Polyhedral Outer Approximations with Application to Natural Language Parsing

André F. T. Martins[1,2] Noah A. Smith[1] Eric P. Xing[1]

[1]Language Technologies Institute
School of Computer Science
Carnegie Mellon University
Pittsburgh, PA, USA

[2]Instituto de Telecomunicações
Instituto Superior Técnico
Lisboa, Portugal

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- Structured prediction: models interdependence among outputs
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- This paper: LP-relaxed inference and max-margin learning
  - Guarantees for algorithmic separability
  - New interpretation: balancing accuracy and computational cost
  - Learning bounds via polyhedral characterizations
- Application: dependency parsing with rich features

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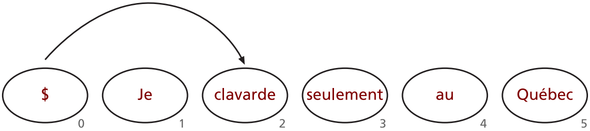

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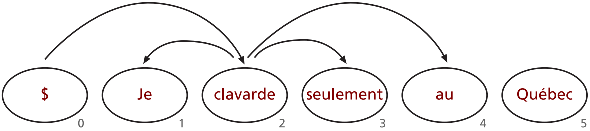

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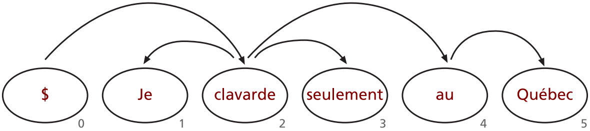

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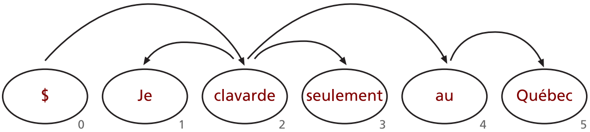

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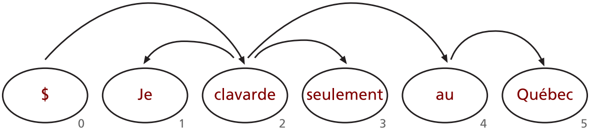

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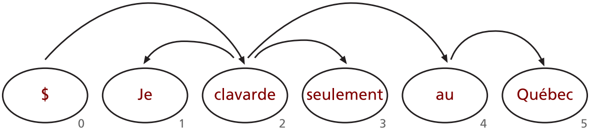

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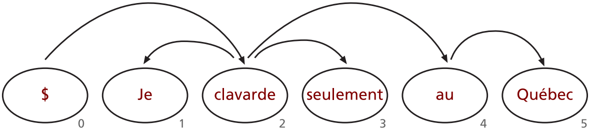

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- This is a structured classification problem involving non-local interactions among output variables

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- Here: linear classifiers

$$h_{\mathbf{w}}(x) = \arg\max_{y \in \mathcal{Y}} \mathbf{w}^\top \mathbf{f}(x, y)$$

- Hypothesis space $\mathcal{H} \triangleq \{h_{\mathbf{w}} \mid \mathbf{w} \in \mathcal{W}\}$, $\mathcal{W} \subseteq \mathbb{R}^d$ convex

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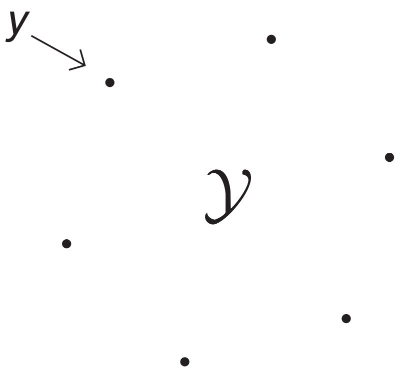
- Assumption: features decompose over the parts

$$\mathbf{f}(x, y) \triangleq \sum_{r \in y} \mathbf{f}_r(x) = \sum_{r \in \mathcal{R}} z_r \mathbf{f}_r(x) = \mathbf{F}(x)\mathbf{z},$$

- $\mathbf{F}(x) \triangleq (\mathbf{f}_r(x))_{r \in \mathcal{R}}$ is a feature matrix (d-by-$|\mathcal{R}|$)

