Comparing Peano Arithmetic, Basic Law V, and Hume's Principle:

Models of Hume's Principle

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Hume's Principle codifies a basic principle about cardinalities, namely: any two properties have the same cardinality if and only if they are bijective.

Here we are conceiving of cardinality as a function *from* properties to *objects* e.g. as in the following simple model (from Boolos):

Consider the structure $(\omega, P(\omega), \#)$, where $\# : P(\omega) \to \omega$ such that $\#(X) = 0$ if X infinite, and $\#(X) = |X| + 1$ otherwise.

It is clear that this structure is a model of Hume's Principle (HP):

$$\#(X) = \#(Y) \Longleftrightarrow \exists \text{ bijection } f : X \to Y$$

Contemporary interest of HP arises from Frege's Theorem, which says that HP interprets the Peano axioms for arithmetic.

Here the sense of interpretability is the sense in which ZFC set theory interprets the Peano axioms for arithmetic: there are set-theoretic definitions of zero, successor, natural number etc. relative to which the Peano axioms are theorems of ZFC.

More generally, T^* *interprets* T if there are formulas in signature of T^* corresponding to primitives of T such that translation φ^* of theorems φ are themselves theorems:

$$T \vdash \varphi \Longrightarrow T^* \vdash \varphi^*$$

Frege's Theorem says that HP interprets the Peano axioms.

In particular, Frege showed how to produce cardinality-theoretic definitions of zero, successor, natural number etc. relative to which the Peano axioms are theorems of HP.

His definitions were the following, e.g. for successor and number:

- $S(n,m)$ iff $\exists\, X, Y\ \#X = n\ \&\ \#Y = m\ \&\ \exists\, y \in Y\ X = Y \setminus \{y\}$
- F is *hereditary* iff Fn and $S(n,m)$ implies Fm
- F is *closed* iff $S(\#\emptyset, m)$ implies Fm
- n is a *number* iff $n = \#\emptyset$ or contained in all hereditary, closed F.

Outline

II. Briefly recall connection of Frege's theorem to logicism in the philosophy of mathematics, which prompts the main question of whether there are predicative versions of Frege's Theorem.

III. Briefly indicate why, using basic tools from model theory of fields, there is no predicative version of Frege's Theorem in this sense.

IV. Indicate a further question– potentially relevant to viability of one variety of logicism– left unresolved by research presented here.

Frege articulated and discussed HP in his *Grundlagen* (1884), and it played an important role in his logicism (more on this soon).

Part of the historical context here is that HP variants were used in the arithmetization of analysis by Frege's contemporaries.

Here is Weierstrass in Winter Semester 1876:

> "When two series of homogenous elements are compared with each other, so that one assigns, as far as possible, to each element of the first series an element of the second series, then there are three possibilities. First, by this operation no element of the second remains left over and each element of the first corresponds to one of the second; then one says that the number quantity represented through the second series is *equal* to that represented through the first [...]"

The logicist' idea is that arithmetical knowledge is grounded in logical knowledge, *viz.* Frege on mathematical induction:

"One will be able to see from this essay that even inferences which are apparently particular to mathematics, like the inference from n to $n + 1$, are based on general logical laws, so that they do not require particular laws of aggregative thought" (*Grundlagen* p. iv)

Wright is an important contemporary advocate of logicism:

"Anyone who accepts the Peano axioms as truths 'not of our making' must recognise the question of what account should be given of our ability to apprehend their truth. If Frege's attempt to ground that apprehension in pure logic were to succeed, we should have an answer [. . .] " (*Frege's Conception of Numbers as Objects* p. 131, cf. p. xiv).

One way to make out connection between HP and arithmetic is:

The Logicist Template

Base Premise: Hume's principle is known.

Interpretability Premise: It is known that the Peano axioms are interpretable in Hume's Principle.

Preservation Premise: If it is known that principles P are interpretable in principles P^*, and principles P^* are known, then principles P are known.

