Kurt Gödel Legacy and the Current Trends in Logic

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Vienna, April 29, 2011

Gödel's Legacy in Action

According to Google Scholar of April 29, 2011, the list of ten most cited papers in mathematical logic published after 2000 in

Annals of Pure and Applied Logic
The Journal of Symbolic Logic
The Bulletin of Symbolic Logic
Archive for Mathematical Logic
Mathematical Logic Quarterly
Fundamenta Mathematicae

includes direct descendants of two Gödel's papers, and these papers are:

Gödel's Legacy in Action

Eine Interpretation des intuitionistischen Aussagenkalkuls. *Ergebnisse Math. Kolloq.*, 14: 39–40, 1933.

Über eine bisher noch nicht benütztwe Erweiterung des finiten Standpunktes, *Dialectica*, 12: 280–287, 1958.

The latter has been already covered beautifully in Ulrich Kohlenbach's talk yesterday. So, we will concentrate on the former.

BHK semantics

The intended semantics of intuitionistic logic is the semantics of proofs, also known as Brouwer-Heyting-Kolmogorov (BHK) semantics.

- a proof of $A \wedge B$ consists of a proof of A and a proof of B,
- a proof of $A \vee B$ is given by presenting either a proof of A or a proof of B,
- a proof of $A \to B$ is a construction which, given a proof of A, returns a proof of B,
- a proof of $\forall x A(x)$ is a function converting c into a proof of $A(c)$,
- a proof of $\exists x A(x)$ is a pair (c, d) where d is a proof of $A(c)$.

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Brouwer, Heyting: intuitionistic foundations
Kolmogorov: classical proofs (problem solutions) to interpret intuitionistic logic
Gödel (1933): classical provability to interpret intuitionistic logic

Gödel's embedding

In 1933 Gödel embedded IPC into modal logic $\mathsf{S4}$, viewed as a modal logic for classical provability, in a way that respects the informal provability reading of $\mathsf{S4}$:

$$\mathsf{IPC} \vdash F \quad \textit{iff} \quad \mathsf{S4} \vdash tr(F),$$

where $tr(F)$ is obtained from F by prefixing each subformula of F with $\Box$. When parsing Gödel's translation $tr(F)$ of some formula F, we encounter a provability modality before each subformula, which forces us to read said subformula as provable rather then true. Therefore, Gödel's translation reflects the fundamental intuitionistic paradigm that intuitionistic truth is provability. Gödel's and Kolmogorov's approach views intuitionstic truth as *classical provability* thus making this version of BHK a non-circular semantics for intuitionistic logic. A similar position was taken by P.S. Novikov in his book "Constructive mathematical logic from the viewpoint of the classical one" (in Russian).

Logic of Proofs as BHK

At that stage, the problem of finding a BHK-type semantics of proof for IPC was reduced to developing such a semantics for $\mathsf{S4}$. The next step was taken in the propositional Logic of Proofs LP with new atoms $t{:}F$ for

$$t \textit{ is a proof of } F$$

was introduced. The Realization Theorem demonstrated that each $\mathsf{S4}$ theorem conceals an explicit statement about proofs, e.g.,

$$\Box F \rightarrow \Box G$$

reads as

$$u{:}F \rightarrow t(u){:}G$$

i.e., *if* u *is a proof of* F, *then* $t(u)$ *is a proof of* G, for an appropriate proof term $t(u)$. The Realization Theorem allows for the extension of this kind of explicit reading of modalities to all theorems of $\mathsf{S4}$, so $\mathsf{S4}$ has a semantics of LP proofs as anticipated by Gödel. Since proof terms in LP can be naturally interpreted as mathematical proofs, e.g., in Peano Arithmetic PA, $\mathsf{S4}$ and IPC received an exact provability semantics consistent with BHK-requirements.

Lessons to learn from LP

Proofs are represented in LP by *proof terms* constructed from *proof variables* and *proof constants* by means of functional symbols for elementary computable operations on proofs, binary $\cdot$, $+$, and unary $!$. The formulas of LP are the usual propositional formulas and those of the form $t{:}F$ where t is a *proof term* and F is a formula. The operations of LP are specified by the following schemas:

$$t{:}(A \rightarrow B) \rightarrow (s{:}A \rightarrow (t \cdot s){:}B) \qquad \textit{application}$$

$$t{:}A \rightarrow (t+s){:}A, \quad s{:}A \rightarrow (t+s){:}A \qquad \textit{choice}$$

$$t{:}A \rightarrow\ !t{:}t{:}A \qquad \textit{proof checker.}$$

LP is axiomatized over the classical propositional calculus by the above schemas, the principle

$$t{:}A \rightarrow A \qquad \textit{reflexivity}$$

and the *axiom necessitation rule*, which allows for the specification of proof constants as proofs of the concrete axioms

$$\vdash c{:}A, \quad \text{where } c \text{ is an axiom constant, } A \text{ is an axiom of } \mathsf{LP}.$$

Lessons to learn from LP

The intended semantics for LP is provided by proof predicates in Peano Arithmetic PA. The proof terms of the LP-language are interpreted by codes of arithmetical derivations. Operations $\cdot$, $+$, and unary ! become total recursive functions on such codes. Formulas of LP are interpreted by closed arithmetical formulas; and $t{:}F$ is interpreted by an arithmetical proof predicate in PA. LP is complete with respect to such provability semantics.

