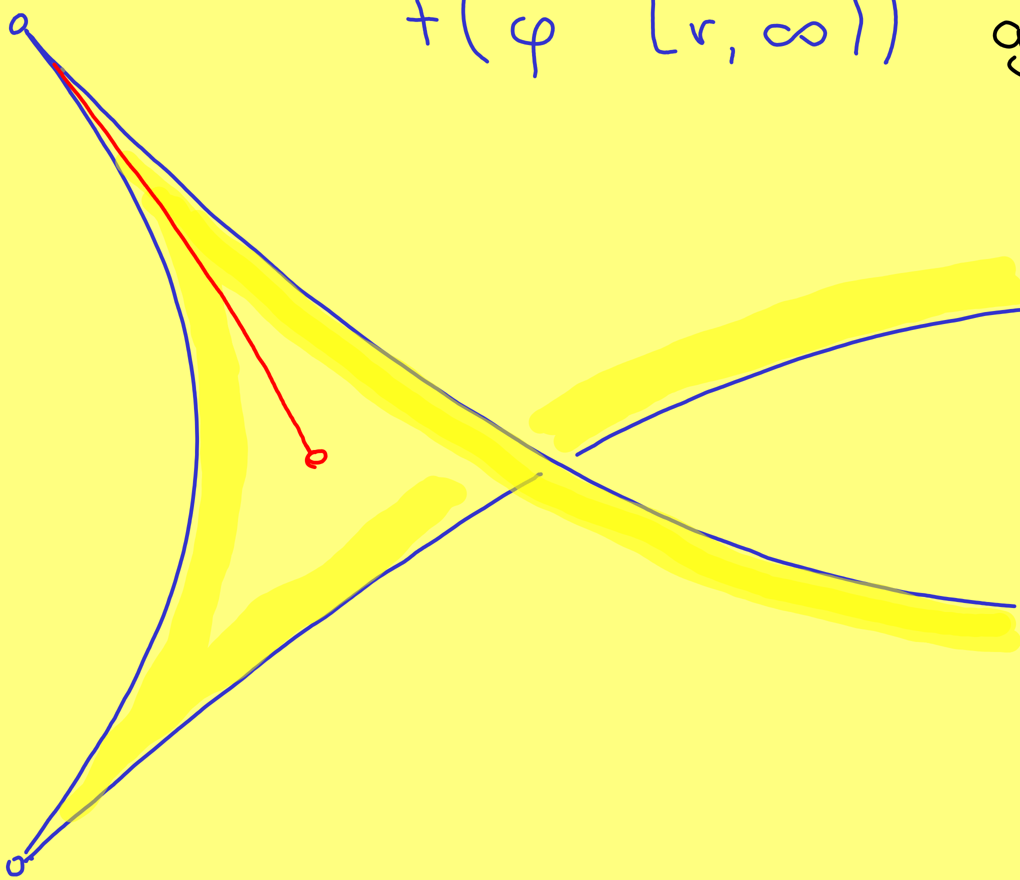


III.3 Erosion

$f(\varphi^{-1}[r, \infty))$ generalizes erosion



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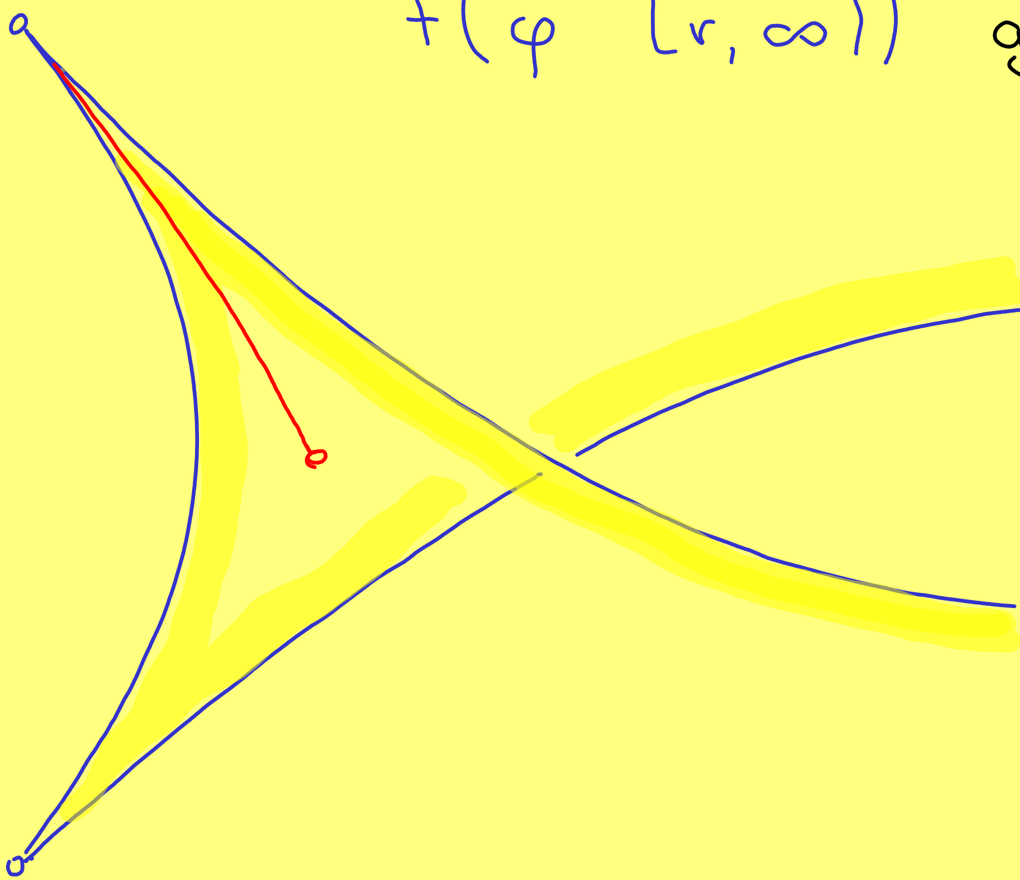

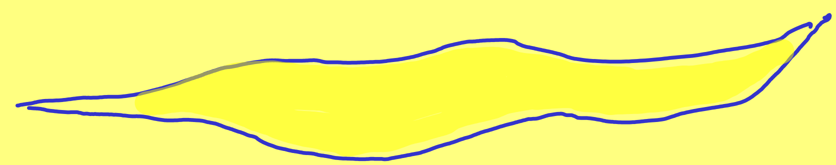

III.4 Global Contour Stability

$f, g : \mathbb{M} \to \mathbb{R}^2$; $\quad \varepsilon = \max_{x \in \mathbb{M}} \|f(x) - g(x)\|_2 .$

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Global Contour Theorem

$\exists$ subsets $\varphi^{-1}[\varepsilon, \infty) \subseteq \mathbb{M}_f \subseteq \mathbb{M}$

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Global Contour Theorem

$\exists$ subsets $\varphi^{-1}[\varepsilon, \infty) \subseteq \mathbb{M}_f \subseteq \mathbb{M}$

$\gamma^{-1}[\varepsilon, \infty) \subseteq \mathbb{M}_g \subseteq \mathbb{M}$

and a bijection $\iota : \mathbb{M}_f \to \mathbb{M}_g$

s.t. $f = g \circ \iota$ and $g = f \circ \iota^{-1}$ if defined.

THANK YOU