

# Empirical portfolio selections

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  - empirical (nonparametric statistics, machine learning)

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asymptotic growth rate

$$W^{(j)} = \lim_{n \rightarrow \infty} W_n^{(j)} = \lim_{n \rightarrow \infty} \frac{1}{n} \ln S_n^{(j)}$$

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we can do much better, applying dynamic portfolio selection

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$\mathbf{b}$  is the constant portfolio vector for each trading day

for the first day  $S_0$  denotes the initial capital

$$S_1 = S_0 \sum_{j=1}^d b^{(j)} x_1^{(j)} = S_0 \langle \mathbf{b}, \mathbf{x}_1 \rangle$$

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with the average growth rate

$$W_n(\mathbf{b}) = \frac{1}{n} \sum_{i=1}^n \ln \langle \mathbf{b}, \mathbf{x}_i \rangle.$$

Special market process:  $\mathbf{X}_1, \mathbf{X}_2, \dots$  is independent and identically distributed (i.i.d.)

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Best Constantly Re-balanced Portfolio (BCRP)

for non i.i.d. market process we can do even better

gambling, horse racing, information theory

Kelly (1956)

Latané (1959)

Breiman (1961)

Finkelstein and Whitley (1981)

Barron and Cover (1988)

Chapter 15 of D. G. Luenberger, *Investment Science*. Oxford University Press, 1998.

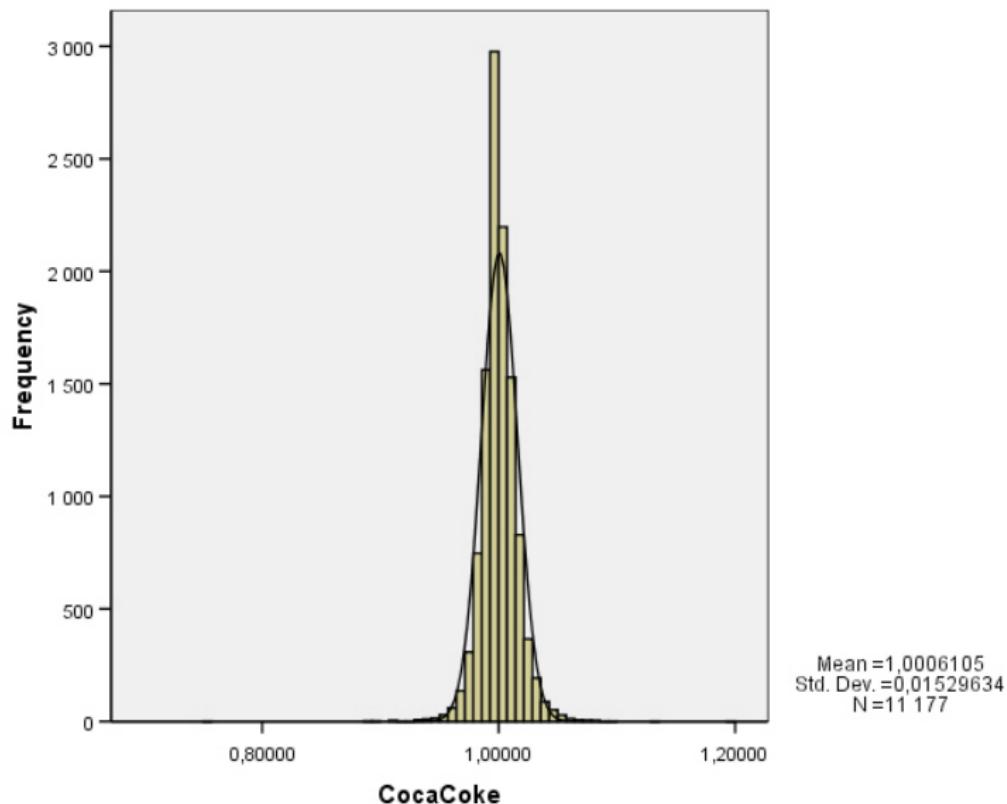
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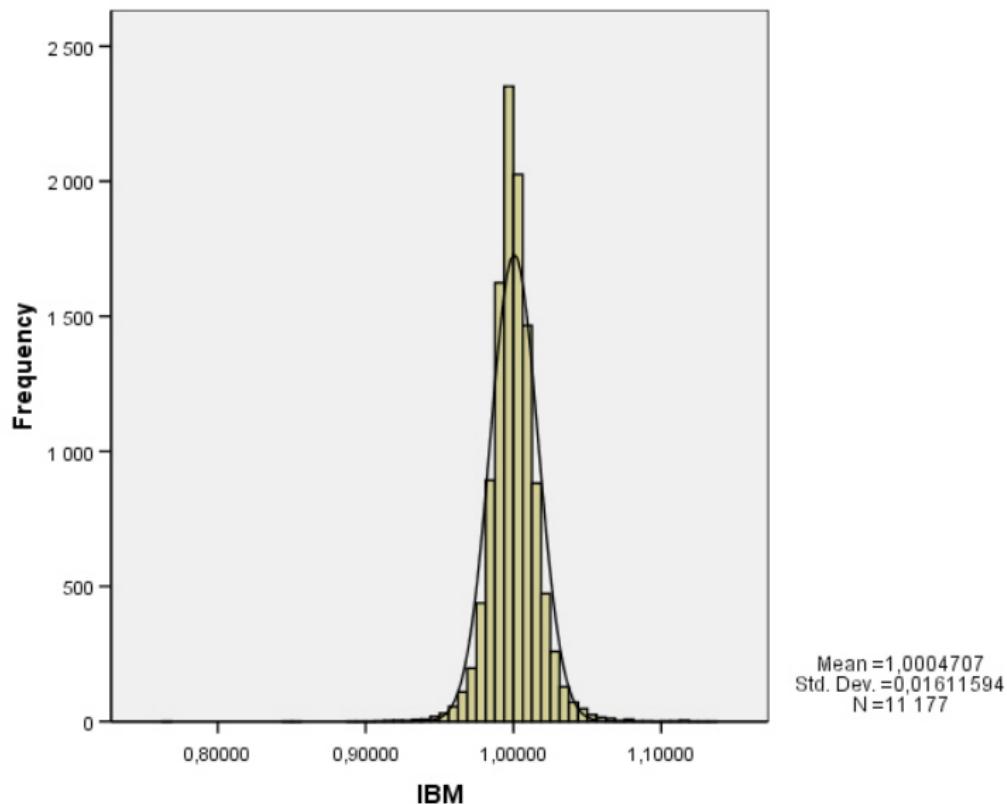
"Conclusions about multiperiod investment situations are not mere variations of single-period conclusions – rather they offer *reverse* those earlier conclusions. This makes the subject exiting, both intellectually and in practice. Once the subtleties of multiperiod investment are understood, the reward in terms of enhanced investment performance can be substantial."