From the Output Set to a Polytope



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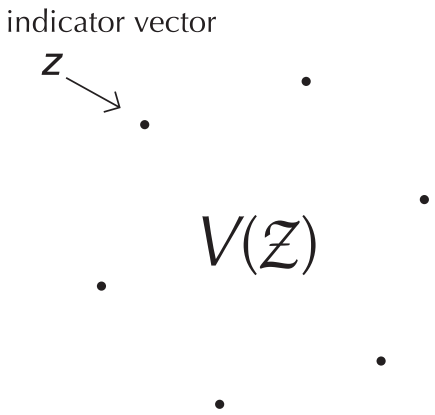

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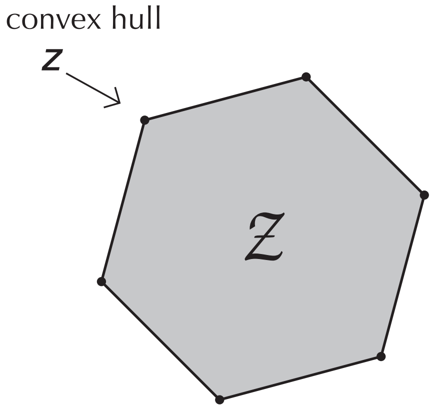

Inference

- Minkowski-Weyl theorem: there is a representation

$$\mathcal{Z} = \{\mathbf{z} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{z} \leq \mathbf{b}\}$$

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- Inference becomes an LP [Taskar et al., 2004]:

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- In general $p = O(\exp(n))$ (exponentially many constraints)

LP-Relaxed Inference and Outer Polytope

- Often: a concise representation of an outer polytope $\bar{\mathcal{Z}} \supseteq \mathcal{Z}$ such that $\bar{\mathcal{Z}} \cap \mathbb{Z}^n = V(\mathcal{Z})$

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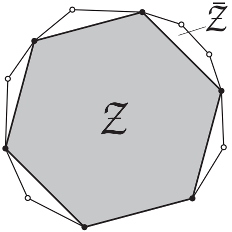

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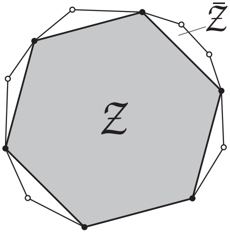

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- Example: Hamming loss

$$\begin{aligned}\ell(y'; y) &\triangleq \sum_{r \in \mathcal{R}} \left(\mathbb{I}(r \in y')\mathbb{I}(r \notin y) + \mathbb{I}(r \notin y')\mathbb{I}(r \in y)\right) \\ &= \|\mathbf{z}' - \mathbf{z}\|_1 \\ &= \mathbf{p}^\top \mathbf{z}' + q \quad \text{where } \mathbf{p} \triangleq \mathbf{1} - 2\mathbf{z} \text{ and } q \triangleq \mathbf{1}^\top \mathbf{z}\end{aligned}$$

- Hamming loss is an affine function of $\mathbf{z}$

Learning

- Structured SVM:

$$\min_{\mathbf{w}} \frac{\lambda}{2}\|\mathbf{w}\|^2 + \frac{1}{m}\sum_{t=1}^{m} r_t(\mathbf{w})$$

where the slack $r_t(\mathbf{w})$ is the solution of the loss-augmented inference (LAI) problem

$$r_t(\mathbf{w}) = \max_{y_t' \in \mathcal{Y}} \mathbf{w}^\top \underbrace{\mathbf{f}(x_t, y_t')}_{\mathbf{F}_t \mathbf{z}_t'} - \mathbf{w}^\top \underbrace{\mathbf{f}(x_t, y_t)}_{\mathbf{F}_t \mathbf{z}_t} + \underbrace{\ell(y_t'; y_t)}_{\mathbf{p}_t \mathbf{z}_t' + q_t}$$

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- Relax.

