Foundational concerns and mathematical concerns

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April 2011

Introduction

What I want to convey is my lack of sympathy with foundational activity that is not informed by the evolution of the organism of Mathematics. On the other hand, I must have felt differently in my youth, since I did two philosophy degrees, and almost embarked on a Philosophy PhD before I returned to Mathematics.

Whatever foundational concerns I had have abated,and indeed I have no sense that there are now any foundational issues.

At the risk of appearing arrogant, and with the excuse that I have reached an age when one must take stock, I want to cast a cold eye on what has happened in foundations of mathematics in the modern era (19th century till now).

In a sense I speculate as to how foundational activity till now will be viewed by later generations of mathematicians. I find it hard to imagine a future where the likes of Hilbert, Brouwer, Poincare or Weyl will engage with foundational issues of the generality common in the classical era.

What do I understand by a foundational concern?

These will vary from generation to generation,and depend on at least educational background. I have neither the time, nor the inclination, nor indeed the ability, to give a historical account.

Rather I will consider matters which were concerns of my adolescent self, but which do not trouble me now (and in some cases give me great mathematical pleasure).
That they faded as foundational concerns seems to me due neither to my being lazy nor to my losing sophistication.

Geometry

One type, probably appearing in many generations, concerns the evolution from geometric ideas to ideas of analysis and calculus.

This is conventionally taken as a move towards clarity, but with age I have come to question this. Note that the trends which have virtually ousted geometry from elementary education may cause such concerns to vanish.

This would surely be to the detriment of mathematics, and to the public's conception of mathematics.

Examples

1. Lengths, areas and volumes of curvilinear figures ;
2. "Meaning" of angle ;
3. Irrationals from elementary geometric constructions ;
4. Trisection of angles ;
5. The manifestations of π, and squaring of the circle ;
6. Calculus and its precursors ;
7. Axiomatization.

Numbers

All of these in their time were foundational concerns (of course of different weight), and have ceased to be, except for issues around axiomatization.

In the Greek beginning there was an evolving stock of "numbers" (all real) emerging from geometry, presumably with a sense that many others might emerge from lengths, areas or volumes of more general constructions.

The most important of all theorems in Euclidean geometry, Pythagoras' Theorem, concerns the arithmetic of areas. 3. is also covered by a fundamental theorem, but of arithmetic.

4. and 5. got impossibility proofs in the 19 th century, the former via abstract algebra (group theory, henceforward to be ubiquitous in mathematics) and the latter via arithmetic.

Mathematics without Foundations

To obtain "foundations" removing the classical concerns about numbers (and geometry) took 2000 years, but note that calculus (and the related analysis of infinite sums and products) flourished before these foundations were laid (and before the foundational ingredients were known..).

Various anomalies were detected and corrected.

The Fundamental Theorem of Algebra was proved, as was much of complex analysis, before the foundations were found.

The exponential function, logarithms, e, and Euler's magical formula, all predate foundations.

Moreover they predate all notions of dimension (a subject where foundational issues persisted after those of calculus had faded).

The various effects of foundational efforts

The foundational efforts of Dedekind and Frege gave definitions, implicit or explicit, of the fundamental mathematical structures.

Cantor's work was not foundational,and all the more important for mathematics for that reason. He gave new concepts and new arguments, and captured the imaginations of many mathematicians (and the suspicions of a few).

The inconsistency in Frege has proved of little mathematical significance - in contrast, his definition of the ancestral is of enduring interest, and is a part of the theory of ordinals in von Neumann style.

Set theory and the rest of mathematics

The advanced definitions and constructions of set theory have been peripheral to later mathematics, though the basic Cantorian arguments around countability are part of everyday culture.

To a large extent the more elaborate ideas about the uncountable have flourished mainly in the search for pathologies in topology and analysis, or algebra far from the mainstream. Transfinite induction (in uncountable settings) has had little impact on later mainstream mathematics.

How many Fields' Medallists have ever used it at all? (One at least..) It is of course crucial for various deep results in set theory and model theory.

One should note too that not long after the pseudo-crises in foundations, the geometers were beginning to put in new ideas (of analytic and topological origin) that would really dominate the next hundred years, such a homology and cohomology, and homotopy theory.

Concerns around Axioms and Incompleteness

Could one have reasonably expected completeness for a system like *ZFC*?

What the axioms clearly did was to codify a number of principles adequate for the formalization of known set-theoretic arguments, in particular those providing definitions of the fundamental structures.