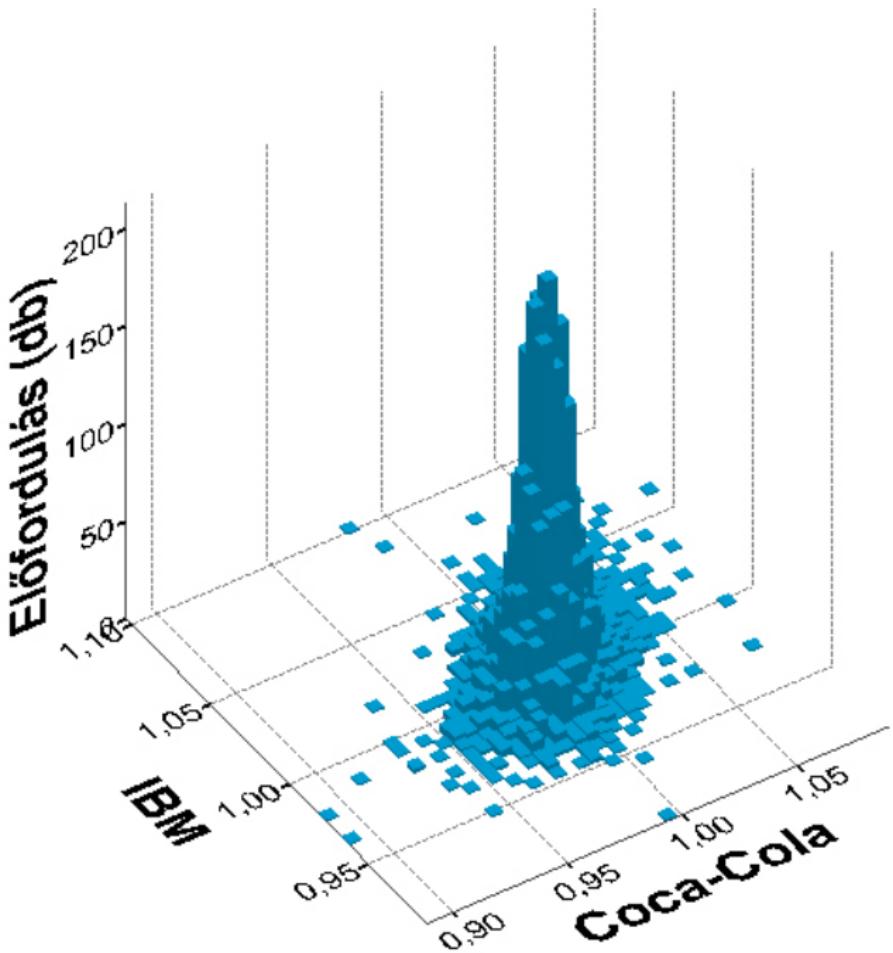
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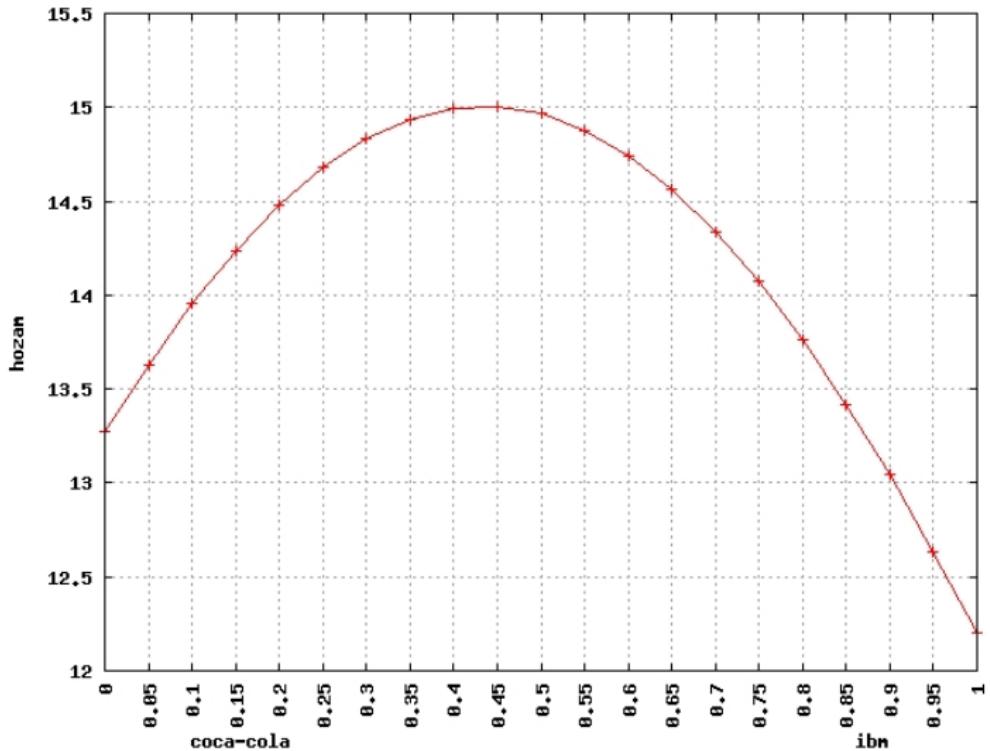
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"Fortunately the concepts and the methods of analysis for multiperiod situation build on those of earlier chapters. Internal rate of return, present value, the comparison principle, portfolio design, and lattice and tree valuation all have natural extensions to general situations. But conclusions such as volatility is "bad" or diversification is "good" are no longer universal truths. The story is much more interesting."