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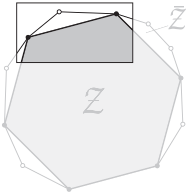
- Remark: $\mathcal{L}$ algorithmically separable $\implies$ $\mathcal{L}$ separable
- Margin of separation: Minimal γ s.t. $\forall (x_t, y_t) \in \mathcal{L}, y'_t \in \mathcal{Y}(x_t)$:

$$\mathbf{w}^\top \mathbf{f}(x_t, y_t) \geq \mathbf{w}^\top \mathbf{f}(x_t, y'_t) + \gamma \ell(y_t, y'_t) \quad \text{with } \|\mathbf{w}\| = 1.$$

A Sufficient Condition for Algorithmic Separability

- Zooming around a fractional vertex of the outer polytope:



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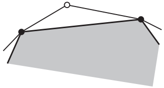

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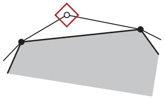
- Let L be the radius of the largest loss ball centered at a fractional vertex which does not contain any integer vertex



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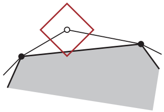

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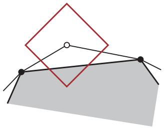

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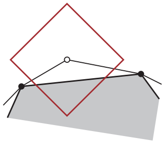

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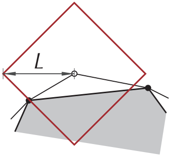

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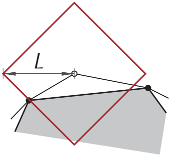

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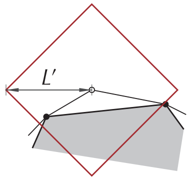

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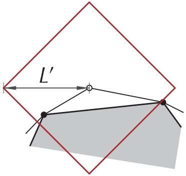

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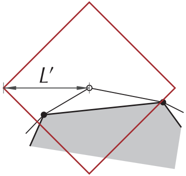

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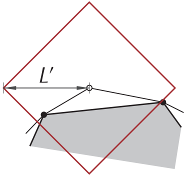

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  - In the paper: bounds for the nonseparable case
  - Also: bounds for ϵ-approximate algorithms

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- Recall that exact inference in our setting is cast as an ILP:

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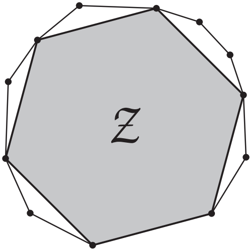

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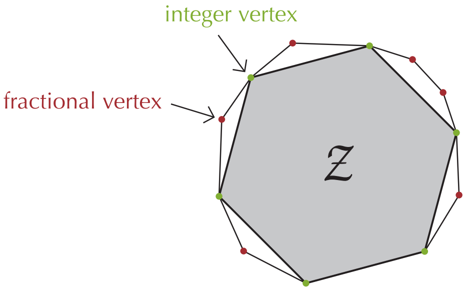

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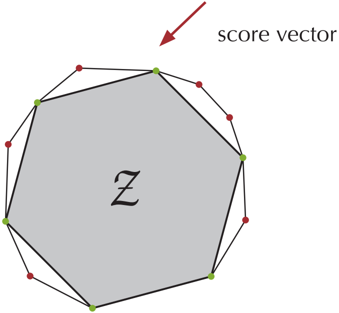

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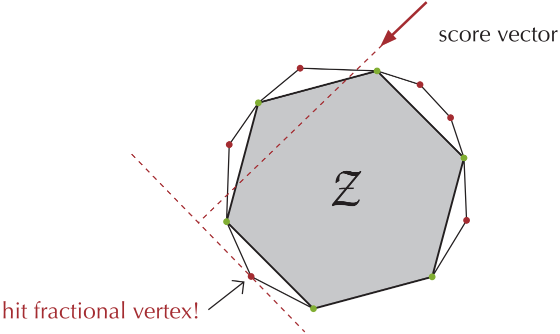

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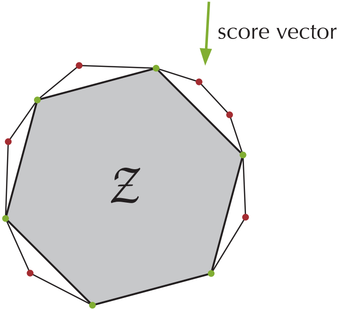