The following Realization Theorem shows that LP is an exact counterpart of Gödel's provability logic S4.

> *A modal formula F is provable in* S4 *iff there exists an assignment (called a "realization") of proof terms to all occurrences of $\Box$ in F such that the resulting formula is provable in* LP.

The proof of the Realization Theorem treats $\Box$ in the style of Skolem as the existential quantifier on proofs. Negative occurrences of $\Box$'s are assumed to hide universal quantifiers and hence are realized by proof variables, and positive occurrences of $\Box$'s are realized as existential quantifiers, i.e., by proof terms depending on these variables.

The Realization Theorem provides S4, and therefore IPC, with the exact BHK-style provability semantics, thus completing Gödel's project of 1933.

Kleene, Martin-Löf

Kleene realizability disclosed a fundamental **computational content** of intuitionistic derivations which is however quite different from the provability semantics.

Martin-Löf: comprehensive **computational semantics** of intuitionistic derivations - not semantics of proofs...

BHK: proofs vs. programs

Kleene realizers are programs rather then proofs. The predicate *r realizes F* is not decidable whereas *p proves F* is decidable.

Realizability logic is different from the logic of proof and from IPC.

Computational semantics does not fit to the original BHK; the well-known disjunction clause for realizability:
a proof of P v *Q is a pair <a,b> where a is 0 and b is a proof of P, or a is 1 and b is a proof of Q*
is a **doctored version** of the original BHK clause.

Kleene in 1945 did not mention BHK and later denied any connection of his realizability with BHK interpretation.

Proof-based BHK

Proof-based BHK relates logic to epistemology. The celebrated account of

Knowledge as Justified True Belief

commonly attributed to Plato, has been a subject a broad discussion in epistemology. The standard epistemic modal logic has represented the True Belief components of Plato's analysis. However, the notion of justification, which has been an essential component of epistemic studies, was conspicuously absent in the mathematical models of knowledge within the epistemic logic framework.

Logic of Proofs and its successor, Justification Logic, supply the missing third component of Plato's characterization.

Quantification and LP

The arithmetical provability semantics for the Logic of Proofs LP, naturally generalizes to a first-order version with conventional quantifiers, and to a version with quantifiers over proofs. In both cases, axiomatizability questions were answered negatively.

> *The first-order logic of proofs is not recursively enumerable* (Artemov & Yavorskaya, 2001. *The logic of proofs with quantifiers over proofs is not recursively enumerable* (Yavorsky 2001).

Earlier this year, Artemov & Yavorskaya found the first-order logic of proofs FOLP capable of realizing first-order modal logic FOS4 and, therefore, the first-order intuitionistic logic HPC. Two kinds of proof semantics for FOLP have been offered: *parametric semantics*, in which proof objects are interpreted as derivations with parameters, and *generic semantics* with proof terms interpreted as provably computable functions from parameters to formal derivations. Both provide semantics of proofs for first-order S4 and a first-order Brouwer-Heyting-Kolmogorov-style semantics for HPC.

FOS4 may be viewed as a general purpose first-order justification logic; it opens the door to a general theory of first-order justification.

First-order LP: format

In the language FOLP, the proof predicate is represented by formulas of the form

$$t{:}_X A$$

where X is the set of individual variables that are considered global parameters. Variables from X and only them are free in $t{:}_X A$. All occurrences of variables from X that are free in A are also free in $t{:}_X A$. All other free variables of A are considered local and hence bound in $t{:}_X A$.

Proofs are represented by proof terms which do not contain individual variables. An arithmetical interpretation $*$, commutes with the Boolean connectives and quantifiers and

$$(t{:}_X F)^* = \mathit{Prof}\,(t^*(\underline{X}), F^*(\underline{X})),$$

i.e., $(t{:}_X F)^*$ is evaluated by the natural arithmetical formula asserting that t is a proof of F with global variables X.

First-order LP: axioms

FOLP is axiomatized by the following schemas. Here A, B are formulas, s, t are terms, X is a set of individual variables, and y is an individual variable.