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log-optimum portfolio  $\mathbf{B}^* = \{\mathbf{b}^*(\cdot)\}$

$$\mathbf{E}\{\ln \langle \mathbf{b}^*(\mathbf{X}_1^{n-1}), \mathbf{X}_n \rangle \mid \mathbf{X}_1^{n-1}\} = \max_{\mathbf{b}(\cdot)} \mathbf{E}\{\ln \langle \mathbf{b}(\mathbf{X}_1^{n-1}), \mathbf{X}_n \rangle \mid \mathbf{X}_1^{n-1}\}$$

Algoet and Cover (1988): If  $S_n^* = S_n(\mathbf{B}^*)$  denotes the capital after day  $n$  achieved by a log-optimum portfolio strategy  $\mathbf{B}^*$ , then for any portfolio strategy  $\mathbf{B}$  with capital  $S_n = S_n(\mathbf{B})$  and for any stationary ergodic process  $\{\mathbf{X}_n\}_{-\infty}^{\infty}$ ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln \frac{S_n}{S_n^*} \leq 0 \quad \text{almost surely}$$

# Optimality

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and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln S_n^* = W^* \quad \text{almost surely},$$

where

$$W^* = \mathbf{E} \left\{ \max_{\mathbf{b}(\cdot)} \mathbf{E} \{ \ln \langle \mathbf{b}(\mathbf{X}_{-\infty}^{-1}), \mathbf{X}_0 \rangle \mid \mathbf{X}_{-\infty}^{-1} \} \right\}$$

is the maximal growth rate of any portfolio.

# Proof

$$\frac{1}{n} \ln S_n = \frac{1}{n} \sum_{i=1}^n \ln \left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i \right\rangle$$

# Proof

$$\begin{aligned}\frac{1}{n} \ln S_n &= \frac{1}{n} \sum_{i=1}^n \ln \left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i \right\rangle \\&= \frac{1}{n} \sum_{i=1}^n \mathbf{E} \{ \ln \left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i \right\rangle | \mathbf{X}_1^{i-1} \} \\&\quad + \frac{1}{n} \sum_{i=1}^n \left( \ln \left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i \right\rangle - \mathbf{E} \{ \ln \left\langle \mathbf{b}(\mathbf{X}_1^{i-1}), \mathbf{X}_i \right\rangle | \mathbf{X}_1^{i-1} \} \right)\end{aligned}$$

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and

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These limit relations give rise to the following definition:

## Definition

A portfolio strategy  $\mathbf{B}$  is called **universally consistent with respect to a class  $\mathcal{C}$  of stationary and ergodic processes  $\{\mathbf{X}_n\}_{-\infty}^{\infty}$** , if for each process in the class,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln S_n(\mathbf{B}) = W^* \quad \text{almost surely.}$$

# Empirical portfolio selection

$$\mathbf{E}\{\ln \langle \mathbf{b}^*(\mathbf{X}_1^{n-1}), \mathbf{X}_n \rangle \mid \mathbf{X}_1^{n-1}\} = \max_{\mathbf{b}(\cdot)} \mathbf{E}\{\ln \langle \mathbf{b}(\mathbf{X}_1^{n-1}), \mathbf{X}_n \rangle \mid \mathbf{X}_1^{n-1}\}$$

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$$\begin{aligned}\mathbf{b}^*(\mathbf{x}_1^{n-1}) &= \arg \max_{\mathbf{b}(\cdot)} \mathbf{E}\{\ln \langle \mathbf{b}(\mathbf{X}_1^{n-1}), \mathbf{X}_n \rangle \mid \mathbf{X}_1^{n-1} = \mathbf{x}_1^{n-1}\} \\ &= \arg \max_{\mathbf{b}(\cdot)} \mathbf{E}\{\ln \langle \mathbf{b}(\mathbf{x}_1^{n-1}), \mathbf{X}_n \rangle \mid \mathbf{X}_1^{n-1} = \mathbf{x}_1^{n-1}\} \\ &= \arg \max_{\mathbf{b}} \mathbf{E}\{\ln \langle \mathbf{b}, \mathbf{X}_n \rangle \mid \mathbf{X}_1^{n-1} = \mathbf{x}_1^{n-1}\},\end{aligned}$$

fixed integer  $k > 0$

$$\mathbf{E}\{\ln \langle \mathbf{b}(\mathbf{X}_1^{n-1}), \mathbf{X}_n \rangle \mid \mathbf{X}_1^{n-1}\} \approx \mathbf{E}\{\ln \langle \mathbf{b}(\mathbf{X}_{n-k}^{n-1}), \mathbf{X}_n \rangle \mid \mathbf{X}_{n-k}^{n-1}\}$$