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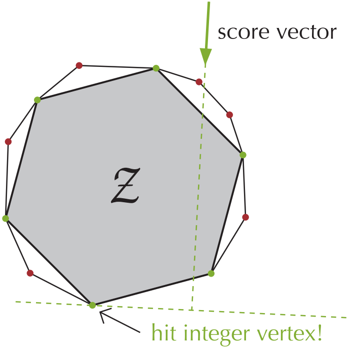

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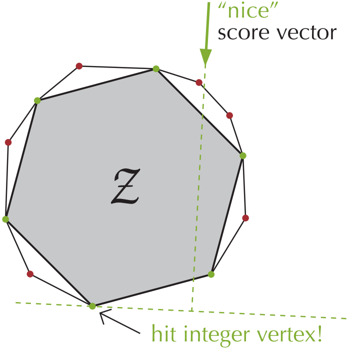
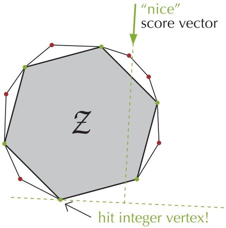


Nice Score Vectors and Low-Cost Hypotheses

- A "nice" score vector $\mathbf{s}$ is one which hits an integer vertex



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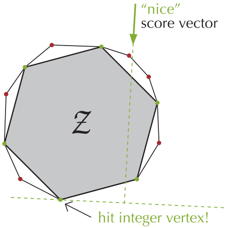

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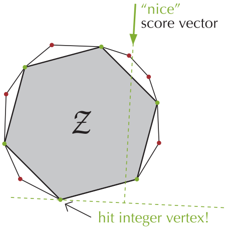

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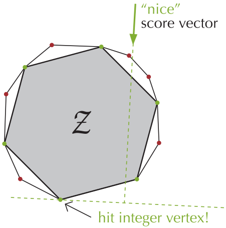

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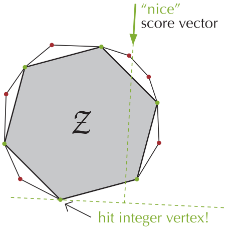

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  - Most ILP solvers (branch-and-bound, Gomory's cuts) converge faster as this gap is smaller

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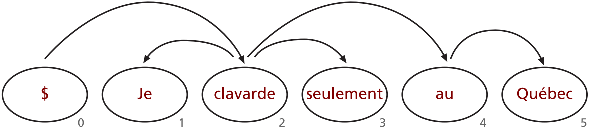
- In the paper: a stochastic adaptation of the online subgradient algorithm [Ratliff et al., 2006]
- A PAC bound with respect to the best exact learner
  - It measures the impact of the approximation in learning
  - Previous bounds were in terms of the approximate learner [Kulesza and Pereira, 2007]

Outline

Experiments

- Dependency parsing for seven languages
  - Danish, Dutch, Portuguese, Slovene, Swedish, Turkish, English



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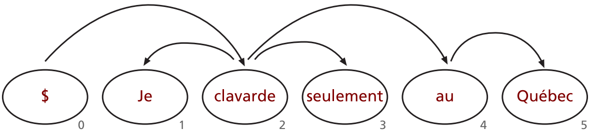

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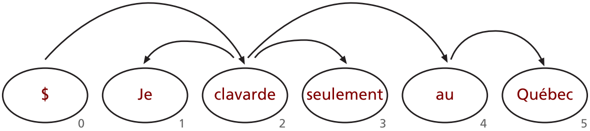

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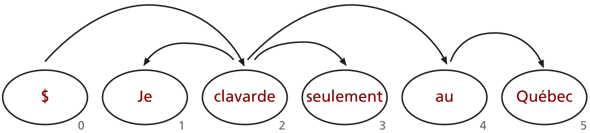

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- Beyond that: NP-hard [McDonald and Satta, 2007]
- Our model: a ILP formulation with non-arc-factored features
  - Models grandparents/siblings interactions
  - Models valency and nonprojective arcs
  - Only $O(n^3)$ variables and constraints
  - More details: [Martins et al., 2009]