A1 classical axioms of first-order logic

A2 $t{:}_{Xy}A \rightarrow t{:}_X A$, $\quad y$ is not free in A

A3 $t{:}_X A \rightarrow t{:}_{Xy}A$

B1 $t{:}_X A \rightarrow A$

B2 $s{:}_X(A \rightarrow B) \wedge t{:}_X A \rightarrow (s \cdot t){:}_X B$

B3 $t{:}_X A \rightarrow (t+s){:}_X A$, $\quad s{:}_X A \rightarrow (t+s){:}_X A$

B4 $t{:}_X A \rightarrow !t{:}_X t{:}_X A$

B5 $t{:}_X A \rightarrow \Box_x(t){:}_X \forall x A$, $x \notin X$

FOLP has the following inference rules:

R1 $\vdash A, A \rightarrow B \ \rightarrow \ \vdash B$ *Modus Ponens*

R2 $\vdash A \ \rightarrow \ \vdash \forall x A$ *generalization*

R3 $\vdash c{:}A$, where A is an axiom, c is a proof constant *axiom necessitation.*

First-order LP: example

Deriving an explicit converse Barcan Formula $\Box\forall xA \rightarrow \forall x\Box A$.

1. $\forall xA \rightarrow A$ - logical axiom;
2. $c{:}(\forall xA \rightarrow A)$ - axiom necessitation;
3. $c{:}_{\{x\}}(\forall xA \rightarrow A)$ - from 2, by axiom A3;
4. $c{:}_{\{x\}}(\forall xA \rightarrow A) \rightarrow (u{:}_{\{x\}}\forall xA \rightarrow (c\cdot u){:}_{\{x\}}A)$ - axiom B2;
5. $u{:}_{\{x\}}\forall xA \rightarrow (c\cdot u){:}_{\{x\}}A$ - from 3, 4, by Modus Ponens;
6. $u{:}\forall xA \rightarrow u{:}_{\{x\}}\forall xA$ - by axiom A3;
7. $u{:}\forall xA \rightarrow (c\cdot u){:}_{\{x\}}A$ - from 5, 6;
8. $\forall x[u{:}\forall xA \rightarrow (c\cdot u){:}_{\{x\}}A]$ - from 7, by generalization;
9. $u{:}\forall xA \rightarrow \forall x(c\cdot u){:}_{\{x\}}A$ - since x is not free in the antecedent.

FOS4 = projection of FOLP

Corollary 1 FOS4 *is the forgetful projection of* FOLP.

Corollary 2 F *is derivable in* HPC *if and only if its Gödel translation is realizable in* FOLP.

Example 1 Consider formula

$$\neg\forall x A(x) \rightarrow \exists x \neg A(x) \quad \text{where } A(x) \text{ is atomic.} \tag{1}$$

This is not derivable in intuitionistic first-order logic HPC. Its Gödel translation (in an equivalent simplified form $(\cdot)^\circ$, cf. [18], Section 9.2.1) is

$$\Box\neg\Box\forall x A(x) \rightarrow \exists x \Box\neg\Box A(x). \tag{2}$$

By Corollary 2, modal formula (2) is not realizable in FOLP.

Example

Consider intuitionistic theorem

$$\exists x A(x) \rightarrow \neg\forall x \neg A(x) \quad \text{(where } A(x) \text{ is atomic).}$$

Its simplified Gödel translation is

$$\Box\exists x \Box A(x) \rightarrow \neg\Box\forall x \neg\Box A(x),$$

which is provable in FOS4. By the Realization Theorem, there is its realization provable in FOLP. We leave it as an exercise to derive in FOLP the following realization

$$u{:}\exists x v{:}_{\{x\}} A(x) \rightarrow \neg w{:}\forall x \neg v{:}_{\{x\}} A(x).$$

It is easy to see that with $F = \forall x \neg v :_{\{x\}} A(x)$, the latter formula states $u{:}\neg F \rightarrow \neg w{:}F$ which is obviously provable in FOLP.

Example: Barcan formula

Provability semantics: reflexivity is a major problem, e.g., $\mathit{Proof}(x, \bot) \to \bot$ cannot be provable, since then $\forall x[\neg \mathit{Proof}(x, \bot)]$ would be provable, but the latter is a consistency formula. So a special theory of how to interpret proof terms t in $t{:}_X\ F$ was developed.

The principal example is the Barcan formula

$$\forall x \Box A \to \Box \forall x A,$$

which is not a provability tautology: take A to be $\neg \mathit{Proof}(x, \bot)$.

We offer a provability semantics for FOLP which does exactly this.

Conclusions

On the theoretical side, FOLP answers a cluster of long standing foundational questions, e.g., a BHK semantics for first-order intuitionistic logic, a provability semantics for first-order S4, a general logic of proofs and propositions.

In addition, FOLP may be viewed as a general purpose justification logic; it opens the door to a general theory of first-order justification in which we anticipate a variety of FOLP-like systems equipped with appropriate epistemic semantics.

Thank You!