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which is the maximization of the regression function

$$m_{\mathbf{b}}(\mathbf{x}_1^k) = \mathbf{E}\{\ln \langle \mathbf{b}, \mathbf{X}_{k+1} \rangle \mid \mathbf{X}_1^k = \mathbf{x}_1^k\}$$

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$Y$  real valued

$X$  observation vector

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L. Györfi, M. Kohler, A. Krzyzak, H. Walk (2002) *A Distribution-Free Theory of Nonparametric Regression*, Springer-Verlag, New York.

*Springer Series in Statistics*

László Györfi Michael Kohler  
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# Correspondence

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# Kernel-based portfolio selection

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for  $n > k + 1$ , define the expert  $\mathbf{b}^{(k,\ell)}$  by

$$\mathbf{b}^{(k,\ell)}(\mathbf{x}_1^{n-1}) = \arg \max_{\mathbf{b}} \sum_{\{k < i < n : \|\mathbf{x}_{i-k}^{i-1} - \mathbf{x}_{n-k}^{n-1}\| \leq r_{k,\ell}\}} \ln \langle \mathbf{b}, \mathbf{x}_i \rangle,$$

if the sum is non-void, and  $\mathbf{b}_0 = (1/d, \dots, 1/d)$  otherwise.

# Combining elementary portfolios

for fixed  $k, \ell = 1, 2, \dots,$

$\mathbf{B}^{(k,\ell)} = \{\mathbf{b}^{(k,\ell)}(\cdot)\}$ , are called elementary portfolios

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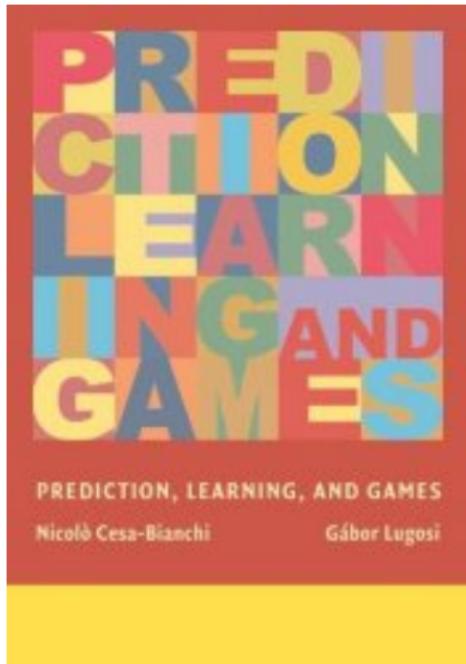
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Machine learning: combination of experts

N. Cesa-Bianchi and G. Lugosi, *Prediction, Learning, and Games*.  
Cambridge University Press, 2006.



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the combined portfolio  $\mathbf{b}$ :

$$\mathbf{b}_n(\mathbf{x}_1^{n-1}) = \sum_{k,\ell} v_{n,k,\ell} \mathbf{b}_n^{(k,\ell)}(\mathbf{x}_1^{n-1}).$$

$$S_n(\mathbf{B}) = \prod_{i=1}^n \left\langle \mathbf{b}_i(\mathbf{x}_1^{i-1}), \mathbf{x}_i \right\rangle$$

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 S_n(\mathbf{B}) &= \prod_{i=1}^n \left\langle \mathbf{b}_i(\mathbf{x}_1^{i-1}), \mathbf{x}_i \right\rangle \\
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&= \sum_{k,\ell} q_{k,\ell} S_n(\mathbf{B}^{(k,\ell)}),
\end{aligned}$$

The strategy  $\mathbf{B}$  then arises from weighing the elementary portfolio strategies  $\mathbf{B}^{(k,\ell)} = \{\mathbf{b}_n^{(k,\ell)}\}$  such that the investor's capital becomes

$$S_n(\mathbf{B}) = \sum_{k,\ell} q_{k,\ell} S_n(\mathbf{B}^{(k,\ell)}).$$

# Theorem

The kernel-based portfolio scheme is universally consistent with respect to the class of all ergodic processes such that  
 $\mathbf{E}\{|\ln X^{(j)}|\} < \infty$ , for  $j = 1, 2, \dots, d$ .