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- Strong baselines:
  - [MP06] — approximate second-order parser [McDonald and Pereira, 2006]
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- Our models:
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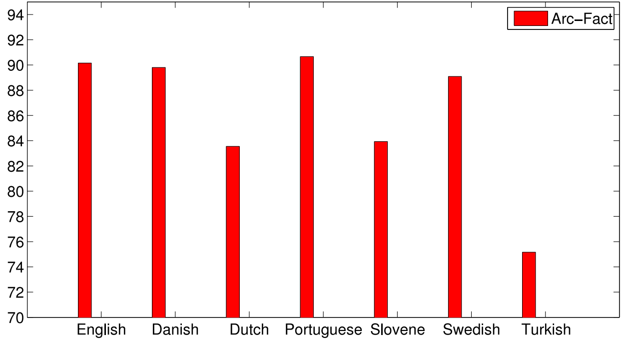

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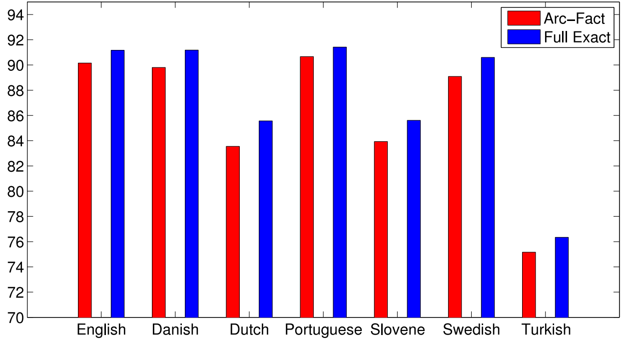

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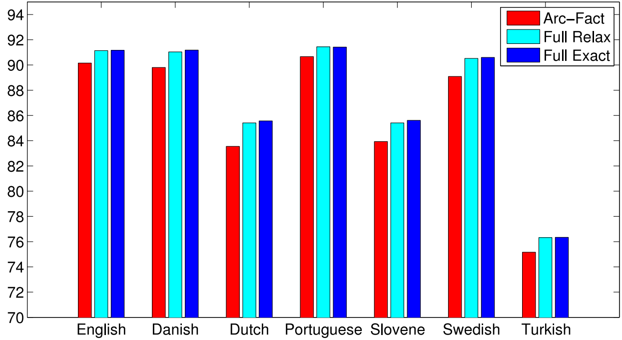
  - Approximate decoding — did not considerably affect accuracy



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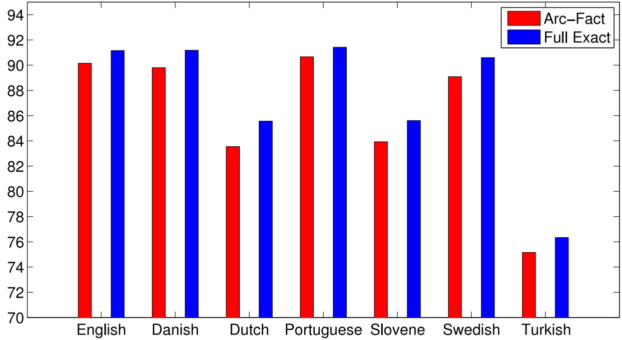

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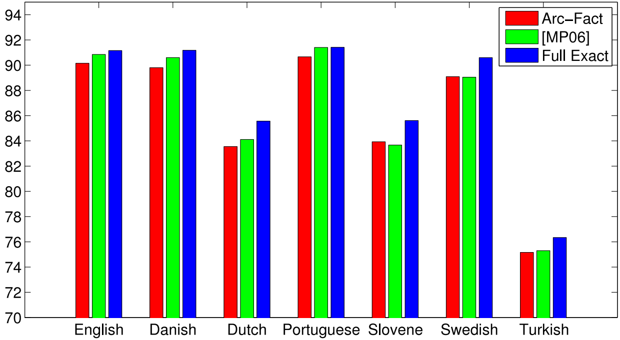