Conclusion: The Peano axioms are known.

Main Question: is the interpretability premise sensitive to the distinction between predicative and impredicative versions of HP and Peano axioms?

By predicative versions of HP and Peano axioms, I mean: these theories but with comprehension replaced by Δ^1_1-comprehension:

$$[\forall\, a\, \varphi(a) \leftrightarrow \psi(a)] \rightarrow [\exists\, F\, \forall\, a\, (Fa \leftrightarrow \varphi(a))]$$

where $\varphi(x)$ is Σ^1_1 and $\psi(x)$ is Π^1_1; this means: both have only one property quantifier, $\varphi(x)$'s existential, $\psi(x)$'s universal.

Intuitively, predicative HP and predicative arithmetic differ from impredicative versions only by being more circumspect in the kinds of properties they suppose to exist.

Main Answer: predicative interpretability premise is false, e.g. predicative arithmetic is *not* interpretable in predicative HP.

First, we show that, *given* Frege's particular definition of zero, successor, and number, the system of predicative HP does not prove the successor axiom, e.g. that every number has a successor.

Second, we show that *no* definitions of number in predicative HP can yield predicative arithmetic, i.e. there is no interpretation.

The idea in both is to build models of predicative HP of the form:

$$(M, D(M), D(M^2), \ldots, \#)$$

where M is L-structure with definable sets $D(M^n) \subseteq P(M^n)$.

Definition (Uniformly Definable Maps)

Let M be an L-structure, with definable subsets $D(M^n) \subseteq P(M^n)$.

Then $\# : D(M) \rightarrow M$ is uniformly definable if for all L-formula $\theta(x, \overline{y})$ with all free variables displayed and with non-empty parameter tuple $\overline{y}$, there is L-formula $\theta'(x, \overline{y})$ in same variables such that:

$$\overline{a}, b \in M \Longrightarrow [\#(\theta(M, \overline{a})) = b \Longleftrightarrow M \models \theta'(b, \overline{a})]$$

Here (and in what follows): $\theta(M, \overline{a}) = \{c \in M : M \models \theta(c, \overline{a})\}$.

Intuitively, $\#(M) \rightarrow M$ uniformly definable yields motto:

"If you know definition of X, then you know definition of $\#(X)$."

Metatheorem (Generalized Barwise-Schlipf/Ferreira-Wehmeier)

Let M be an L-structure, with definable subsets $D(M^n) \subseteq P(M^n)$.

Suppose $\# : D(M) \to M$ is B-computably uniformly definable,

where $B \in 2^\omega$; and suppose M is B-computably saturated.

Then $N = (M, D(M), D(M^2), \ldots, \#)$ models Δ^1_1-comprehension.

. . . The idea is that saturation just means M^n is definably compact.

To show that Δ^1_1-definable $Z \subseteq M^n$ is in $D(M^n)$,

note that $Z, M^n \setminus Z$ being Σ^1_1 yield cover of M^n by a computable class of definable subsets, which hence has a finite subcover.

First Application: Successor Axiom is Unprovable from Predicative HP, Part 1

First, we show that, *given* Frege's particular definition of zero, successor, and number, the system of predicative HP does not prove the successor axiom, e.g. that every number has a successor.

Consider the structure of the complex numbers $(\mathbb{C}, +, \times)$.

Turns out that every definable subset of $\mathbb{C}$ is finite or cofinite.

Set $\#X = k$ if $|X| = k$, and $\#X = -(k+1)$ if $|\mathbb{C} \setminus X| = k$.

By saturation, the map $\# : D(\mathbb{C}) \to \mathbb{Z} \subseteq \mathbb{C}$ is uniformly definable.