That the system had various models was clear, but even the strength over arithmetic was unclear. One of the most productive foundational initiatives was that of Hilbert, to clarify the issue of strength here.

Concerns around Axioms and Incompleteness,continued

The history of the isolation of the axioms is very interesting, especially for Replacement (cf Kanamori's paper, and discussion of uses of the axiom).

The class/set distinction is formally worthwhile, but not in the end powerful axiomatically.

The work on AC is frequently beautiful,and sometimes enchanting (as on Lebesgue measurability). For me, however, it is more fruitful mathematically to do as Kreisel did and provide a metatheorem easing concerns of some of the very best mathematicians.

What did Formalization bring?

It led to Metamathematics, Proof Theory and Model Theory,and enormous areas of Computer Science, and thus extended the mathematical and scientific repertoire.

Whether or not it eased foundational concerns is less clear to me, particularly if those were about what there is. Evidently formalization consolidated a sense of the objectivity of mathematics.

What did Incompleteness bring?

Computabilty Theory (idealized,and thus all the more compelling in negative results), and an unsuspected fact of life about formalized proof (and then later more refined positive proof theory). I have argued in detail elsewhere that this is not now inhibiting Mathematics as an organism going where structure and connections are to be found.

We have no real idea of how Mathematics will look in hundreds of years time (there has been little success till now in predicting singular events), and it seems to me foolish to make "foundational" predictions.

Predictions

Most people will agree on the value(in Jacobi's sense) of the Clay Prize Problems. None is foundational per se.

One hopes for a mixture of powerful abstraction and refined special techniques such as those that have led to diverse advances on the Langlands Programme, or the efforts of Hamilton and Perelman leading to the solution of the Poincare Conjecture(and more) by essential use of non-topological ideas.

I see no indications that foundational considerations
will be relevant to such syntheses, which seem generally to have a geometric component.

Abstraction, Category Theory, Cohomology

The abstraction provided by category theory is evidently less than that of set theory, but it has the advantage of coming from geometry.

Set-theoretic formalization nowadays only rarely avoids obscuring the central ideas.

Typical of the enormous, and yet valuable , abstraction in category theory is in the theory of derived categories, giving the deep structure of cohomology. The latter, coming to us via Poincare and Kolmogorov, is increasingly present in central mathematics, in a way that no set-theoretic technique is.

The point about Foundations of Algebraic Geometry is that they have been changed several times, and in the last reorganization created ideas that are radiating through mathematics,and may change even logic.

The present place of set theory

It remains a convenient basis for definitions, on the one hand, and on the other is a highly sophisticated subject in its own right.

One waits patiently for some sign that it will meet up with the mathematics of the mainstream.

Is what Woodin is doing foundational? It is certainly great mathematics.

Is CH a mathematical or foundational concern?

In Friedman's work, what deep naturality underlies the great formal simplicity of his Boolean Relational Statements?

Guessing at future crises or dramas

If ZFC turned out to be inconsistent, Mathematics would surely not peter out.

Very few parts of the subject use set theory except as a foundational formalism.

While it is true that consistency statements are natural enough,and arithmetic, they are not natural diophantine statements, being of a complexity not yet contemplated in number theory.

I have argued elsewhere that there is simply no reason to expect our current technology for Incompleteness to be relevant for "natural number theory" (despite the enormous sophistication of that subject), and thus one should not expect any logical dramas.

For "finite combinatorics" the situation is less clear (Friedman).

Categoricity

One of the most attractive features of early foundations is the emphasis on categorical axiomatizations for classical structures.

That these cannot in general be done in first-order logic is of course a boon for model theory, but does not at all detract from the satisfying nature of the results. It is true that the first-order cases, rare in classical mathematics, have determined the course of model theory for many years. I think one has come to expect that new , central examples would not be found.

But Zilber has found some,and indeed it may be the case that the complex exponential is one. Zilber claims that such a phenomneon deserves to be called perfection, and is a sure sign of the mathematical signicficance of the subject matter.

Categoricity Continued

For me the problem has always been, in cases where one does not have unique isomorphism, that the theorems depend on the axiom of choice,and in interesting cases the mediating isomorphisms are nonmeasurable.

Whatever deep structure is being revealed depends on "invisible" maps . My shift of emphasis is that in all cases we must understand a lot about definitions in the structure in order for there to be any chance of such a theorem.

In fact, I see most foundational advances as being advances in the scope of our definitions, and therefore in the flexibility of our arguments. Schemes seem to me a particularly beautiful example.

What the set-theoretic definitions have lacked is not beauty, but flexibility.