L. Györfi, G. Lugosi, F. Udina (2006) "Nonparametric kernel-based sequential investment strategies", *Mathematical Finance*, 16, pp. 337-357

[www.sztit.bme.hu/~gyorfi/kernel.pdf](http://www.sztit.bme.hu/~gyorfi/kernel.pdf)

# Proof

We have to prove that

$$\liminf_{n \rightarrow \infty} W_n(\mathbf{B}) = \liminf_{n \rightarrow \infty} \frac{1}{n} \ln S_n(\mathbf{B}) \geq W^* \quad \text{a.s.}$$

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Thus

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Because of  $\lim_{\ell \rightarrow \infty} r_{k,\ell} = 0$ , we have that

$$\sup_{k,\ell} \epsilon_{k,\ell} = \lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} \epsilon_{k,\ell} = W^*.$$

# Semi-log-optimal portfolio

empirical log-optimal:

$$\mathbf{b}^{(k,\ell)}(\mathbf{x}_1^{n-1}) = \arg \max_{\mathbf{b}} \sum_{i \in J_n} \ln \langle \mathbf{b}, \mathbf{x}_i \rangle$$

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# Semi-log-optimal portfolio

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$$\mathbf{b}^{(k,\ell)}(\mathbf{x}_1^{n-1}) = \arg \max_{\mathbf{b}} \sum_{i \in J_n} \ln \langle \mathbf{b}, \mathbf{x}_i \rangle$$

Taylor expansion:  $\ln z \approx h(z) = z - 1 - \frac{1}{2}(z - 1)^2$  empirical  
semi-log-optimal:

$$\tilde{\mathbf{b}}^{(k,\ell)}(\mathbf{x}_1^{n-1}) = \arg \max_{\mathbf{b}} \sum_{i \in J_n} h(\langle \mathbf{b}, \mathbf{x}_i \rangle) = \arg \max_{\mathbf{b}} \{ \langle \mathbf{b}, \mathbf{m} \rangle - \langle \mathbf{b}, \mathbf{C}\mathbf{b} \rangle \}$$

Connection to the Markowitz theory.

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L. Györfi, A. Urbán, I. Vajda (2007) "Kernel-based semi-log-optimal portfolio selection strategies", *International Journal of Theoretical and Applied Finance*, 10, pp. 505-516.  
[www.szit.bme.hu/~gyorfi/semi.pdf](http://www.szit.bme.hu/~gyorfi/semi.pdf)

# Conditions of the model:

Assume that

- the assets are arbitrarily divisible,
- the assets are available in unbounded quantities at the current price at any given trading period,
- there are no transaction costs,
- the behavior of the market is not affected by the actions of the investor using the strategy under investigation.

At [www.szit.bme.hu/~otil/portfolio](http://www.szit.bme.hu/~otil/portfolio) there are two benchmark data set from NYSE:

- The first data set consists of daily data of 36 stocks with length 22 years.
- The second data set contains 23 stocks and has length 44 years.

# NYSE data sets

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- The second data set contains 23 stocks and has length 44 years.

Our experiment is on the second data set such that left four assets (SHERW, KODAK, COMME, KINAR) having small capitalization (less than  $10^{10}$  dollars)

# Experiments on average annual yields (AAY)

Kernel based semi-log-optimal portfolio selection with  
 $k = 1, \dots, 5$  and  $\ell = 1, \dots, 10$

$$r_{k,\ell}^2 = 0.0002 \cdot d \cdot k + 0.00002 \cdot d \cdot k \cdot \ell,$$

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the BCRP had average AAY 21%

The average annual yields of the individual experts, for the 19 large assets.

$k$	1	2	3	4	5
$\ell$					
1	31%	30%	24%	21%	26%
2	34%	31%	27%	25%	22%
3	35%	29%	26%	24%	23%
4	35%	30%	30%	32%	27%
5	34%	29%	33%	24%	24%
6	35%	29%	28%	24%	27%
7	33%	29%	32%	23%	23%
8	34%	33%	30%	21%	24%
9	37%	33%	28%	19%	21%
10	34%	29%	26%	20%	24%