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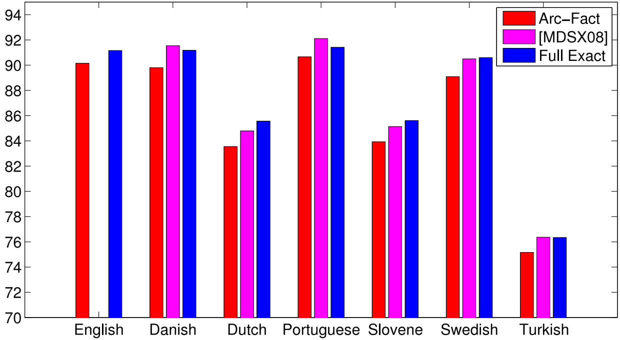

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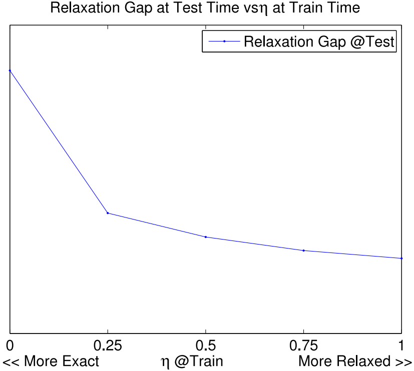

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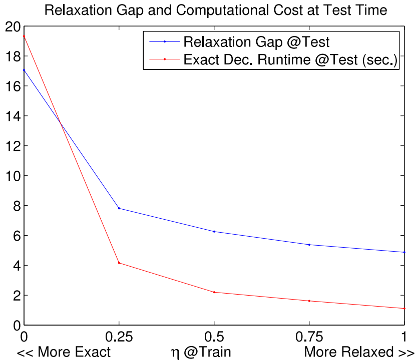

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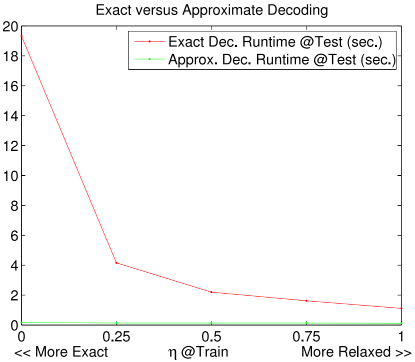

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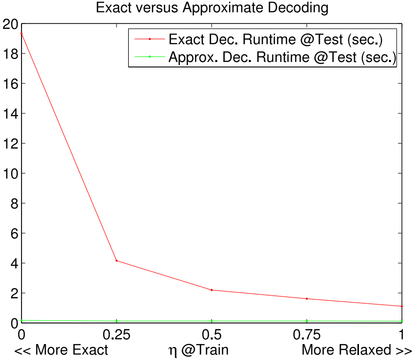

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- Our full model: Same order of magnitude as the baselines ($\approx$ 0.632 sec.)

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- Future work: polyhedral characterizations that guarantee tighter bounds

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- We demonstrated the effectiveness of these techniques in dependency parsing with non-arc-factored features
- Future work: polyhedral characterizations that guarantee tighter bounds
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Conclusions and Future Work

- We studied the impact of LP relaxed inference in max-margin learning
- We established sufficient conditions for algorithmic separability
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- Future work: polyhedral characterizations that guarantee tighter bounds
- Conditions for vanishing relaxation gap in online learning
- Connections with regularization

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Balancing Accuracy and Runtime

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end for
Return the averaged model $\hat{\mathbf{w}} \leftarrow \frac{1}{m}\sum_{t=1}^{m} \mathbf{w}_t$.

Generalization Bound

Proposition

Setting $\alpha_t = 1/(\lambda t)$, $\lambda = \Theta\left(\sqrt{\frac{1+\log m}{m}}\right)$ *and* $\eta_t = \Theta(t^{-1/2})$*:*

$$\mathbb{E}\ell(h_{\hat{\mathbf{w}}}(X), Y) \leq \frac{1}{m}\sum_{t=1}^{m} r_t(\mathbf{w}^*) + \frac{L \cdot o(m)}{m} + O\left(\sqrt{\frac{1}{m}\ln\frac{1}{\delta}}\right)$$

holds with probability $\geq 1 - \delta$.

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