One can show, using the metatheorem and Ax's Theorem, that

$$(\mathbb{C}, D(\mathbb{C}), D(\mathbb{C}^2), \ldots, \#) \text{ models predicative HP.}$$

Recall Frege's definition of zero, successor, and number:

- $S(n, m)$ iff $\exists\, X, Y\ \#X = n\ \&\ \#Y = m\ \&\ \exists\, y \in Y\ X = Y \setminus \{y\}$
- F is *hereditary* iff Fn and $S(n, m)$ implies Fm
- F is *closed* iff $S(\#\emptyset, m)$ implies Fm
- n is a *number* iff $n = \#\emptyset$ or contained in all hereditary, closed F.

Recall: $\#X = k$ if $|X| = k$, and $\#X = -(k+1)$ if $|\mathbb{C} \setminus X| = k$.

Hence: $S(0, 1), S(1, 2), S(2, 3), \ldots, S(-2, -1), S(-3, -2), \ldots$

Recall: $\#X = k$ if $|X| = k$, and $\#X = -(k+1)$ if $|\mathbb{C} \setminus X| = k$.

Hence: $S(0,1), S(1,2), S(2,3), \ldots, S(-2,-1), S(-3,-2), \ldots$

Proposition. In $(\mathbb{C}, D(\mathbb{C}), D(\mathbb{C}^2), \ldots, \#)$, -1 is a number.

Suppose F is hereditary, closed. Suffices to show that $-1 \in F$.

Since F is hereditary closed, it is infinite and hence cofinite.

Then for sufficiently large positive number k, we have $-k \in F$.

Suppose e.g. $-3 \in F$. Then $S(-3,-2), S(-2,-1)$ imply $-1 \in F$.

. . . so we are done: -1 is a number with no successor.

Second, we show that *no* definitions of number in predicative HP can yield predicative arithmetic, i.e. there is no interpretation.

Consider recursively saturated extension $(R, +, \times)$ of real numbers.

Every definable set is finite disjoint union of points, open intervals.

That is $X = (a_1, b_1) \sqcup \cdots \sqcup (a_n, b_n) \sqcup \{c_1\} \sqcup \cdots \sqcup \{c_m\}$

Let $\dim(X) = 1$ if X contains an interval, $\dim(X) = 0$.

Let $E(X) = m - n$, number of points minus number of intervals.

X, Y definably bijective iff $\dim(X) = \dim(Y)$ and $E(X) = E(Y)$.

Example: consider $X = (-2, -1) \sqcup \{0\} \sqcup (1, 2)$ and $Y = (-1, 1)$

Then $E(X) = 1 - 2 = -1$ and $E(Y) = 0 - 1 = -1$, and definable bijection given by $(-2, -1) \mapsto (-1, 0)$ and $(1, 2) \mapsto (0, 1)$.

How does this yield that: no definitions of number in predicative HP can yield all the axioms of predicative arithmetic?

Well . . . we can prove all the facts on previous slide in predicative arithmetic. Hence, using metatheorem (which is also provable in predicative arithmetic), we can show that predicative arithmetic shows that there is a model N of predicative HP.

Suppose, for sake of contradiction, that some definition of number in predicative HP yielded all axioms of predicative arithmetic, then the model N would define a model M of predicative arithmetic.

But this would mean that we proved the consistency of predicative arithmetic inside predicative arithmetic, and this in turn would contradict . . . Gödel's second incompleteness theorem.

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Main Answer: predicative interpretability premise is false, e.g. predicative arithmetic is *not* interpretable in predicative HP.

First, we showed that, *given* Frege's particular definition of zero, successor, and number, the system of predicative HP does not prove the successor axiom, e.g. that every number has a successor.

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HP is but one of many so-called abstraction principles:

$$X \sim Y \iff \partial(X) = \partial(Y)$$

generated by equivalence relations $\sim$ on properties.

The natural initial question to ask at this point is the following:

Is there a predicative abstraction principle that interprets predicative arithmetic?

A positive answer to this question would allow the logicist to complete her project, at least if all such abstraction principles are thought to be epistemically on par.

A negative answer to the question would underscore the need to clarify the epistemic status of impredicativity.